CHAPTER III

FUZZY FILTERS AND ULTRAFUZZY FILTERS

Let $L$ be a complete and distributive lattice, and $X$ be any set. In this chapter fuzzy filters on $X$ are defined on the lines of definition given by A.K. KATSARAS [16] and P. SRIVASTAVA and R.L. GUPTA [23], by taking $L$ to be the membership set, instead of the closed unit interval $[0,1]$. Ultrafuzzy filters are defined and characterized in terms of properties of the membership lattice. Study is extended to the case when the membership lattice is further, complemented as well.

We denote the complement of an element $a$ by $a'$, and $I$ is commonly used to denote an arbitrary index set with $i$ denoting a general element in it.

3.1 Definitions:

A nonempty subset $F$ of $L(X)$ is said to be a fuzzy filter if

i) $0 \in F$

ii) $a, b \in F$ implies $a \land b \in F$

and iii) $a \in F$, $b \in L(X)$ and $b \geq a$, imply $b \in F$. 

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A nonempty subset \( B \) of \( L(X) \) is said to be a fuzzy filterbase if i) \( 0 \not\supseteq B \), and ii) \( a, b \in B \) implies that there exists \( c \in B \) such that \( c \subseteq a \land b \).

A subset \( B \) of \( L(X) \) is said to be a base for a fuzzy filter \( F \), if \( F = \{ a \in L(X) : a \supseteq b, \text{ for some } b \in B \} \).

Let \( x \in X \) and \( l \in L \), be such that \( l \neq 0 \). Let \( s(x, l) \), denote the fuzzy subset defined by, for \( y \in X \),

\[
s(x, l) = \begin{cases} 
0 & \text{if } y \neq x \\
1 & \text{if } y = x.
\end{cases}
\]

\( s(x, l) \) is called the fuzzy singleton at \( x \).

A fuzzy filter is said to be an ultrafuzzy filter if it is not properly contained in any other fuzzy filter.

Let \( a \) be a nonzero fuzzy subset of \( X \). Then the subset \( P(a) \) of \( L(X) \) defined by \( P(a) = \{ b \in L(X) : b \supseteq a \} \), is a fuzzy filter on \( X \), called the principal fuzzy filter at \( a \).

### 3.2 Existence of ultrafuzzy filters

#### 3.2.1 Theorem

Every fuzzy filterbase \( B \), determines a fuzzy filter \( F \), uniquely such that \( B \) is a base for the fuzzy filter \( F \).
Proof

Let \( B \) be a fuzzy filterbase and

\[
F = \{ a \in L(X) : a \geq b, \text{ for some } b \in B \}.
\]

Clearly, \( F \) is nonempty and \( 0 \notin F \), as \( B \) is a subset of \( F \) and 
\( 0 \notin B \). Further, if \( a, b \in F \), then there exists \( c, d \) in \( B \) such that 
\( c \leq d \) and \( d \leq b \). But then there exists an \( e \) in \( B \) such that 
\( e \leq c \land d \). Thus \( e \leq c \land d \leq a \land b \), and therefore, \( a \land b \)
belongs to \( F \).

Trivially, for every \( a \) in \( F \), if \( b \) in \( L(X) \) is such that \( b \geq a \), then \( b \in F \). Thus \( F \) is a fuzzy filter and \( B \) is a base for \( F \). Uniqueness is immediate from the definition of \( F \).

3.2.2 Theorem

Every fuzzy filter is contained in an ultrafuzzy filter.

Proof

Let \( F \) be a fuzzy filter and \( \mathcal{B} \) be the set of all fuzzy filters containing \( F \). \( \mathcal{B} \) is nonempty, as \( F \) belongs to it. \( \mathcal{B} \) is partially ordered under set inclusion. Let \( \{ U(i) : i \in I \} \) be a chain in \( \mathcal{B} \). Let \( U = \bigcup U(i) \). We claim that \( U \) is an upper bound for the chain.
i) \( O \subseteq U \), since \( O \subseteq U(i) \), for each \( i \).

ii) If \( a, b \in U \), then \( a \in U(i) \) and \( b \in U(j) \), for some \( i, j \) in \( I \). Then either \( U(i) \subseteq U(j) \) or \( U(j) \subseteq U(i) \), since \( U(i), U(j) \) belong to a chain. Let us assume that \( U(i) \subseteq U(j) \). Thus \( a, b \in U(j) \), and hence \( a \wedge b \in U \).

iii) Let \( a \in U \) and \( b \in L(X) \) be such that \( b \geq a \). Then \( a \) is in \( U(i) \) for some \( i \) in \( I \). But then \( b \) belongs to that \( U(i) \) and hence \( b \in U \).

From i), ii) and iii), \( U \) is a fuzzy filter. Clearly \( U \) is bigger than \( U(i) \) for each \( i \) in \( I \) and \( U \) contains \( F \). Hence \( U \) is an upper bound for the chain. Thus we have proved that every chain in \( \mathcal{A} \) has an upper bound, hence by Zorn's lemma, \( \mathcal{A} \) contains maximal elements. i.e., there exist ultrafuzzy filters containing \( F \).

