Chapter 5  
An Inventory Model For Deteriorating Products with Inflation, Lost Sales and Time Dependent Demand Under Trade-Credit

5.1 Introduction

As we all know, the capacity of any warehouse is limited. In reality, there usually exist various factors that induce the decision-maker of the inventory system to order more items that can be held in his own warehouse. There is often a limitation on available storage space or a fixed budget on investment for raw materials. Furthermore, various researchers have discussed a two-warehouse inventory system. Therefore, due to the limited capacity of the available showroom facility (existing storage, owned warehouse (OW)), an additional storage which is assumed to be available with abundant space, is required to hold a large stock. This additional storage facility may be a rented warehouse (RW) with better preserving facility. This type of situation generally arise when the acquisition costs are higher than maintaining a RW on the supply. Lead time is so large that stocks are receivable only at fixed times and no procurement is possible until the next instant of supply on when the item under consideration is a seasonal product such as output of harvest on when the management gets an attractive price discount for purchasing grain.

A number of mathematical models have been developed for these deteriorating items. But constant deterioration is that concept, which cannot justify itself under any circumstances. Deterioration depends on a lot of factors, but the main factor is time. It is a very logical reasoning that decay rate of any item increases with time.

In the present chapter, a two warehouse inventory model with linear trend in demand under the inflationary conditions has been developed. Two cases are discussed here: one is two warehouse inventory model without trade credit and other is two warehouse inventory model with trade credit. A rented warehouse (RW) is used to store the excess units over the capacity of the own warehouse (OW). In this chapter, we have developed an EOQ model for deteriorating items considering time dependent deterioration rate in both RW and OW, inflation and demand is stimulated by time above a certain stock level and, thereafter, it becomes constant, partially backlogged shortages and allowable delay in payment facility considered. The backlogging rate is dependent on the waiting time up to the
starting of next replenishment. Finally, the results are illustrated with the help of numerical examples. Also, the effects of changes of different parameters are studied graphically on the average profit.

5.2 Assumptions and Notations

The mathematical models of the two warehouse inventory problems are based on the following assumptions:

(i) The demand rate is dependent on time down to a level \( S_0 \), where \( S_0 \) is given and fixed, beyond which it is assumed to be constant, that is, when the demand rate of the item is considered to be of the form \( R[f(t)] \) is a function of the linear trend in demand for simplicity, we assumed that \( R[f(t)] = a + bt \), where \( a, b \), are non-negative constants and \( a \geq b \).

(ii) Shortages, if any, are allowed and partially backlogged and the demand rate \( R \) is given by \( R = a \), where some of the unsatisfied demand is backlogged, and the fraction of shortages backordered is \( 1/(1+\delta x) \), where \( x \) is the waiting time up to the next replenishment and \( \delta \) is a positive constant.

(iii) The effect of trade credits is also considered.

(iv) The owned warehouse (OW) has a fixed capacity of \( W \) units; the rented warehouse (RW) has unlimited capacity.

(v) The goods of OW are consumed only after consuming the goods kept in RW.

(vi) The unit inventory costs (including holding and deterioration cost) per unit time in RW are higher than those in OW; that is, \( C_1 + \theta_1 C_5 > C_2 + \theta_2 C_5 \).

(vii) Replenishment rate is infinite, lead time is assumed to be negligible.

(viii) The time horizon of the inventory system is finite.

(ix) Rate of deterioration is taken to be different in different warehouses where \( \theta_1(t) = \theta_1 t \) for RW and \( \theta_2(t) = \theta_2 \) for OW, \( \theta_1, \theta_2 \) is a positive constant where \( 0 < \theta_1, 0_2 < 1 \).

The following notations are used in this model:

- \( A \): The replenishment cost per order.
- \( s \) : The selling price per unit, where \( s > c_5 \)
- \( W \) : The capacity of the owned warehouse.
- \( q_r \) : Maximum inventory level in RW.
Q: The ordering quantity per cycle.

B: The maximum inventory level per cycle.

c_1: The holding cost per unit per unit time in RW.

c_2: The holding cost per unit per unit time in OW, where \( c_1 > c_2 \)

\( c_3 \): The backordered cost per unit per unit time.

\( c_4 \): The opportunity cost (i.e. goodwill cost) per unit.

\( c_5 \): The purchasing cost per unit.

i_e: Interest earned per unit time.

i_c: Interest charged per unit time.

M: Permissible delay in settling the accounts.

t_w: The time at which the inventory level reaches zero in RW.

\( t_1 \): The time at which the inventory level reaches \( S_0 \) in OW, where \( S_0 \) is given.

\( t_2 \): The time at which the inventory level reaches zero in OW.

\( t_3 \): The length of period during which shortages are allowed.

T: The length of the inventory cycle, hence \( T = t_2 + t_3 \)

\( q_1(t) \): The level of positive inventory in RW at time \( t \), where \( 0 \leq t \leq t_w \)

\( q_2(t) \): The level of positive inventory in OW at time \( t \), where \( 0 \leq t \leq t_w \)

\( q_3(t) \): The level of positive inventory in OW at time \( t \), where \( t_w \leq t \leq t_1 \)

\( q_4(t) \): The level of positive inventory in OW at time \( t \), where \( t_1 \leq t \leq t_2 \)

\( q_5(t) \): The level of negative inventory at time \( t \), where \( t_2 \leq t \leq T \)

\( P(t_1, t_2) \): The total profit per unit time with time-dependent demand rate.

5.3 Mathematical Formulation

We have discussed two cases: First is two warehouse inventory model without trade credits and second is two warehouse inventory model with trade credits.
Case I: An inventory model without trade credits

![Graphical representation of the two-warehouse inventory system](image)

Using above assumptions, the inventory level follows the pattern represented in Figure 5.1. To establish the total relevant profit function, we consider the following time-intervals separately, $[0,t_w]$, $[t_w,t_1]$, $[t_1,t_2]$ and $[t_2,T]$. During the interval $[0,t_w]$, the inventory levels are positive at RW and OW. At OW, the inventory is only depleted by the effect of deterioration. At RW, the inventory is depleted due to the combined effects of demand and deterioration on time and on-hand inventory level reaches $S_0$ at time $t = t_1$. During the time interval $[t_w,t_1]$, the inventory in OW is depleted due to the combined effects of demand and deterioration same as RW on the time and reaches $S_0$ at time $t = t_1$. Hence, the inventory level at RW and OW are governed by the following differential equations:

- $q_1'(t) + \theta_1(t)q_1(t) = -(a + bt) \quad , \quad 0 \leq t \leq t_w \quad (5.1)$
- $q_2'(t) + \theta_2(t)q_2(t) = 0 \quad , \quad 0 \leq t \leq t_w \quad (5.2)$
- $q_3'(t) + \theta_3(t)q_3(t) = -(a + bt) \quad , \quad t_w \leq t \leq t_1 \quad (5.3)$

with the boundary conditions $q_1(0) = q_r$, $q_2(0) = W$, $q_3(t_1) = S_0$.

