CHAPTER II

UNSTEADY FREE CONVECTIVE MHD FLOW OF A VISCO-ELASTIC FLUID PAST AN INFINITE VERTICAL PLATE WITH CONSTANT SUCTION AND HEAT SINK

2.1 Introduction

Convective heat transfer past an infinite plate is a great interest of subjects motivated by several important applications in geophysics, astrophysics and fluid engineering. The unsteady free convective flow past an infinite plate and heat sources has been studied by Pop and Soundalgekar (1974). Singh and Cowling (1963) have studied the effect of magnetic field on free convective flow of electrically conducting fluids past semi-infinite plate. An exact solution for the unsteady MHD problem has been investigated by Sacheti, Chandran and Singh (1994). MHD free convective flow with hall current in a porous medium for electrolytic solution (viz. salt water) has been studied by Sattar and Alam (1995). Sahoo et al. (2003) have studied the heat transfer in mercury (Pr=0.025) and electrolytic solution (Pr=1.0) past an infinite porous plate with constant suction in the presence of uniform transverse magnetic field and heat sink.

In this chapter, the unsteady free convective magnetohydrodynamic flow of a visco-elastic, electrically conducting fluid past an infinite vertical porous plate with constant suction and heat absorbing sinks has been investigated. Approximate solutions of the equations governing the flow have been derived for the velocity and temperature fields, mean skin friction and mean rate of heat transfer by using multi-parameter perturbation technique. The profiles of mean velocity, the mean temperature, the temperature and the skin friction have been presented graphically for different values of the visco-elastic parameter. The mean skin friction and the mean rate of heat transfer with corresponding amplitudes and phases have been numerically worked out for

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different values of the parameters involved in the solution to observe the effects of visco-elastic parameter encountered in the problem under investigation.

2.2 Mathematical Analysis

Consider the unsteady flow of an incompressible second-order fluid past an infinite vertical porous plate. The flow is considered in the presence of a uniform magnetic field with constant suction and heat sink. The \( \bar{x} \)-axis is taken in the vertically upward direction (along the infinite vertical plate) and \( \bar{y} \)-axis normal to it. Denoting velocity components \( \bar{u}, \bar{v} \) in the directions \( \bar{x}, \bar{y} \) respectively and temperature by \( \bar{T} \) and neglecting the induced magnetic field of strength \( B_0 \), the flow is governed by the following equations under the usual Boussinesq’s approximation:

Equation of Continuity:

\[
\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad (2.2.1)
\]

i.e.

\[
\bar{v} = -\bar{v}_0 \text{ (constant)} \quad (2.2.2)
\]

Momentum equation:

\[
\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = g\beta(\bar{T} - \bar{T}_0) - \frac{\sigma B_0^2}{\rho} \bar{u} + \nu_1 \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \nu_2 \left[ \frac{1}{4} \frac{\partial^2}{\partial \bar{y}^2} \left( \frac{\partial \bar{u}}{\partial \bar{t}} \right) + \bar{v} \frac{\partial^3 \bar{u}}{\partial \bar{y}^3} \right] \quad (2.2.3)
\]

Energy equation:

\[
\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v} \frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\kappa_0}{\rho C_p} \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{S}{C_p} (\bar{T} - \bar{T}_0) + \nu_1 \frac{\partial \bar{u}}{\partial \bar{y}} \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right) - \frac{\nu_2}{2} \left( \frac{\partial \bar{u}}{\partial \bar{y}} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \bar{v} \frac{\partial^3 \bar{u}}{\partial \bar{y}^3} \right) \quad (2.2.4)
\]

where \( g \) is the acceleration due to gravity, \( \bar{T} \) is the temperature of the fluid, \( \nu_1, \nu_2 \) are the coefficients of viscosity and elasticity respectively, \( \beta \) is the coefficient of the volume expansion of heat transfer, \( \kappa_0 \) is the thermal diffusivity, \( S \) is the heat sink parameter, \( \sigma \) is the electric conductivity and \( \rho \) is the density of the fluid.

The relevant boundary conditions are
\( \ddot{u} = 0, \ddot{v} = -\ddot{v}_0, \ddot{T} = \ddot{T}_\infty + \varepsilon (\ddot{T}_\omega - \ddot{T}_\infty) e^{i\omega t} \) at \( y = 0 \)

\( \ddot{u} = 0, \ddot{T} = \ddot{T}_\infty \) as \( y \to 0 \) \hspace{1cm} (2.2.5)

Here \( \ddot{v}_0 > 0 \) is a constant and the negative sign indicates that the suction is towards the plate.

### 2.3 Method of solution

Let us introduce the following non-dimensional variables and parameters

\[
y = \frac{\ddot{y} \ddot{v}_0}{\nu_1}, \quad t = \frac{\ddot{t} \ddot{v}_0^2}{4 \nu_1}, \quad u = \frac{\ddot{u}}{\ddot{v}_0}, \quad T = \frac{\ddot{T} - \ddot{T}_\infty}{\ddot{T}_\omega - \ddot{T}_\infty}, \quad P_r = \frac{\nu_1}{\kappa}, \quad G_r = \frac{\nu_1 g \beta (\ddot{T}_\omega - \ddot{T}_\infty)}{\ddot{v}_0^3}, \quad \quad S = \frac{4 \ddot{S} \nu_1}{\ddot{v}_0^2}, \quad E_c = \frac{\ddot{E}_c}{\ddot{c}_p (\ddot{T}_\omega - \ddot{T}_\infty)}, \quad M = \frac{\ddot{M} \nu_1}{\ddot{v}_0^2}, \quad \alpha = \frac{\ddot{\alpha} \ddot{v}_0^2}{\nu_1} \quad (2.3.1)
\]

where \( P_r, G_r, S, E_c, M \) and \( \alpha \) are respectively the Prandtl number, Grashof number, sink strength, Eckert number, Hartmann number and visco-elastic parameter.