### 3.2.3 Theorem

A fuzzy filter \( U \) is an ultrafuzzy filter iff \( a \in L(X) \) and \( a \wedge u \neq O \), for all \( u \) in \( U \), imply that \( a \) is in \( U \).

**Proof**

Necessary: Let \( U \) be an ultrafuzzy filter and \( a \) in \( L(X) \) be such that \( a \wedge u \neq O \) for all \( u \) in \( U \). Let \( B = \{ a \wedge u : u \in U \} \). Now \( B \) is nonempty, \( O \subseteq B \) and \( B \) is closed for meet operation, since, for \( u, v \) in \( U \), \( u \wedge v \) is in \( U \) and \( (a \wedge u) \wedge (a \wedge v) = a \wedge (u \wedge v) \).
Thus $B$ is a fuzzy filterbase and contains $a$, since $1$ is in $U$. Hence by the theorem (3.2.1), there exists a fuzzy filter $F$ containing $B$. But then $U$ is a subset of $F$, since for every $u$ in $U$, $u$ is $\geq a \land u$. $U$ being an ultrafuzzy filter, this implies that $U = F$. Thus $a$ is in $U$.

Sufficiency: Let $U$ be a fuzzy filter containing all $a$ in $L(X)$ such that $a \land u \neq 0$, for all $u$ in $U$. Suppose $F$ is a fuzzy filter containing $U$. Then for every $f \in F$, $fu \neq 0$, for all $u$ in $U$, since $u \in F$, also. But then by the hypothesis, $f$ is in $U$, for every $f$ in $F$. Thus $F$ is a subset of $U$ and Therefore, $F = U$. Since, $F$ is arbitrary, $U$ is an ultrafuzzy filter. This completes the proof.

3.2.4 Theorem

If $F$ is a fuzzy filter such that for all $a$ in $L(X)$, either $a$ or $a'$ (if exists), belong to $F$, then $F$ is an ultrafuzzy filter on $X$.

Proof

Suppose $G$ is a fuzzy filter containing $F$. Let $a \in G$. Then either i) $a'$ exists, or ii) $a'$ does not exists.

Case i) By hypothesis, either $a$ is in $F$ or $a'$ is in $F$. $a'$ cannot be in $F$, for otherwise, $a'$ will also belong to $G$, which is impossible, as $a \in G$. Thus $a$ is in $F$.
Case ii). By hypothesis, a belongs to F. Thus in either case, a \in G implies a \in F. Therefore, F contains G, and hence F = G. Since G is an arbitrary fuzzy filter containing F, F is an ultrafuzzy filter. The proof is complete.

3.2.5 Remark

If there exists a, b in L(X), such that a \wedge b = 0 and a' and b' do not exist, then no fuzzy filter on X can satisfy the hypothesis of the theorem (3.2.4). This would imply that the hypothesis is not a necessary condition for a fuzzy filter to be an ultrafuzzy filter.

3.2.6 Theorem

If L is not complemented and X contains at least two elements, then no ultrafuzzy filter on X, satisfies the hypothesis of theorem (3.2.4).

Proof

Let 1 in L be such that 1' does not exist. Let x, y \in X be such that x \neq y. Consider the fuzzy singletons s(x,1) and s(y,1), of X. Clearly s(x,1) and s(y,1) do not have complements in L(X), and s(x,1) \wedge s(y,1) = 0. Hence by the remark (3.2.5), we have the theorem.
3.2.7 Remark

Suppose L is not complemented and F is a fuzzy filter on X such that for all complemented elements a in L(X), either a or a' is in F. This does not imply that F is an ultrafuzzy filter on X.

Let L = [0,1] and X be any set. Let, for a fixed z in X, F = {a ∈ L(x); a ≥ s(z,1)} for a fixed z in X. Then F is a fuzzy filter on X. Let a be a complemented fuzzy subset of X. Then a ∧ a' = 0 and a ∨ a' = 1, i.e., for every x in X, a(x) ∧ a'(x) = 0 and a(x) ∨ a'(x) = 1. Thus for each x in X either a(x) = 1 and a'(x) = 0, or a(x) = 0 and a'(x) = 1.

Thus, either a ≥ s(z,1) or a' ≥ s(z,1). Therefore, either a or a' is in F. But F is not an ultrafuzzy filter, since {a ∈ L(x); a ≥ s(z,0.5)} is a fuzzy filter bigger than F.

3.2.8 Theorem

Let F be an ultrafuzzy filter on X, and a in L(X) be a complemented fuzzy subset. Then either a ∈ F or a' ∈ F.

Proof

Let a' ∉ F. Then we claim that a ∧ b ≠ 0 for all b in F. For if there exists b in F such that a ∧ b = 0, then

b = b ∧ (a ∨ a') = (b ∧ a) ∨ (b ∧ a') = b ∧ a'.

Thus a' ≥ b, and hence a' ∉ F, a contradiction. Hence the claim.
Let \( B = \{ a \wedge b : b \in F \} \). Then \( B \) is a fuzzy filterbase, containing \( a \). Let \( U \) be the fuzzy filter generated by \( B \) (given by the theorem 3.2.1). Then clearly \( F \) is contained in \( U \), and since \( F \) is an ultrafuzzy filter, \( F = U \). Therefore, \( a \in F \). Hence the theorem.