Solving the differential equation (5.1), (5.2) and (5.3), we get the inventory level as:

- $q_1(t) = q_r \left(1 - \frac{\theta_1 t^2}{2}\right) - \left[at + \frac{bt^2}{2} - \frac{a\theta_1 t^3}{3} - \frac{b\theta_1 t^4}{8}\right] \quad , \quad 0 \leq t \leq t_w \quad (5.4)$
- $q_2(t) = W e^{-\frac{\theta_2 s^2}{2}} \quad , \quad 0 \leq t \leq t_w \quad (5.5)$
\[ q_i(t) = \int_0^{t_i} \frac{e^{\frac{b}{a}t_i^2-t^2}}{\sqrt{\pi}} \left[ a \ t_1 - t + \frac{b}{2} t_1^2 - t^2 + \frac{a\theta_2}{6} t_1^3 - t^3 + \frac{b\theta_2}{8} t_1^4 - t^4 \right. \]
\[- \frac{a\theta_2}{2} t_1 t^2 - t^3 - \frac{b\theta_2}{4} t_1^2 t^2 - t^4 \right] \right], \quad t_w \leq t \leq t_1 \tag{5.6}

After the time \( t = t_1 \), the demand rate becomes constant, and the inventory level falls to zero at time \( t = t_2 \). During the interval \([t_1, t_2]\), the inventory in OW is depleted due to the combined effects of demand and deterioration. Hence, the inventory level at OW is governed by the following differential equation:

\[ q_i'(t) + \theta_2(t)q_i(t) = -a \quad , \quad t_1 \leq t \leq t_2 \tag{5.7} \]

with the boundary condition \( q_i(t_2) = 0 \). Solving the differential equation (5.7), we obtain the inventory level as

\[ q_i(t) = \left[ a \ t_2 - t + \frac{a\theta_2}{6} t_2^3 - t^3 - \frac{a\theta_2}{2} t_2 t^2 - t^3 \right] \quad , \quad t_1 \leq t \leq t_2 \tag{5.8} \]

Furthermore, at time \( t_2 \), shortage occurs and the inventory level starts dropping below 0. During \([t_2, T]\), the inventory level only depends constant demand, and a fraction \( \frac{1}{1 + \delta} \frac{T-t}{T-t} \) of the demand is backlogged, where \( t \in [t_2, T] \). The inventory level is governed by the following differential equation:

\[ q_i'(t) = -a / \left[ 1 + \delta \ T - t \right] \quad , \quad t_2 \leq t \leq T \tag{5.9} \]

with the boundary condition and \( q_i(t_2) = 0 \). Solving the differential equation (5.9), we obtain the inventory level as:

\[ q_i(t) = \left[ a \ 1 - \delta T \ t_2 - t + \frac{a\delta}{2} t_2^2 - t^2 \right] \quad , \quad t_2 \leq t \leq T \tag{5.10} \]

Therefore, the ordering quantity over the replenishment cycle can be determined as:

\[ Q = q_i(0) + q_i(0) - q_i(T) \]

\[ Q = q_i + W - \left[ a \ 1 - \delta T \ t_2 - T + \frac{a\delta}{2} t_2^2 - T^2 \right] \tag{5.11} \]

And the maximum inventory level per cycle is:
Based on Eqs. (5.4), (5.5), (5.6), (5.8) and (5.10), the total profit per cycle consists of the following elements:

Ordering cost per cycle = \( A \) (5.13)

Holding cost per cycle in RW = \( c_1 \int_0^{t_w} q_i(t)e^{-r_t}dt \)

\[
= c_1 \left[ q_i \left( t_w - r \frac{t_w^2}{2} \right) - a \left( \frac{t_w^2}{2} - r \frac{t_w^3}{3} \right) - b + q_i \theta_1 \left( \frac{t_w^3}{6} - r \frac{t_w^4}{8} \right) \right] + a \theta_1 \left( \frac{t_w^4}{12} - r \frac{t_w^5}{15} \right) + b \theta_1 \left( \frac{t_w^5}{20} - r \frac{t_w^6}{24} \right)
\] (5.14)

Holding cost per cycle in OW = \( c_2 \left[ \int_0^{t_w} q_i(t)e^{-r_t}dt + \int_{t_w}^{t_i} q_i(t)e^{-r_t}dt + \int_{t_i}^{t_i^t} q_i(t)e^{-r_t}dt \right] \)