Introducing the non-dimensional variables (2.3.1), the equations (2.2.3) and (2.2.4) become

\[
\frac{1}{4} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial y} = G_r T - M u + \frac{\partial^2 u}{\partial y^2} + \alpha \left[ \frac{1}{4} \frac{\partial^2}{\partial y^2} \left( \frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial y^3} \right) \right] \quad (2.3.2)
\]

and

\[
\frac{P_r}{4} \frac{\partial T}{\partial t} - P_r \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} + \frac{P_r S}{4} T + P_r E_c \left[ \frac{\partial u}{\partial y} \right]^2 + \alpha \left[ \frac{1}{4} \frac{\partial}{\partial y} \frac{\partial^2 u}{\partial y \partial t} + \frac{\partial u}{\partial y} \frac{\partial^3 u}{\partial y^3} \right] \quad (2.3.3)
\]

subject to the boundary conditions

\[
u = 0, \quad T = 1 + \varepsilon e^{i\omega t} \quad \text{at} \quad y = 0
\]

and

\[
u \to 0, \quad T \to 0, \quad \text{as} \quad y \to \infty \quad (2.3.4)
\]

To solve the equations (2.3.2) and (2.3.3), we assume \( \varepsilon \) to be very small, and write the velocity and temperature in the neighbourhood of the plate as

\[
u(y, t) = \nu_0(y) + \varepsilon e^{i\omega t} \nu_1(y),
\]

\[
T(y, t) = T_0(y) + \varepsilon e^{i\omega t} T_1(y)
\]

\hspace{1cm} (2.3.5)
Putting (2.3.5) in the equations (2.3.2) and (2.3.3), equating harmonic and non-harmonic terms and neglecting the coefficients of \( \varepsilon^2 \), we get the equations

\[
\begin{align*}
\alpha u_0'' - u_0'' - u_0' + Mu_0 &= G_r T_0, \\
\alpha u_1'' - \left(1 + \frac{i \omega}{4} \alpha \right)u_1'' - u_1' + \left(\frac{i \omega}{4} + M\right)u_1 &= G_r T_1,
\end{align*}
\]

\[
T_0'' + P_r T_0' + \frac{P_c S}{4} T_0 = -P_r E_c \left[\left(\frac{\partial^2 u_0}{\partial y^2}\right)^2 + \alpha \frac{\partial u_0}{\partial y} \frac{\partial^2 u_0}{\partial y^2}\right],
\]

\[
T_1'' + P_r T_1' - \frac{P_c}{4} (i \omega - S) T_1 = -P_r E_c \left[2 \frac{\partial u_0}{\partial y} \frac{\partial^2 u_1}{\partial y^2} + \alpha \left\{ \frac{\partial u_0}{\partial y} \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_0}{\partial y^2} \frac{\partial u_1}{\partial y} \right\} \right]
\]

(2.3.6)

subject to the boundary conditions

\[
u_0 = 0 = u_1, \quad T_0 = 1 = T_1 \quad \text{at} \quad y = 0
\]

and \( u_0, \ u_1, \ T_0, \ T_1 \to 0 \) as \( y \to \infty \) (2.3.7)

Using multi-parameter perturbation technique and assuming \( E_c \approx \varepsilon_c \ll 1 \), we write

\[
u_0 = u_{00} + \varepsilon_c u_{01}, \quad T_0 = T_{00} + \varepsilon_c T_{01};
\]

and

\[
u_1 = u_{10} + \varepsilon_c u_{11}, \quad T_1 = T_{10} + \varepsilon_c T_{11}
\]

(2.3.8)

Using (2.3.8) in the equation (2.3.6) and equating the coefficients of \( \varepsilon_c \) and \( \varepsilon_c^2 \), we get the following sets of differential equations for \( u_{00}, \ u_{10}, \ T_{00}, \ T_{10} \) and \( u_{01}, \ u_{11}, \ T_{01}, \ T_{11} \)

\[
\alpha u_{00}'' - u_{00}'' - u_{00}' + Mu_{00} = G_r T_{00},
\]

\[
\alpha u_{10}'' - \left(1 + \frac{i \omega}{4} \alpha \right)u_{10}'' - u_{10}' + \left(\frac{i \omega}{4} + M\right)u_{10} = G_r T_{10},
\]

\[
T_{00}'' + P_r T_{00}' + \frac{P_c S}{4} T_{00} = 0,
\]

\[
T_{10}'' + P_r T_{10}' - \frac{P_c}{4} (i \omega - S) T_{10} = 0.
\]

(2.3.9)
and \[ au_{01}'' - u_{01}'' - u_{01} + Mu_{01} = G, T_{01}, \]
\[ au_{11}'' - (1 + \frac{i\omega}{4} \alpha)u_{11}'' - u_{11}' + (\frac{i\omega}{4} + M)u_{11} = G, T_{11}, \]
\[ T_{01}'' + P_{r}T_{01}' + \frac{P_{r}S}{4} T_{01} = -P_{r} \left[ (\frac{\partial u_{00}}{\partial y})^2 + \alpha \frac{\partial^2 u_{00}}{\partial y^2} \right], \]
\[ T_{11}'' + P_{r}T_{11}' - \frac{P_{r}}{4} (i\omega - S)T_{11} = -P_{r} \left[ (\frac{\partial u_{00}}{\partial y})^2 + \alpha \frac{\partial^2 u_{00}}{\partial y^2} \right] \] (2.3.10)

with the corresponding boundary conditions

\[ u_{00} = u_{01} = u_{10} = u_{11} = 0, \ T_{00} = 1 = T_{10}, T_{01} = T_{11} = 0 \text{ at } y = 0 \]

and \[ u_{00}, u_{01}, u_{10}, u_{11}, T_{00}, T_{01}, T_{10}, T_{11} \rightarrow 0 \text{ as } y \rightarrow \infty \] (2.3.11)