\[
= c_2 \left[ W \left( t_w - \frac{\theta t_w^3}{6} - r \frac{t_w^2}{2} \right) + \right. \\
\left. \begin{aligned}
S_0 t_1 - t_w \left( 1 - \frac{r}{2} t_1 + t_w \right) + a \left( \frac{t_1^2}{2} - t_1 t_w + \frac{t_w^2}{2} \right) + b + S_0 \theta_2 \left( \frac{2t_1^3}{3} - t_1^2 t_w + \frac{t_w^3}{3} \right) \\
- ar \left( \frac{t_1^3}{6} - t_1^2 t_w^3 + \frac{t_w^3}{3} \right) + a \theta_2 \left( \frac{3t_1^4}{4} - t_1^3 t_w + \frac{t_w^4}{4} \right) + a \theta_2 \left( \frac{t_1^4}{12} - t_1^2 t_w^4 + \frac{t_w^5}{4} \right) \\
- \frac{S_o r \theta_2}{4} \left( \frac{t_1^4}{2} - t_1^2 t_w^2 + \frac{t_w^4}{2} \right) + b \theta_2 \left( \frac{4t_1^5}{5} - t_1^4 t_w + \frac{t_w^5}{5} \right) \\
- \frac{b \theta_2}{4} \left( \frac{t_1^6}{15} - t_1^4 t_w^3 + \frac{t_w^6}{3} \right) + a \theta_2 \left( \frac{3t_1^5}{10} - t_1^3 t_w^2 + \frac{t_w^5}{2} + \frac{t_w^5}{5} \right) + a \theta_2 \left( \frac{t_1^5}{20} - t_1^3 t_w^4 + \frac{t_w^5}{4} + \frac{t_w^5}{5} \right) \\
- \frac{b r \theta_2}{8} \left( \frac{t_1^6}{3} - t_1^4 t_w^2 + \frac{t_w^6}{6} \right) + b r \theta_2 \left( \frac{t_1^6}{6} - t_1^4 t_w^4 + \frac{t_w^6}{2} \right) \\
+ \left. \left[ a \left( \frac{t_1^2}{2} - t_1 t_w + \frac{t_w^2}{2} \right) + a \theta_2 \left( \frac{3t_1^4}{4} - t_1^2 t_w^3 + \frac{t_w^4}{4} \right) - a \theta_2 \left( \frac{t_1^4}{12} - t_1^2 t_w^3 + \frac{t_w^4}{4} \right) \right] \right]
\]
\] (5.15)
Backorder cost per cycle = $c_b \int_{t_1}^{T} \left[ -q_s(t) \right] e^{-rt} dt$

$$= c_b \left[ a \left( 1 - \delta T \left( \frac{t_2^2}{2} - t_2 T + \frac{T^2}{2} \right) \right) \right. \left. - r \left( \frac{t_2^3}{6} - t_2^2 \frac{T^2}{2} + \frac{T^3}{3} \right) + a \delta \left( \frac{2t_2^3}{3} - t_2^2 T + \frac{T^3}{3} \right) \right. \left. - r \left( \frac{t_4^4}{4} - t_2^2 \frac{T^2}{2} + \frac{T^4}{4} \right) \right]$$  \hspace{1cm} (5.16)

Opportunity cost due to lost sales per cycle = $c_b \int_{t_1}^{T} a \left[ 1 - \frac{1}{1 + \delta (T - t)} \right] e^{-rt} dt$

$$= c_b \left( \frac{T^2}{2} - Tt_2 + \frac{t_2^2}{2} \right) a \delta r \left( \frac{T^3}{6} - T \frac{t_2^2}{2} + \frac{t_2^3}{3} \right)$$  \hspace{1cm} (5.17)

Purchase cost per cycle = $c_p Q$

$$= c_p \left[ q_s + W - \left( a \left( 1 - \delta T \right)t_2 - T + \frac{a \delta}{2} t_2^2 - T^2 \right) \right]$$  \hspace{1cm} (5.18)

Sales revenue per cycle

$$= s \int_0^{t_w} \left[ a + bt \right] e^{-rt} dt + \int_{t_w}^{t_1} \left[ a + bt \right] e^{-rt} dt + \int_{t_1}^{t_5} ae^{-rt} dt + \int_{t_5}^{T} \frac{a}{1 + \delta (T - t)} e^{-rt} dt$$

$$= s \left[ at_w - \frac{1}{2} ar - b t_w^2 - \frac{bracht_w^3}{3} \right]$$

$$+ \left( a \ t_1 - t_w - \frac{1}{2} ar - b t_1^2 - t_w^2 - \frac{bracht_w^3}{3} \right) + \left( a \ t_2 - t_1 - \frac{ar}{2} t_2^2 - t_2^2 \right)$$

$$+ \left( a \ T - t_2 - \frac{ar}{2} T^2 - t_2^2 - \alpha \delta \left( \frac{T^2}{2} - Tt_2 + \frac{t_2^2}{2} \right) + \alpha \delta \left( \frac{T^3}{6} - T \frac{t_2^2}{2} + \frac{t_2^3}{3} \right) \right]$$  \hspace{1cm} (5.19)

Therefore, the total profit per unit time of our model is obtained as follows:

$$P(t_1, t_2) = \frac{1}{T} \{ \text{Sales revenue-purchase cost-ordering cost-holding cost in RW - holding cost in OW - backorder cost -opportunity cost} \}$$  \hspace{1cm} (5.20)
\[
\pi(t_1, t_2) = \frac{1}{T} \left[ \left( a t_w - \frac{1}{2} ar - b t_w^2 + \frac{br t_w^3}{3} \right) + \left( a t_1 - t_w - \frac{1}{2} ar - b t_1^2 - t_w^2 + \frac{br}{3} t_1^3 - t_w^3 \right) \right] \\
+ \left[ a t_2 - t_1 - \frac{ar}{2} t_2^2 - t_1^2 \right] + \left[ a T - t_2 - \frac{ar}{2} T^2 - t_2^2 - a \delta \left( \frac{T^2}{2} - T t_2 + t_2^2 \right) + a \delta \left( \frac{T^3}{6} - \frac{T t_2^2}{2} + \frac{t_2^3}{3} \right) \right] \\
- A - c_1 \left( q_t \left( t_w - r \frac{t_w^2}{2} \right) - a \left( \frac{t_w^2}{2} + \frac{t_w^3}{3} \right) \right) - b + q_1 \left( \frac{t_w^3}{6} - r \frac{t_w^4}{8} \right) \\
+ a \theta \left( \frac{t_w^4}{12} - r \frac{t_w^5}{15} \right) + \frac{b \theta}{6} \left( \frac{t_w^5}{20} - r \frac{t_w^6}{24} \right) - c_2 \left( W - \frac{t_w - r \frac{t_w^3}{3} - r \frac{t_w^2}{2}}{6} \right) \\
\left[ S_0 t_1 - t_w \left( \frac{1 - r}{2} t_1 + t_w \right) + a \left( \frac{t_1^2}{2} - t_1 t_w + \frac{t_w^2}{2} \right) + \frac{1}{2} b + S_0 \theta \left( \frac{2 t_1^3}{3} - t_1^2 t_w + \frac{t_w^3}{3} \right) \right] \\
- a r \left( \frac{t_1^2}{6} - t_1 \frac{t_1^2}{2} + \frac{t_w^3}{3} \right) + \frac{a \theta_2}{6} \left( \frac{3 t_1^4}{4} - t_1^3 t_w + \frac{t_w^4}{4} \right) - \frac{a \theta_2}{2} \left( \frac{t_1^4}{12} - t_1 t_w^3 + \frac{t_w^4}{4} \right) \\
- S_0 \theta \left( \frac{3 t_1^5}{2} - t_1^3 t_w + \frac{t_w^5}{5} \right) + \frac{a r \theta_2}{6} \left( \frac{3 t_1^5}{10} - t_1^3 t_w + \frac{t_w^5}{5} \right) + \frac{a \theta_2}{2} \left( \frac{t_1^5}{20} - t_1 t_w^3 + \frac{t_w^5}{5} \right) \\
- \frac{b \theta}{8} \left( \frac{t_1^6}{3} - t_1^4 t_w^2 + \frac{t_w^6}{6} \right) + \frac{b \theta}{8} \left( \frac{t_1^6}{6} - t_1 t_w^3 + \frac{t_w^6}{3} \right) + \left( \frac{a \theta_2}{2} - t_1 t_2 + \frac{t_1^3}{2} \right) \\
+ \frac{a \theta_2}{6} \left( \frac{3 t_1^4}{4} - t_1^3 t_w + \frac{t_w^4}{4} \right) - \frac{a \theta_2}{2} \left( \frac{t_1^4}{12} - t_1 t_2^3 + \frac{t_w^4}{4} \right) \\
- a r \theta_2 \left( \frac{3 t_1^5}{2} - t_1^3 t_w + \frac{t_w^5}{5} \right) - \frac{a \theta_2}{6} \left( \frac{3 t_1^5}{10} - t_1^3 t_w + \frac{t_w^5}{5} \right) + \frac{a \theta_2}{2} \left( \frac{t_1^5}{20} - t_1 t_w^3 + \frac{t_w^5}{5} \right) \right] \\
= c_3 \left[ a 1 - \delta T \left( \frac{t_2^2}{2} - t_2 T + \frac{T^2}{2} \right) - r \left( \frac{t_2^3}{6} - t_2 T + \frac{T^3}{3} \right) \right] + a \delta \left( \frac{2 t_2^3}{3} - t_2^2 T + \frac{T^3}{3} \right) \\
- r \left( \frac{t_2^4}{4} - t_2^2 \frac{T^2}{2} + \frac{T^4}{4} \right) - c_4 \left[ a \delta \left( \frac{T^2}{2} - T t_2 + \frac{t_2^2}{2} \right) - a \delta r - b \delta \left( \frac{T^3}{6} - T \frac{t_2^2}{2} + \frac{t_2^3}{3} \right) - b \delta r \left( \frac{T^4}{12} - T \frac{t_2^3}{3} + \frac{t_2^4}{4} \right) \right] \\
- c_5 \left[ q_w - W - \left( a 1 - \delta T - t_2 - T + \frac{a \delta}{2} t_2^2 - T^2 \right) \right] \right] \\
\]

To maximize total average profit per unit time (P), the optimal values of t_1 and t_2 can be obtained by solving the following equations simultaneously.
\[ \frac{\partial P}{\partial t_1} = 0 \]  
(5.21)

and \[ \frac{\partial P}{\partial t_2} = 0 \]  
(5.22)

provided, they satisfy the following conditions

\[ \frac{\partial^2 P}{\partial t_1^2} > 0, \frac{\partial^2 P}{\partial t_2^2} > 0 \]  
(5.23)

and \[ \left( \frac{\partial^2 P}{\partial t_1^2} \right) \left( \frac{\partial^2 P}{\partial t_2^2} \right) - \left( \frac{\partial^2 P}{\partial t_1 \partial t_2} \right)^2 > 0 \]  
(5.24)

Equations (5.21) and (5.22) are highly non-linear and hence are solved with the help of mathematical software \textit{MATHEMATICA 5.2}. With the use of these optimal values equation (5.20) provides maximum total average profit per unit time of the system in consideration.

5.4 Numerical Examples

Numerical example is used to illustrate all results in this chapter. The necessary parameters and the optimal solutions are presented respectively and the data which is taken in this chapter is based on the previous study similar to [26].

\textbf{Example :} a = 800, b = 0.008, \( \theta_1 = 0.09 \), \( \theta_2 = 0.06 \), \( t_w = 20 \), \( T = 80 \), \( r = 0.06 \), \( S_0 = 100 \), \( q_r = 200 \), \( W = 600 \), \( A = 100 \), \( \delta = 0.7 \), \( c_1 = 0.6 \), \( c_2 = 0.3 \), \( c_3 = 0.5 \), \( c_4 = 5 \), \( c_5 = 12 \), \( s = 14 \), \( t_1 = 50.6455 \), \( t_2 = 67.9233 \), \( H_{RW} = 44.9501 \), \( H_{OW} = 2218.28 \), \( BC = 122889 \), \( OC = 47938 \), \( Q = 283353 \),

Total profit (\( \pi \)) = 60440.5

\begin{figure}[h]
    
    
    \begin{center}
    
    \includegraphics[width=\textwidth]{fig5_2}
    
    \caption{Graphical representation of convexity of the total profit (P)}
    \end{center}
    
    
\end{figure}
5.5 Sensitivity Analysis

We now study the effects of changes in the values of the system parameters $r$, $\delta$ on the different costs, ordering quantity and optimal total profit.

Table 5.1 Effects of inflation rate parameter in demand ($r$)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>% Change in Different Costs and Total Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-20</td>
</tr>
<tr>
<td>$R$</td>
<td></td>
</tr>
<tr>
<td>$H_{OW}$</td>
<td>-59.7362</td>
</tr>
<tr>
<td>$OC$</td>
<td>-30.7006</td>
</tr>
<tr>
<td>Total profit</td>
<td>-46.6841</td>
</tr>
</tbody>
</table>

Table 5.2 Effects of backlogging parameter ($\delta$)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>% Change in Different Costs, Ordering Quantity and Total Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-20</td>
</tr>
<tr>
<td>$\Delta$</td>
<td></td>
</tr>
<tr>
<td>$OC$</td>
<td>-20.0000</td>
</tr>
<tr>
<td>$Q$</td>
<td>49.0038</td>
</tr>
<tr>
<td>Total profit</td>
<td>-12.9137</td>
</tr>
</tbody>
</table>

5.6 Observations

(i) It is observed from table 5.1, that the total average profit of the system increases rapidly as inflation parameter increases. Also, it has been observed that if we increase the value of inflation parameter in “$r$” then the value of holding cost in OW, backlogging cost, opportunity cost, ordering cost increases. Whereas holding cost in RW decreases with increment in inflation parameter.
It is also concluded from Table 5.2, that if we increase the backlogging parameter “δ”, then the value of backlogging cost, opportunity cost, and total profit increases but ordering cost decreases with the increase in the backlogging parameter.