Solving the differential equations (2.3.9) and (2.3.10) with the aid of the corresponding boundary conditions (2.3.11) and then substituting the values in the relations (2.3.8), we obtain the mean velocity \( u_0 \) and mean temperature \( T_0 \) as well as \( u_1 \), \( T_1 \) as

\[ u_0 = \frac{G}{a_1^2(1 + 4M)(a_1^2 + a_1 - 1)} \left[ A_1 e^{\alpha y} - A_2 e^{-\alpha y} + A_3 e^{-2\alpha y} \right] \]
\[ + \frac{4P_{r}e_1^2 G}{a_1^2(1 + 4M)^2(a_1^2 + a_1 - 1)} \left[ A_4 e^{\alpha y} - A_5 e^{\alpha y} + A_6 e^{\alpha y} - A_7 e^{-2\alpha y} - A_8 e^{-2\alpha y} + A_9 e^{\alpha y} - A_{10} e^{-(\alpha + \alpha) y} - A_{11} e^{-\alpha y} \right] \] (2.3.12)

\[ T_0 = e^{-\alpha y} + \frac{4P_{r}e_1^2 G}{a_1^2(1 + 4M)^2(a_1^2 + a_1 - 1)} \left[ B_1 e^{\alpha y} - B_2 e^{-\alpha y} \right. \]
\[ - B_3 e^{-2\alpha y} - B_4 e^{-2\alpha y} - B_5 e^{\alpha y} - B_6 e^{-2\alpha y} - B_7 e^{\alpha y} \] (2.3.13)

\[ u_1 = \frac{G}{a_1(1 + 4M)(a_1^2 + a_1 - 1)} \left[ C_1 e^{\alpha y} - C_2 e^{-\alpha y} + C_3 e^{-2\alpha y} \right] \]
\[ + \frac{2P_{r}e_1^2 G}{a_1 a_1(1 + 4M)(a_1^2 + a_1 - 1)(a_1^2 + a_1 - 1)(1 + 4M)(1 + 4M)} \left[ C_4 e^{\alpha y} + C_5 e^{\alpha y} + C_6 e^{-\alpha y} \right. \]
\[ + C_7 e^{\alpha y} + C_8 e^{\alpha y} + C_9 e^{-(\alpha + \alpha) y} - C_{10} e^{-(\alpha + \alpha) y} - C_{11} e^{-\alpha y} - C_{12} e^{-\alpha y} \] (2.3.14)
\[ T_1 = e^{-\alpha\gamma} + \frac{2P_2G_2}{\alpha a_a(1+M)\left(\alpha a_a^2 + \alpha - 1\right)} \left[ D_1 e^{-\alpha\gamma} \right] \]

\[ + D_2 e^{-\alpha\gamma} + D_3 e^{-\alpha\gamma} + D_4 e^{-\alpha\gamma} - D_5 e^{-\alpha\gamma} - D_6 e^{-\alpha\gamma} + D_7 e^{-\alpha\gamma} + D_8 e^{-\alpha\gamma} + D_9 e^{-\alpha\gamma} + D_10 e^{-\alpha\gamma} \]

where the constants are given as follows:

\[ a_1 = \frac{1}{2} \left[ P_r + \sqrt{P_r^2 - P_S} \right] a_2 = -\frac{M}{1+M}, a_3 = \frac{1}{2} \left[ P_r + \sqrt{P_r^2 - P_r^2(i\omega - S)} \right] \]

\[ a_4 = \frac{i\omega + M}{1+\frac{i\omega}{4}M(1+\frac{i\omega}{4})}, m_1 = \frac{1-\sqrt{1+4\alpha}}{2\alpha}, m_2 = m_3 = \frac{1-\sqrt{1+4\alpha}}{\alpha}, \]

\[ m_4 = \frac{1+\sqrt{1+4\alpha}}{2\alpha} - a_1, A_1 = \frac{1-\sqrt{1+4\alpha}}{2\alpha - \sqrt{1+4\alpha}}, A_2 = \frac{a_1}{a_1 + a_2}, A_3 = A_2 - A_1, \]

\[ b_1 = \frac{(1-\sqrt{1+4\alpha})^2}{2\alpha(2-2a_2 - \sqrt{1+4\alpha})}, b_2 = \frac{a_2(1-\sqrt{1+4\alpha})}{1-2a_2 - \sqrt{1+4\alpha}}, b_3 = \frac{b_2}{(1-2a_2 - \sqrt{1+4\alpha})}, b_3 = \frac{b_2}{4m_2^2 + 4m_2 P_r + P_S}, \]

\[ b_4 = \frac{2b_2a_2}{a_1 + a_2 4m_2^2 + 4m_2 P_r + P_S}, b_3 = 2b_2 \left( \frac{a_1 a_2}{a_1 + a_2} - b_2 \right) \frac{1}{4(m_1 - a_2)^2 + 4(m_1 - a_2)P_r + P_S}, \]

\[ b_5 = \frac{a_1^4}{(a_1 + a_2)^2(16a_1^2 - 8a_1 P_r + P_S)} + \frac{(a_1 a_2 - b_2)^2}{(a_1 + a_2)^2 - 4(a_1 + a_2)P_r + P_S} + b_4 \]

\[ b_6 = b_3 + \frac{a_1^4}{(a_1 + a_2)^2(16a_1^2 - 8a_1 P_r + P_S)} + \frac{(a_1 a_2 - b_2)^2}{(a_1 + a_2)^2 - 4(a_1 + a_2)P_r + P_S} + b_4 \]