Case II: An inventory model with trade credit

We have discussed two-warehouse inventory model with trade credits. The inventory level at RW and OW are governed by the following differential equations:-

\[ q_1'(t) + \theta_1(t)q_1(t) = -(a + bt), \quad 0 \leq t \leq t_w \]  
(5.25)

\[ q_2'(t) + \theta_2(t)q_2(t) = 0, \quad 0 \leq t \leq t \]  
(5.26)

\[ q_3'(t) + \theta_2(t)q_3(t) = -(a + bt), \quad t_w \leq t \leq t_1 \]  
(5.27)

with the boundary conditions \( q_1(0) = q_r, q_2(0) = W, q_3(t_1) = S_0 \). Solving the differential equation (5.25), (5.26) and (5.27), we get the inventory level as:

\[ q_1(t) = q_r \left(1 - \frac{\theta_1^2t^2}{2}\right) - \left[at + \frac{bt^2}{2} - \frac{a\theta_1t^3}{3} - \frac{b\theta_1t^4}{8}\right], \quad 0 \leq t \leq t_w \]  
(5.28)

\[ q_2(t) = We^{\frac{\theta_1^2}{2}}, \quad 0 \leq t \leq t_w \]  
(5.29)

\[ q_3(t) = S_0 e^{\frac{\theta_1^2}{2} t^2} \left[a t_1 - t + \frac{b}{2} t_1^2 - t^2 + \frac{a\theta_1}{6} t_1^3 - t^3 + \frac{b\theta_1}{8} t_1^4 - t^4ight. \]  

\[ \left. - \frac{a\theta_1}{2} t_1^2 t^2 - t^3 - \frac{b\theta_1}{4} t_1^2 t^2 - t^4 \right], \quad t_w \leq t \leq t_1 \]  
(5.30)

After the time \( t = t_1 \), the demand rate becomes constant and the inventory level falls to zero at time \( t = t_2 \). During the interval \([t_1,t_2] \), the inventory in OW is depleted due to the combined effects of demand and deterioration. Hence, the inventory level at OW is governed by the following differential equation:

\[ q_4'(t) + \theta_2(t)q_4(t) = -a, \quad t_1 \leq t \leq t_2 \]  
(5.31)
with the boundary condition $q_1(t_2) = 0$. Solving the differential equation (5.31), we obtain the
inventory level as:

$$q_1(t) = \left[ a - t_2 - t + \frac{a\theta_2}{6} t_2^3 - t^3 - \frac{a\theta_2}{2} t_2 t^2 - t^3 \right], \quad t_1 \leq t \leq t_2 \quad (5.32)$$

Furthermore, at time $t_2$, shortage occurs and the inventory level starts dropping below 0. During $[t_2, T]$, the inventory level only depends on constant demand, and a fraction $\frac{1}{1 + \delta} \frac{t_2}{T - t}$ of the demand
is backlogged, where $t \in [t_2, T]$. The inventory level is governed by the following differential
equation:

$$q_3'(t) = -a \left[ 1 + \delta \left( T - t \right) \right], \quad t_2 \leq t \leq T \quad (5.33)$$

with the boundary condition and $q_3(t_2) = 0$. Solving the differential equation (5.33), we
obtain the inventory level as:

$$q_3(t) = \left[ a - t_2 - t + \frac{a\delta}{2} t_2^2 - t^2 \right], \quad t_2 \leq t \leq T \quad (5.34)$$

Therefore, the ordering quantity over the replenishment cycle can be determined as:

$$Q = q_1(0) + q_2(0) - q_5(T)$$

$$Q = q_1 + W - \left[ a - t_2 - t + \frac{a\delta}{2} t_2^2 - T^2 \right] \quad (5.35)$$

and the maximum inventory level per cycle is

$$B = q_1(0) + q_2(0)$$

$$B = q_1 + W \quad (5.36)$$
Based on Eqs. (5.28), (5.29), (5.30), (5.32) and (5.34), the total profit per cycle consists of the following elements:

Ordering cost per cycle = $A$  \hspace{1cm} (5.37)

Holding cost per cycle in RW = $c_1 \int_0^{t_w} q_1(t)e^{-rt}dt$

\[
= c_1 \left[ q_1 \left( t_w - r \frac{t_w^2}{2} \right) - a \left( \frac{t_w^2}{2} - r \frac{t_w^3}{3} \right) - b + q_1 \theta_1 \left( \frac{t_w^3}{6} - r \frac{t_w^4}{8} \right) + a \theta_1 \left( \frac{t_w^4}{12} - r \frac{t_w^5}{15} \right) + \frac{b \theta_1}{2} \left( \frac{t_w^5}{20} - r \frac{t_w^6}{24} \right) \right] \hspace{1cm} (5.38)
\]

Holding cost per cycle in OW = $c_2 \left[ \int_0^{t_w} q_2(t)e^{-rt}dt + \int_{t_w}^{t_i} q_3(t)e^{-rt}dt + \int_{t_i}^{t} q_4(t)e^{-rt}dt \right]

\[
= c_2 \left[ W \left( t_w - \frac{\theta_2 t_w^3}{6} - r \frac{t_w^2}{2} \right) + \left[ S_0 t_1 - t_w \left( 1 - \frac{r}{2} t_1 + t_w \right) + a \left( \frac{t_1^2}{2} - t_1 t_w + \frac{t_w^2}{2} \right) + b + S_0 \theta_2 \left( \frac{2t_1^3}{3} - t_1^2 t_w + \frac{t_w^3}{3} \right) \right] - a \theta_2 \left( \frac{t_1^3}{6} - t_1 \frac{t_2^2}{2} + \frac{t_w^3}{3} \right) + a \theta_2 \left( \frac{3t_1^4}{4} - t_1^3 t_w + \frac{t_w^4}{4} \right) - a \theta_2 \left( \frac{t_1^4}{12} - t_1^3 \frac{t_w^3}{3} + \frac{t_w^4}{4} \right) \right] - \frac{b \theta_2}{4} \left( \frac{2t_1^5}{15} - t_1^2 \frac{t_w^3}{3} + \frac{t_w^5}{5} \right) - a \theta_2 \left( \frac{3t_1^5}{10} - t_1^4 \frac{t_w^3}{2} + \frac{t_w^5}{5} \right) + \frac{b \theta_2}{2} \left( \frac{t_1^6}{20} - t_1^5 \frac{t_w^3}{4} + \frac{t_w^5}{5} \right) \right] - b r \theta_2 \left( \frac{t_1^6}{3} - t_1^4 \frac{t_w^3}{2} + \frac{t_w^6}{6} \right) + b r \theta_2 \left( \frac{t_1^6}{6} - t_1^5 \frac{t_w^3}{2} + \frac{t_w^6}{3} \right) \right] + \left[ a \left( \frac{t_1^2}{2} - t_1 \frac{t_1^2}{2} + \frac{t_w^2}{2} \right) + a \theta_2 \left( \frac{3t_1^4}{4} - t_1^3 t_w + \frac{t_w^4}{4} \right) - a \theta_2 \left( \frac{t_1^4}{12} - t_1^3 \frac{t_w^3}{3} + \frac{t_w^4}{4} \right) \right] \right] - a \theta_2 \left( \frac{3t_1^5}{10} - t_1^4 \frac{t_w^3}{2} + \frac{t_w^5}{5} \right) + \frac{b \theta_2}{2} \left( \frac{t_1^6}{20} - t_1^5 \frac{t_w^3}{4} + \frac{t_w^5}{5} \right) \right] \hspace{1cm} (5.39)
Backorder cost per cycle = \( c_3 \int_{t_2}^{T} \left[ -q_s(t) \right] e^{-rt} dt \)

\[
= c_3 \left[ a \left( 1 - \delta T \right) \left( \frac{t_2^2}{2} - t_2 T + \frac{T^2}{2} \right) - r \left( \frac{t_2^3}{6} - t_2^2 \frac{T^2}{2} + \frac{T^3}{3} \right) \right] + \frac{a \delta}{2} \left( \frac{2t_2^3}{3} - t_2^2 T + \frac{T^3}{3} \right)
\]

\(-r \left( \frac{t_2^4}{4} - t_2^3 \frac{T^2}{2} + \frac{T^4}{4} \right) \right] \right) \right) \right)
\]

(5.40)

Opportunity cost due to lost sales per cycle = \( c_4 \int_{t_2}^{T} a \left[ 1 - \frac{1}{1 + \delta} \left( T - t \right) \right] e^{-rt} dt \)

\[
= c_4 \left[ a \delta \left( \frac{T^2}{2} - T t_2 + \frac{t_2^2}{2} \right) - a \delta r \left( \frac{T^3}{6} - T \frac{t_2^2}{2} + \frac{t_2^3}{3} \right) \right]
\]

(5.41)

Purchase cost per cycle = \( c_5 Q \)

\[
= c_5 \left[ q_s + W - \left( a \left( 1 - \delta T \right) \frac{a \delta}{2} t_2^2 - T^2 \right) \right] \]

(5.42)

Sales revenue per cycle

\[
= s \left[ \int_{0}^{t_w} \left[ a + bt \right] e^{-rt} dt + \int_{t_w}^{t_2} \left[ a + bt \right] e^{-rt} dt + \int_{t_2}^{T} ae^{-rt} dt + \int_{t_2}^{T} \frac{a}{1 + \delta} \left( T - t \right) e^{-rt} dt \right]
\]

\[
= s \left[ \left\{ at_w - \frac{1}{2} ar - b t_w^2 + \frac{br t_w^3}{3} \right\} + \left\{ a t_1 - t_2 - \frac{1}{2} ar - b t_1^2 - t_2 + \frac{br t_1^3}{3} - t_2^3 \right\} \right]
\]

\[
+ \left\{ a \left( T - t_2 - \frac{ar}{2} t_2^2 - t_1^2 \right) \right\}
\]

\[
+ \left\{ a \left( T - t_2 - \frac{ar}{2} T^2 - t_2^2 - a \delta \left( \frac{T^2}{2} - T t_2 + \frac{t_2^2}{2} \right) + a r \delta \left( \frac{T^3}{6} - T \frac{t_2^2}{2} + \frac{t_2^3}{3} \right) \right) \right\} \right] \right)
\]

(5.43)

Now, according to the situation of trade-credit period there may arise three different cases:

**Case 1: (t_w ≤ M ≤ t_1)** In this case, retailer can sell the items and earn interest with rate \( i_s \) until the end of the credit period \( M \).

Thus, interest earned per cycle = \( i_s \left[ s \int_{0}^{M} \left( a + bt \right) e^{-rt} dt \right) \)

\[
= \frac{i_s s}{12} \left[ 2a \left( 3M^2 - 2rM^3 \right) + b \left( 4M^3 - 3rM^4 \right) \right]
\]

(5.44)
On the other hand, the retailer still has some inventory on hand when paying the total purchasing amount to the supplier. Hence, for the items still in stock, retailer has to pay interest at a rate of $i_c$.

Interest charged per cycle $= i_c s \left[ \int_{t_1}^{t_2} q_s(t) e^{-r t} dt + \int_{t_i}^{t_2} q_s(t) e^{-r t} dt \right]$

$$= i_c s \left[ S_0 \left( 1+ \frac{\theta_1 t_1^2}{2} \right) \left( t_1 - M - \frac{r t_1^2}{2} + \frac{r M^2}{2} \right) - \frac{\theta_2}{6} \left( t_1^3 - M^3 \right) + \frac{r \theta_2}{8} \left( t_1^4 - M^4 \right) \right]$$

$$+ \left( a t_1 + \frac{b t_1^2}{2} + \frac{a \theta_1 t_1^3}{6} + \frac{b \theta_2 t_1^4}{8} \right) \left( t_1 - M - \frac{r t_1^2}{2} + \frac{r M^2}{2} \right) - \frac{a \theta_1 t_1^4}{2}$$

$$+ \frac{1}{6} (b - 2 a r) \left( t_1^3 - M^3 \right) + \frac{1}{24} \left( a \theta_2 - 3 b r \right) \left( t_1^4 - M^4 \right) + \frac{\theta_2}{120} (3 b - 4 a r) \left( t_1^5 - M^5 \right)$$