\[ b_7 = \frac{ab_2}{1-\sqrt{1+4\alpha}}, b_8 = \frac{2ab_4}{1-2a_1 - \sqrt{1+4\alpha}}, b_9 = a_1 b_7(1 - \frac{1-\sqrt{1+4\alpha}}{\alpha}), \]

\[ b_{10} = b_8(1 + a_1 - \frac{1-\sqrt{1+4\alpha}}{2\alpha}), \]

\[ b_{11} = (1 + a_1)b_6 + b_9 - \frac{(1+2a_2)a_1^4}{2(a_1 + a_2)^2(16a_1^2 - 8a_1 P_r + P_S)} \]

\[ - \frac{a_1}{2a_2} \left( \frac{a_1 a_2}{a_1 + a_2} - b_2 \right)^2 \frac{1+2a_2}{(16a_2^2 - 8a_2 P_r + P_S)} + b_{10} \]

\[ - \frac{2a_1^3}{(a_1 + a_2)^2} \left( \frac{a_1 a_2}{a_1 + a_2} - b_2 \right)^2 \frac{1+a_1 + a_2}{4(a_1 + a_2)^2 - 4(a_1 + a_2)P_r + P_S}, \]

\[ b_{12} = m_1 b_{11}, b_{13} = m_2 b_9, b_{14} = m_3 b_{10}, b_{15} = \frac{2ab_2}{1-2a_2 - \sqrt{1+4\alpha}}, A_4 = b_{15}, \]

\[ b_{16} = \frac{ab_3}{1-2a_2 - \sqrt{1+4\alpha}}, b_{17} = \frac{2ab_4}{1-2a_1 - \sqrt{1+4\alpha}}, \]

(2.3.15)
\[A_5 = \frac{a_i(1 + a_i)}{a_i - a_2} b_{24}, A_6 = b_{16}, A_7 = \frac{(1 + 2a_i)a_i^5}{(a_i + a_2)^2 (2a_i + a_2)(16a_i^2 - 8a_i P_r + P_r S)},\]
\[A_8 = \frac{a_i}{a_i + a_2} - b_2)^2}{1 + 2a_i} \left(\frac{1}{16a_i^2 - 8a_i P_r + P_r S}\right), A_9 = a_i b_17,\]
\[A_{10} = \frac{2a_i^2}{(a_i + a_2)^2} - b_2 = \frac{1 + a_i + a_2}{4(a_i + a_2)^2 - 4(a_i + a_2)P_r + P_r S},\]
\[b_{18} = A_4 - A_5 - A_6 + A_7 + A_8 - A_9 + A_{10}, A_{11} = b_{18},\]
\[b_{19} = \frac{1}{2a} \left[1 + \frac{i \alpha \omega}{4}\right] - \sqrt{\left(1 + \frac{i \alpha \omega}{4}\right)^2 + 4\alpha} b_{20} = b_{19} + a_4, b_{21} = \frac{b_{19}}{b_{20}},\]
\[b_{22} = \frac{a_3}{a_3 - a_4} - b_{21}, b_{23} = \frac{G_i b_{22}}{a_3 [\alpha a_i^2 + (1 + i \alpha \omega) a_i - 1][1 + (\alpha \omega / 4) + M(1 + i \alpha \omega / 4)]},\]
\[n_i = m_i, n_2 = m_i - a_1, n_3 = m_i - a_2, B_i = b_{16}, B_2 = b_3, B_3 = \frac{a_i}{(a_i + a_3)^2 (16a_i^2 - 8a_i P_r + P_r S)},\]
\[B_4 = \frac{a_i}{a_i + a_2} - b_2)^2}{1 + 2a_i} \left(\frac{1}{16a_i^2 - 8a_i P_r + P_r S}\right), B_5 = b_4,\]
\[B_6 = \frac{2a_i^2}{(a_i + a_2)^2} - b_2 = \frac{1}{4(a_i + a_2)^2 - 4(a_i + a_2)P_r + P_r S}, B_7 = b_5, s_1 = b_{19},\]
\[s_2 = m_i + b_{19}, s_3 = m_i - a_3, s_4 = m_i - a_4, s_5 = b_{19} - a_i, C_1 = b_{21}, C_2 = \frac{a_3}{a_3 - a_4}, C_3 = C_2 - b_{21},\]
\[b_{24} = \frac{b_{19} b_{21}}{4(m_i + b_{19})^2 + 4P_r (m_i + b_{19}) - P_r (i \omega - S)} - \frac{b_{19} b_{21}}{4(m_i + b_{19})^2 + 4P_r (m_i - a_4) - P_r (i \omega - S)}\]
\[- \frac{b_i a_3^2}{b_i a_1 b_{19} b_{21}}\]
\[+ \frac{a_3}{a_3 - a_4} (4(m_i - a_3)^2 + 4P_r (m_i - a_3) - P_r (i \omega - S))\]
\[+ \frac{a_3 b_{19} b_{21}}{a_i + a_2} (4(b_{19} - a_i)^2 + 4P_r (b_{19} - a_i) - P_r (i \omega - S))\]
\[+ \frac{a_3 a_i^2}{(a_i + a_2)(a_3 - a_4)} (4(a_i + a_3)^2 + 4P_r (a_i + a_3) - P_r (i \omega - S))\]
\[= \frac{a_3 a_i^2}{(a_i + a_3)} (4(a_i + a_4)^2 + 4P_r (a_i + a_4) - P_r (i \omega - S))\]
\[- \frac{b_{19} b_{21}}{a_i + a_2} (4(b_{19} - a_2)^2 + 4P_r (b_{19} - a_2) - P_r (i \omega - S))\]
\[- \frac{b_{19} b_{21}}{a_i + a_2} (a_3 - a_4) (4(a_2 + a_3)^2 + 4P_r (a_2 + a_3) - P_r (i \omega - S))\]
\[- \frac{b_{19} b_{21}}{a_i + a_2} (a_3 - a_4) (4(a_2 + a_4)^2 + 4P_r (a_2 + a_4) - P_r (i \omega - S))\]
\[+ \frac{a_3}{a_i + a_2} b_{21} \]
\[ D_1 = b_{24}, t_1 = m_1 + b_{19}, t_2 = m_1 - a_1, t_3 = m_1 - a_4, t_4 = b_{19} - a_1, \]
\[ D_2 = \frac{b_1 b_9 b_{21}}{4(m_1 + b_{19})^2 + 4P_r(m_1 + b_{19}) - P_r(i\omega - S)}, \]
\[ D_3 = \frac{b_1 a_1^2}{(a_3 - a_4)\{4(m_1 - a_3)^2 + 4P_r(m_1 - a_3) - P_r(i\omega - S)\}}, \]
\[ D_4 = \frac{b_1 a_4 b_{22}}{4(m_1 - a_4)^2 + 4P_r(m_1 - a_4) - P_r(i\omega - S)}, \]
\[ D_5 = \frac{a_1^2 b_{19} b_{21}}{(a_1 + a_2)\{4(b_{19} - a_1)^2 + 4P_r(b_{19} - a_1) - P_r(i\omega - S)\}}, \]
\[ D_6 = \frac{a_1^2 a_3^2}{(a_1 + a_2)(a_3 - a_4)\{4(a_2 + a_3)^2 + 4P_r(a_2 + a_3) - P_r(i\omega - S)\}}, \]
\[ D_7 = \frac{a_2^2 a_4 b_{22}}{(a_1 + a_2)\{4(a_1 + a_3)^2 + 4P_r(a_1 + a_3) - P_r(i\omega - S)\}}, \]
\[ D_8 = \frac{(a_2 - b_2)}{a_1 + a_2} \frac{a_2^2}{(a_3 - a_4)\{4(a_2 + a_3)^2 + 4P_r(a_2 + a_3) - P_r(i\omega - S)\}}, \]
\[ D_9 = \frac{(a_2 - b_2)}{a_1 + a_2} \frac{a_4 b_{22}}{4(a_2 + a_4)^2 + 4P_r(a_2 + a_4) - P_r(i\omega - S)} \]
\[ b_{25} = \frac{b_1 b_9 b_{21}\{-G_r + (1 + \frac{i\omega\alpha}{4})(m_1 + b_{19})\}}{4(m_1 + b_{19})^2 + 4P_r(m_1 + b_{19}) - P_r(i\omega - S)}, \]
\[ b_{26} = \frac{b_1 a_3^2\{-G_r + (1 + \frac{i\omega\alpha}{4})(m_1 - a_3)\}}{(a_3 - a_4)\{4(m_1 - a_3)^2 + 4P_r(m_1 - a_3) - P_r(i\omega - S)\}}, \]
\[ b_{27} = \frac{b_1 a_4 b_{22}\{G_r - (1 + \frac{i\omega\alpha}{4})(m_1 - a_4)\}}{4(m_1 - a_4)^2 + 4P_r(m_1 - a_4) - P_r(i\omega - S)}, \]
\[ b_{28} = \frac{a_1^2 b_{19} b_{21}\{-G_r + (1 + \frac{i\omega\alpha}{4})(b_{19} - a_1)\}}{(a_1 + a_2)\{4(b_{19} - a_1)^2 + 4P_r(b_{19} - a_1) - P_r(i\omega - S)\}}, \]
\[ b_{29} = \frac{a_1^2 a_3^2\{-G_r + (1 + \frac{i\omega\alpha}{4})(a_1 + a_3)\}}{(a_1 + a_2)(a_3 - a_4)\{4(a_1 + a_3)^2 + 4P_r(a_1 + a_3) - P_r(i\omega - S)\}}, \]
\[ b_{30} = \frac{(a_1^2 - b_2)}{a_1 + a_2} \frac{a_2^2}{(a_3 - a_4)\{4(a_2 + a_3)^2 + 4P_r(a_2 + a_3) - P_r(i\omega - S)\}}, \]
\[ b_{31} = \frac{(a_1^2 - b_2)}{a_1 + a_2} \frac{a_4 b_{22}}{4(a_2 + a_4)^2 + 4P_r(a_2 + a_4) - P_r(i\omega - S)} \]
Putting the values of \( u_0, T_0, u_1 \) and \( T_1 \) and separating real and imaginary parts of velocity and temperature expressions (2.3.5) and taking only the real part, we obtain the velocity and temperature fields in terms of fluctuating parts in the form

\[
u = u_0 + e(M_x \cos \omega t - M_y \sin \omega t)
\]

where the constants are given as follows:

\[
M_x = \frac{G_r}{a_1[(aa_2^3 + a_3 - 1)^2 - \frac{\omega^2 a_2^2}{16}][(1 + M)^2 - (\frac{\omega}{4} + \frac{\omega \alpha}{4} M)^2]}
- \frac{\omega \alpha}{4} \left( \frac{\omega}{4} + \frac{\omega \alpha}{4} M \right) [C_4 e^{-\omega t} + C_2 e^{-\omega \alpha t} + C_3 e^{-\omega \alpha^2 t}]
+ 2\rho R e^2 G_r^2 [(aa_2^3 + a_3 - 1)^2 - \frac{\omega^2 a_2^2}{16}] [(1 + M)^2 - (\frac{\omega}{4} + \frac{\omega \alpha}{4} M)^2]
- \frac{\omega \alpha}{4} \left( \frac{\omega}{4} + \frac{\omega \alpha}{4} M \right) [C_4 e^{-\omega \alpha t} + C_6 e^{-\omega \alpha^2 t} + C_2 e^{-\omega \alpha^3 t} + C_4 e^{-\omega \alpha^3 t} e^{-\omega \alpha t} - C_{10} e^{-\omega t - a_3 \omega \alpha t}]
+ C_{12} e^{-\omega t - a_4 \omega \alpha t} - C_{13} e^{-\omega \alpha t}]
\]
The transient velocity and temperature for \( \omega t = \frac{\pi}{2} \) are given by