$$- \frac{b \theta_2 r}{48} \left( t_1^6 - M^6 \right) + \frac{\theta t_2}{12} (2 a + b t_1) \left( t_1^3 - M^3 \right) - \left( \frac{a \theta_1 t_1^4}{8} + \frac{b \theta_2 t_1^5}{20} \right)$$

$$+ \left( \frac{a \theta_2 M^4}{8} + \frac{b \theta_2 M^5}{20} \right) + \frac{a \theta_1 r t_1^4}{24} + \frac{a \theta_2 r t_1 M^3}{24} + \frac{a \theta_2 r}{10} \left( t_1^5 - M^5 \right) - \frac{b \theta_2 r}{16} \left( t_1^6 - t_1^5 M^4 \right)$$

$$+ \frac{b \theta_2 r}{24} \left( t_1^6 - M^6 \right) + a t_2 \left( t_2 - \frac{r t_2^2}{2} \right) - a t_2 \left( t_2 - \frac{r t_1^2}{2} \right) + \frac{a \theta_2}{6} \left( t_2^4 - t_1^4 \right) - \frac{a \theta_2}{5} \left( t_2^5 - t_1^5 \right)$$

$$- \frac{a \theta_2}{30} \left( t_2^6 - t_1^6 \right) + \frac{a r \theta_2}{24} \left( t_2^5 - t_1^5 \right)$$

$$= \frac{i_c s}{12} \left[ 2 a \left( 3 M^2 - 2 r M^3 \right) + b \left( 4 t_1^3 - 3 r t_1^4 \right) \right]$$

(5.45)

Therefore, the total profit per unit time for case 1 is obtained as follows:-

$$\pi_1(t_1, t_2) = \frac{1}{T} \{ \text{Sales revenue} - \text{purchase cost} - \text{ordering cost} - \text{holding cost in RW} - \text{holding cost in OW} - \text{backorder cost} - \text{opportunity cost} - \text{interest charged} + \text{interest earned} \}$$

(5.46)

Case 2: ($t_1 \leq M \leq t_2$) In this case, retailer can sell the items and earn interest with rate $i_e$ during the trade-credit period. Therefore, interest earned per cycle $= i_e s \left[ \int_{0}^{t_1} (a + b t) e^{-r t} dt + \int_{t_1}^{M} a e^{-r t} dt \right]$

$$= \frac{i_e s}{12} \left[ 2 a \left( 3 M^2 - 2 r M^3 \right) + b \left( 4 t_1^3 - 3 r t_1^4 \right) \right]$$

(5.47)
The retailer still has some inventory on hand when paying the total purchasing amount to the supplier. Therefore, for the items still in stock, retailer has to pay interest at a rate of \( i_c \).

Interest charged per cycle = \( i_c t_1 \int_{t_1}^{t_2} q_d(t)e^{-\alpha t}dt \)

\[
= i_c \int_{t_1}^{t_2} \left[ a\left(t_2 - M - \frac{rt_2^2}{2} + \frac{rM^2}{2}\right) - \frac{(t_2^2 - M^2)}{2} + \frac{r(t_2^3 - M^3)}{3} \right] \, dt \\
+ \frac{a\theta_2}{6} \int_{t_1}^{t_2} \left[ t_2 - M - \frac{rt_2^2}{2} + \frac{rM^2}{2}\right] - \frac{(t_2^4 - M^4)}{4} + \frac{r(t_2^5 - M^5)}{5} \, dt \\
- \frac{a\theta_2}{2} \int_{t_1}^{t_2} \left[ \frac{t_2M^3}{3} + \frac{M^4}{4} - \frac{rt_2^5}{20} + \frac{rM^5}{4} - \frac{rt_2^3}{2} \right] \, dt
\]

(5.48)

Therefore, the total profit per unit time for case 2 is obtained as follows:

\[
\pi_2(t_1, t_2) = \frac{1}{T} \{ \text{Sales revenue} - \text{purchase cost} - \text{ordering cost} - \text{holding cost in RW} - \text{holding cost in OW} - \text{backorder cost} - \text{opportunity cost} + \text{interest earned} \} \tag{5.49}
\]

**Case 3:** \((t_2 \leq M)\) In this case, as the permissible delay time expires on or after the inventory is depleted completely, the retailer pays no interest for the purchase items. Through the credit period, retailer sells the items and uses the sales revenue to earn interest at a rate of \( i_e \).

Thus, interest earned per cycle

\[
= i_e \left[ \int_{t_1}^{t_2} (a + bt)e^{-\alpha t}dt + \int_{t_1}^{t_2} ate^{-\alpha t}dt + \int_{t_1}^{t_2} (M - t_2)\left( a + bt \right)e^{-\alpha t}dt + \int_{t_1}^{t_2} ae^{-\alpha t}dt \right]
\]

\[
= i_e \left[ b \left( \frac{t_2^3}{3} - \frac{rt_1^3}{4} \right) + a \left( \frac{t_2^2}{2} - \frac{rt_2^3}{3} \right) + (M - t_2) \left( b \frac{t_1^2}{2} - \frac{rt_1^3}{3} \right) + a \frac{t_2^2}{2} \right]
\]

(5.50)

Therefore, the total profit per unit time for case 3 is obtained as follows:

\[
\pi_3(t_1, t_2) = \frac{1}{T} \{ \text{Sales revenue} - \text{purchase cost} - \text{ordering cost} - \text{holding cost in RW} - \text{holding cost in OW} - \text{backorder cost} - \text{opportunity cost} + \text{interest earned} \} \tag{5.51}
\]

- **Solution Procedure**

Our objective is to maximize the total average profit per unit time \( \pi_i(t_1, t_2) \), for case \( i \) (where \( i=1,2,3 \)) the optimal values of \( t_1 \) and \( t_2 \) can be obtained by solving the following equations simultaneously
\[ \frac{\partial \pi_1}{\partial t_1} = 0, \quad \frac{\partial \pi_1}{\partial t_2} = 0 \] (5.52)

provided, they satisfy the following conditions

\[ \frac{\partial^2 \pi_1}{\partial t_1^2} > 0, \quad \frac{\partial^2 \pi_1}{\partial t_2^2} > 0 \] (5.53)

and

\[ \left( \frac{\partial^2 \pi_1}{\partial t_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial t_2^2} \right) > 0 \] (5.54)

Equations (5.53) and (5.54) are highly non-linear and hence are solved with the help of mathematical software **MATHEMATICA 5.2**. With the use of the optimal values of \( t_1 \) and \( t_2 \) for case 1, 2 and 3 the optimal values of the total profit per unit time for these cases can be obtained from equation (5.46), (5.49) and (5.51) respectively.