\[
T = T_0 + \varepsilon (T_r \cos \omega t - T_i \sin \omega t) \quad (2.3.17)
\]

where

\[
T_r = \frac{2P_1E^2G^2[((a \alpha^2 + a - 1)(1 + M) - \frac{\omega \alpha}{4} (\frac{\omega}{4} + \frac{\omega \alpha}{4} M)]}{a_i a_1 [(a \alpha^2 + a - 1) - \frac{\omega^2 \alpha^2}{16] [(1 + M)^2 - (\frac{\omega}{4} + \frac{\omega \alpha}{4} M)^2]}
\]

\[
T_i = \frac{2P_1E^2G^2[((a \alpha^2 + a - 1)(1 + M) - \frac{\omega \alpha}{4} (\frac{\omega}{4} + \frac{\omega \alpha}{4} M)]}{a_i a_1 [(a \alpha^2 + a - 1) - \frac{\omega^2 \alpha^2}{16] [(1 + M)^2 - (\frac{\omega}{4} + \frac{\omega \alpha}{4} M)^2]}
\]

The transient velocity and temperature for \( \omega t = \frac{\pi}{2} \) are given by

\[
u = u_0 - \varepsilon M \quad \text{and} \quad T = T_0 - \varepsilon T_i \quad (2.3.18)
\]

### 2.4 Skin Friction and Rate of Heat Transfer

The non-dimensional skin friction at the plate is given by

\[
\tau_{\infty} = [u'_0(0) + \alpha u_0^*(0)] + \varepsilon e^{i \alpha}[u'_0(0) + \alpha u_0^*(0)] \quad (2.4.1)
\]

Separating real and imaginary parts of (2.3.16) and taking the real part only, we obtain the mean skin-friction \( \tau_{\infty}^m \) as
\[
\tau_{\omega} = \tau_{\omega}^n + eB|\cos(\omega t + \beta) \tag{2.4.2}
\]

where \(B = \sqrt{B_r^2 + B_i^2} \) and \( \beta = \tan^{-1}\left(\frac{B_r}{B_i}\right) \) are respectively the amplitude and phase of skin-friction and

\[
\tau_{\omega}^n = u_0'(0) + \alpha u_\alpha^2(0)
\]

\[
= \frac{G_r}{a_i(1 + M)(\alpha a_i^2 + a_i - 1)} [m_1A_1 + a_1A_2 - a_2A_3] + \frac{4P_{r_e}G_r}{a_i^3(1 + M)^3(\alpha a_i^2 + a_i - 1)^2} [m_2A_4 - a_1A_5 + m_2A_6 + 2a_1A_7 + 2a_2A_6 + m_2A_9 + (a_1 + a_2)A_{10} + a_2A_{11}] + \alpha \left[ \frac{G_r}{a_i(1 + M)(\alpha a_i^2 + a_i - 1)} \right] \{m_1^2A_1 - a_1^2A_2 + a_2^2A_3\}
\]

\[
+ \frac{4P_{e}\alpha G_r}{a_i^3(1 + M)^3(\alpha a_i^2 + a_i - 1)^2} [m_1^2A_4 - a_1^2A_5 + m_2^2A_6 - 4a_1^2A_7 - 4a_2^2A_8 + m_2^2A_9 - (a_1 + a_2)^2A_{10} - a_2^2A_{11}] ,
\]

\[
B_r = \frac{G_r}{a_i[(a_i^2 + a_i - 1)^2 - \frac{\omega^2 a_i^2}{16}][(1 + M)(\alpha a_i^2 + a_i - 1) - \omega \alpha \left(\frac{\omega}{4} + \frac{\omega \alpha}{4} M\right)] \left[-a_3C_1 + a_1C_2 - a_2C_3\right]
\]

\[
+ \frac{2P_{r_e}G_r}{a_i[(a_i^2 + a_i - 1)^2 - \frac{\omega^2 a_i^2}{16}][(1 + M)^2 - \left(\frac{\omega}{4} + \frac{\omega \alpha}{4} M\right)] \left[-a_3C_4 + s_3C_5\right]
\]

\[
+ s_3C_6 + s_4C_7 + s_5C_8 - (a_1 + a_3)C_9 + (a_2 + a_3)C_{10} - (a_2 + a_4)C_{11} + (a_2 + a_4)C_{12} + a_4C_{13} ,
\]

\[
B_i = \frac{G_r}{a_i[(a_i^2 + a_i - 1)^2 - \frac{\omega^2 a_i^2}{16}][(1 + M)^2 - \left(\frac{\omega}{4} + \frac{\omega \alpha}{4} M\right)] \left[(\frac{\omega}{4} + \frac{\omega \alpha}{4} M)(\alpha a_i^2 + a_i - 1) - \omega \alpha \left(\frac{\omega}{4} + \frac{\omega \alpha}{4} M\right)\right] \left[-a_3C_1 + a_1C_2 - a_2C_3\right]
\]

\[
+ \frac{2P_{e}\alpha G_r}{a_i[(a_i^2 + a_i - 1)^2 - \frac{\omega^2 a_i^2}{16}][(1 + M)^2 - \left(\frac{\omega}{4} + \frac{\omega \alpha}{4} M\right)] \left[-a_3C_4 + s_3C_5\right]
\]

\[
+ s_3C_6 + s_4C_7 + s_5C_8 - (a_1 + a_3)C_9 + (a_2 + a_3)C_{10} - (a_2 + a_4)C_{11} + (a_2 + a_4)C_{12} + a_4C_{13} .
\]
Similarly, the rate of heat transfer at the plate is given by

$$q_\infty = \left( \frac{\partial T}{\partial y} \right)_{y=0} = T_0'(0) + e^{i\omega T}(0)$$