**Numerical Examples**

The data is based on the previous study and is taken similar to [26].

**Example 1.** (case 1: \( t_w \leq M \leq t_1 \))

\( a = 800, b = 0.008, \theta_1 = 0.09, \theta_2 = 0.06, t_w = 20, M = 40, T = 80, r = 0.06, S_0 = 100, q_r = 200, W = 600, A = 100, \delta = 0.7, c_1 = 0.6, c_2 = 0.3, c_3 = 0.5, c_4 = 5, c_5 = 12, i_e = 0.09, i_c = 0.10, s = 14, t_1 = 48.2358, t_2 = 65.6534, H_{RW} = 42.3468, H_{OW} = 2210.34, BC = 122756, OC = 47826, Q = 285478, \text{Total profit } \pi_1(t_1, t_2) = 62440.5 \)

**Example 2.** (case 2: \( t_1 \leq M \leq t_2 \))

\( a = 800, b = 0.008, \theta_1 = 0.09, \theta_2 = 0.06, t_w = 20, M = 55, T = 80, r = 0.06, S_0 = 100, q_r = 200, W = 600, A = 100, \delta = 0.7, c_1 = 0.6, c_2 = 0.3, c_3 = 0.5, c_4 = 5, c_5 = 12, i_e = 0.09, i_c = 0.10, s = 14, t_1 = 49.3856, t_2 = 66.7283, H_{RW} = 44.3638, H_{OW} = 2250.28, BC = 122993, OC = 47958, Q = 285553, \text{Total profit } \pi_2(t_1, t_2) = 64356.7 \)

**Example 3.** (case 3: \( t_2 \leq M \))

\( a = 800, b = 0.008, \theta_1 = 0.09, \theta_2 = 0.06, t_w = 20, M = 70, T = 80, r = 0.06, S_0 = 100, q_r = 200, W = 600, A = 100, \delta = 0.7, c_1 = 0.6, c_2 = 0.3, c_3 = 0.5, c_4 = 5, c_5 = 12, i_e = 0.09, i_c = 0.10, s = 14, t_1 = 50.8973, t_2 = 67.5286, H_{RW} = 46.9482, H_{OW} = 2289.67, BC = 123189, OC = 48138, Q = 285756, \text{Total profit } \pi_3(t_1, t_2) = 66983.4 \)

**Sensitivity Analysis:**

We now study the effects of changes in the values of the system parameters total profit.
Table 5.3 Effects of changes in the values of the system parameters total profit

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<th>$t_2 \leq M$</th>
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Fig. 5.3 Variation in profit w.r.t. $A$ for case $t_w \leq M \leq t_1$

Fig. 5.4 Variation in profit w.r.t. $A$ for case $t_2 \leq M$
Fig. 5.5 Variation in profit w.r.t. $s$ for case $t_w \leq M \leq t_1$

Fig. 5.6 Variation in profit w.r.t. $s$ for case $t_2 \leq M$
Fig. 5.7 Variation in profit w.r.t. $c_3$ for case $t_w \leq M \leq t_1$

Fig. 5.8 Variation in profit w.r.t. $c_3$ for case $t_2 \leq M$
Fig. 5.9 Variation in profit w.r.t. W for case $t_w \leq M \leq t_1$

Fig. 5.10 Variation in profit w.r.t. W for case $t_2 \leq M$
Fig. 5.11 Variation in profit w.r.t. $\delta$ for case $t_w \leq M \leq t_1$

Fig. 5.12 Variation in profit w.r.t. $\delta$ for case $t_2 \leq M$
Fig. 5.13 Variation in profit w.r.t. 'a' for case $t_w \leq M \leq t_1$

Fig. 5.14 Variation in profit w.r.t. 'a' for case $t_2 \leq M$
Observations

(i) From Table 5.3, total profit is increased by increasing the parameter of ordering cost.

(ii) From Table 5.3, selling price gives the reverse effect on total profit.

(iii) From Table 5.3, total profit is increased by increasing the backlogging parameter (δ) (or equivalently decreasing the backlogging rate).
(iv) From Figure 5.9 and Figure 5.10, profit increase with increase in the value of parameter \( W \). A corresponding trend is noticed in the case of parameter \( W \), as an increase in this parameter \( W \), increases the total profit.

(v) From Figures 5.13, 5.14, 5.15 and 5.16, total profit increases with increase in the values of demand parameters ‘a’ and ‘b’ to a greater extent, this shows the system has a greater flexibility towards some other parameters.

5.7 Conclusion

A linearly time-varying demand means a uniform change in the demand rate of the product per unit time. In this chapter, we have developed an inventory model for deteriorating items with linearly time-dependent demand rate followed by finite warehouse capacity, permitting shortage and time-proportional backlogging rate. In particular, the backlogging rate is considered to be a decreasing function in the waiting time until the next replenishment which is more realistic.

It is true that the stock-out is very difficult to measure. In practice, we can observe periodically, the proportion of demand which would accept backlogging and the corresponding waiting time for the next replenishment. Furthermore, we found the following results:

(i) We show that the inventory policy with partial backlogging is more profitable. Since, as discussed previously, constant deterioration is not a viable concept; hence, we have considered an inventory with deterioration increasing linearly with time which is more realistic.

(ii) To make the study more suitable to present-day market, we have done our research in an inflationary environment under the facility of permissible delay in payment. Three different cases are discussed according to the situation of the delay period.

(iii) To reduce the inventory costs, it will be economical for firms to store goods in OW before RW, but clear the stocks in RW before OW. Also, in this study, it is assumed that holding costs and deterioration costs are different in OW and RW due to different preservation environments which are more realistic and profitable. The inventory in RW is assumed to be higher than those in OW. Solution procedure is provided to arrive at the optimal solution. To illustrate the model for different cases, numerical examples and sensitivity analysis are implemented.