(2.4.3)

and on further simplification, we have

$$q_\infty = q_\infty^m + \varepsilon |H| \cos(\omega t + \gamma)$$

(2.4.4)

where $$|H| = \sqrt{H_r^2 + H_i^2}$$ and $$\gamma = \tan^{-1}\left( \frac{H_i}{H_r} \right)$$ are the amplitude and phase of the rate of heat transfer and

$$q_\infty^m = -a_1 + \frac{4P_r \varepsilon^2 G_r^2}{a_1^2 (1+M)^2 ((\alpha \alpha_1^2 + 1)^2)} \left[ -a_1 B_1 + n_1 B_2 + 2a_1 B_3 + 2a_2 B_4 - n_2 B_5 + (a_1 + a_2) B_6 - n_3 B_7 \right],$$

$$H_r = \frac{2P_r E_r^2 G_r^2 [((\alpha \alpha_1^2 + 1)^2 - \frac{\omega \alpha}{4} (1+M)^2 - \frac{\omega \alpha}{4} (1+M)]}{a_1 a_2 (\alpha \alpha_1^2 + 1)((\alpha \alpha_1^2 + 1)^2 - \frac{\omega \alpha}{4} (1+M)^2 - \frac{\omega \alpha}{4} (1+M)]}$$

$$+ t_1 D_2 + t_2 D_3 - t_3 D_4 + t_4 D_5 - (a_1 + a_3) D_6 + (a_1 + a_4) D_7 + (a_2 + a_3) D_8 - (a_2 + a_4) D_9,$$

$$H_i = \frac{2P_r E_r^2 G_r^2 [((\alpha \alpha_1^2 + 1)^2 - \frac{\omega \alpha}{4} (1+M)^2 - \frac{\omega \alpha}{4} (1+M)]}{a_1 a_2 (\alpha \alpha_1^2 + 1)((\alpha \alpha_1^2 + 1)^2 - \frac{\omega \alpha}{4} (1+M)^2 - \frac{\omega \alpha}{4} (1+M)]}$$

$$+ t_1 D_2 + t_2 D_3 - t_3 D_4 + t_4 D_5 - (a_1 + a_3) D_6 + (a_1 + a_4) D_7 + (a_2 + a_3) D_8 - (a_2 + a_4) D_9.$$
friction \( \tau_w \) for various values of Hartman number (M) and sink strength (S) considering the Prandtl number \( Pr = 9 \), Grashof number \( Gr = 5.0 \), Eckert number \( E_c = 0.001 \) with \( \omega = 5.0 \), \( \varepsilon = 0.2 \), \( \omega \alpha = \pi/2 \) to observe the visco-elastic effect. It is observed from the figures that the profiles of \( u_0 \) and \( \tau_w \) decrease, but \( T_0 \) and \( T \) increase in both Newtonian and non-Newtonian cases. It is noted from the Fig.2.1 to Fig.2.3 that \( u_0 \) decreases with the increase of \( |\alpha| \) with \( S = -0.05 \), \( M = 1 \); \( S = -0.05 \), \( M = 5 \) and \( M = 5 \), \( S = -0.10 \) respectively. Fig.2.4 to Fig.2.9 exhibit that the mean temperature \( T_0 \) and the temperature \( T \) increase for \( M = 1 \), \( S = -0.05 \); \( M = 5 \), \( S = -0.05 \) and \( M = 5 \), \( S = -0.10 \) respectively with increasing value of \( |\alpha| \). From Fig.2.10 and Fig.2.11, it is seen that the skin friction \( \tau_w \) decreases as \( M \) and \( S \) increase. Also \( \tau_w \) decreases for increasing values of \( |\alpha| \).

The mean skin friction \( \tau_w^m \) and the mean rate of heat transfer \( q_w^m \) with corresponding amplitudes and phases for \( Pr = 5.0 \) with \( Gr = 5.0 \), \( E_c = 0.001 \), \( \omega = 5.0 \), \( \varepsilon = 0.2 \), \( \omega \alpha = \pi/2 \) have been given in Tables 2.1 and 2.2 to observe the visco-elastic effects. It is notified from the Table 2.1 that the values of \( \tau_w^m \) and \( |\tan \beta| \) decrease and \( |B| \) increases with the increasing value of \( |\alpha| \) but the reverse effect is observed when magnetic field strength is increased with \( |\alpha| \) (cases I and II) for constant value of the sink strength. We also observe that \( \tau_w^m \) and \( |\tan \beta| \) increase and \( |B| \) decreases with the increasing values of \( |\alpha| \) but when the sink strength \( |S| \) also increases with \( |\alpha| \) then \( \tau_w^m \) increases but \( |B| \) and \( |\tan \beta| \) decrease (cases II and III) for constant value of magnetic field strength. Table 2.2 reveals that \( q_w^m \) and \( |H| \) decrease and \( |\tan \gamma| \) increases with increasing values of \( |\alpha| \) but the reverse effect is observed when the magnetic field strength increases with constant sink strength (cases I and II). Again it is seen that \( q_w^m \) and \( |H| \) increase and \( |\tan \gamma| \) decrease with the increasing values of \( |\alpha| \), but when the sink strength \( |S| \) increases with \( |\alpha| \) then \( q_w^m \) with corresponding amplitude \( |H| \) and phase \( |\tan \gamma| \) enhance (cases II and III) for the constant value of magnetic field strength

2.6 Conclusion

The problem of the unsteady free convective magnetohydrodynamic flow of a visco-elastic, electrically conducting fluid past an infinite vertical porous plate with constant
suction and heat absorbing sinks has been investigated. The key observations are:

- The mean velocity diminishes with the variation of the visco-elastic parameter as well as the magnetic and the sink strength parameters.
- The temperature field is noticeably affected by the variation of the visco-elastic parameter in combination with the other flow parameters.
- The skin friction is reduced considerably with the increasing values of magnetic and sink strength parameters. Again, a rising trend of the skin friction is observed by the growth of the absolute values of the visco-elastic parameter.
- The mean skin friction and the mean rate of heat transfer with their respective phases and the amplitudes are seen to be affected by the visco-elastic parameter with the variation of magnetic and sink strength parameters.
Fig. 2.1  Variation of $u_0$ against $y$ for $P_r=9$, $M=1$, $S=\pm 0.05$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\varepsilon=0.2$, $\omega t=\pi/2$

Fig. 2.2  Variation of $u_0$ against $y$ for $P_r=9$, $M=5$, $S=\pm 0.05$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\varepsilon=0.2$, $\omega t=\pi/2$
Fig. 2.3  Variation of $u_0$ against $y$ for $P_r=9$, $M=5$, $S=0.10$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\varepsilon=0.2$, $\omega t=\pi/2$

Fig. 2.4  Variation of $T_0$ against $y$ for $P_r=9$, $M=1$, $S=0.05$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\varepsilon=0.2$, $\omega t=\pi/2$
Fig. 2.5 Variation of $T_0$ against $y$ for $P_r=9$, $M=5$, $S=-0.05$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\epsilon=0.2$, $\omega t=\pi/2$.

Fig. 2.6 Variation of $T_0$ against $y$ for $P_r=9$, $M=5$, $S=-0.10$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\epsilon=0.2$, $\omega t=\pi/2$. 
Fig. 2.7  Variation of $T$ against $y$ for $P_r=9$, $M=1$, $S=-0.05$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\epsilon=0.2$, $\omega t=\pi/2$

Fig. 2.8  Variation of $T$ against $y$ for $P_r=9$, $M=5$, $S=-0.05$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $\epsilon=0.2$, $\omega t=\pi/2$
Fig. 2.9  Variation of $T$ against $y$ for $Pr=9$, $M=5$, $S=-0.10$, $Gr=5.0$, $Ec=0.001$, $\omega=5.0$, $\epsilon=0.2$, $\omega t=\pi/2$

Fig. 2.10  Variation of $\tau_w$ against $M$ for $Pr=9$, $S=-0.50$, $Gr=5.0$, $Ec=0.001$, $\omega=5.0$, $\epsilon=0.2$, $\omega t=\pi/2$
Fig. 2.11  Variation of $\tau_w$ against $S$ for $P_r=9$, $M=5$, $G_r=5.0$, $E_c=0.001$, $\omega=5.0$, $e=0.2$, $\omega t=\pi/2$
TABLE 2.1

Values of mean skin friction ($\tau_{\omega}^m$), amplitude ($|B|$), phase (tan $\beta$) for $Pr=5$, $Gr=5.0$, $Ec=0.001$, $\omega = 5.0$, $\epsilon = 0.2$, $\omega t = \pi/2$

| Case | $M$ | $S$ | $\alpha$ | $\tau_{\omega}^m$ | $|B|$ | tan $\beta$ |
|------|-----|-----|-----------|------------------|-------|-------------|
| I    | 1.0 | -0.05 | 0         | 2.7559           | 1.2347 | -1.4999     |
|      |     |       | -0.01     | 0.8424           | 1.4234 | -1.3693     |
|      |     |       | -0.015    | 0.7102           | 1.8637 | -0.2605     |
| II   | 5.0 | -0.05 | 0         | 2.2630           | 2.6531 | -0.8779     |
|      |     |       | -0.01     | 3.8908           | 1.4791 | -1.6913     |
|      |     |       | -0.015    | 8.9705           | 1.0027 | -2.1271     |
| III  | 5.0 | -0.10 | 0         | 1.7104           | 2.5926 | -0.7314     |
|      |     |       | -0.01     | 3.0734           | 2.0417 | -0.6879     |
|      |     |       | -0.015    | 3.6372           | 1.1947 | -0.5127     |

TABLE 2.2

Values of mean rate of heat transfer ($q_{\omega}^m$), amplitude ($|H|$), phase (tan $\gamma$) for $Pr=5$, $Gr=5.0$, $Ec=0.001$, $\omega = 5.0$, $\epsilon = 0.2$, $\omega t = \pi/2$

| Case | $M$ | $S$ | $A$ | $q_{\omega}^m$ | $|H|$ | tan $\gamma$ |
|------|-----|-----|-----|--------------|-------|-------------|
| I    | 1.0 | -0.05 | 0   | -0.2729      | 1.9376 | 0.9923      |
|      |     |       | -0.01| -0.2194      | 1.8725 | -1.3849     |
|      |     |       | -0.015| -0.2185     | 1.3448 | -6.6525     |
| II   | 5.0 | -0.05 | 0   | -1.1117      | 5.6668 | 0.8867      |
|      |     |       | -0.01| -1.9129      | 6.8073 | -0.7241     |
|      |     |       | -0.015| -2.6897    | 7.8881 | -3.0891     |
| III  | 5.0 | -0.10 | 0   | -1.4357      | 3.0632 | 2.0267      |
|      |     |       | -0.01| -2.0721      | 3.2635 | -3.3023     |
|      |     |       | -0.015| -2.1907    | 3.5606 | -3.3842     |