3.2.9 Theorem

Let \( U \) be an ultrafuzzy filter on \( X \). Then for \( a, b \) in \( L(X) \), if \( a \vee b \subseteq U \), implies either \( a \subseteq U \) or \( b \subseteq U \).

Proof

Let \( a, b \) in \( L(X) \) be such that \( a \vee b \) is in \( U \). Let \( a \not\subseteq U \). Then we claim that \( b \wedge u \neq 0 \), for all \( u \) in \( U \). For otherwise, there exists some \( u \) in \( U \) such that \( b \wedge u = 0 \). But then \((a \vee b) \wedge u \subseteq U \), would imply that

\[
(a \vee b) \wedge u = (a \wedge u) \vee (b \wedge u) = a \wedge u \subseteq U.
\]

Thus \( a \) is in \( U \), as \( a \not\subseteq a \wedge u \), a contradiction. Therefore, \( b \wedge u \neq 0 \) for all \( u \) in \( U \). Consider \( B = \{ b \wedge u : u \subseteq U \} \). \( B \) is a fuzzy filterbase containing \( b \). Clearly the fuzzy filter \( W \), generated by \( B \) contains \( U \). Thus \( U = U \), as \( U \) is an ultrafuzzy filter. Therefore, \( b \subseteq U \). Hence the theorem.

3.3 Principal fuzzy filter.

3.3.1 Therem

Let \( a \in L(X) \). A principal fuzzy filter at \( a \) on \( X \),
is an ultrafuzzy filter iff \(a\) is a fuzzy singleton \(s(x, l)\), for some \(x\) in \(X\), such that \(l\) is an atom in \(L\).

Proof

**Necessary:** Let \(P\) be the principal fuzzy filter at \(a\). Suppose \(P\) is an ultrafuzzy filter, then there exists a unique \(x\) in \(X\) such that \(a(x) > 0\). For otherwise, the fuzzy singleton \(s(x, a(x)) \ll a\), and hence the principal fuzzy filter at \(s(x, a(x))\), will be bigger than \(P\), a contradiction. Further \(a(x)\) must be an atom, for otherwise, there exists an \(m\) in \(L\) such that \(0 < m < a(x)\). Then again the principal fuzzy filter at \(s(x, m)\) would be bigger than \(P\), a contradiction. Thus \(a = s(x, a(x))\), where \(a(x)\) is an atom.

**Sufficiency:** Let \(l\) be an atom in \(L\) and \(x\) be a fixed element in \(X\). Let \(P\) be the principal fuzzy filter at \(s(x, l)\). Suppose, \(U\) is a fuzzy filter containing \(P\). As \(s(x, l) \in U\), for every \(u\) in \(U\), \(s(x, l) \land u = s(x, l \land u(x)) \in U\). Thus \(1 \land u(x) \neq 0\), therefore, \(1 \land u(x) = l\), since \(l\) is an atom. Thus \(u(x) \succeq l\), i.e., \(u \succeq s(x, l)\). Therefore, \(u\) is in \(P\). Hence \(P = U\). Since \(U\) is arbitrary, \(P\) is an ultrafuzzy filter. The proof of the theorem is complete.

3.3.2 Remark

If a principal fuzzy filter at a fuzzy singleton is not an ultrafuzzy filter then there exists a finer
principal fuzzy filter. If in \( L \), every nonzero element, either is an atom or has an atom below it, then every principal fuzzy filter is contained in a principal ultrafuzzy filter. Unlike in the ordinary set theory, principal fuzzy filters on fuzzy singletons are not maximal. One may note that principal fuzzy filter at every fuzzy singleton is an ultrafuzzy filter iff \( L = \{ 0,1 \} \).

3.4 Fuzzy filters with complemented membership lattice.

3.4.1 Theorem

If \( L \) is complemented, then for every fuzzy filter \( F \) on \( X \), the following are equivalent.

i) \( F \) is an ultrafuzzy filter.

ii) For \( a \) in \( L(X) \), either \( a \) is in \( F \) or \( a' \) is in \( F \).

iii) For \( a, b \) in \( L(X) \), if \( a \lor b \in F \) implies \( a \in F \) or \( b \in F \).

Proof

i) implies ii): Suppose \( F \) is an ultrafuzzy filter, \( a \in L(X) \) and \( a' \not\in F \). Then \( a \land b \neq 0 \), for all \( b \not\in F \), or otherwise, let \( a \land b = 0 \) for some \( b \) in \( F \). Then \( b = b \land (a \lor a') = (b \land a) \lor (b \land a') \);

\( (b \land a') = b \land a' \). Which implies \( a' \geq b \). Hence \( a' \) is in \( F \), a contradiction. Thus \( a \land b \neq 0 \) for all \( b \) in \( F \). Hence by the theorem (3.2.3), \( a \in F \), i.e., for all \( a \) in \( L(X) \) either \( a \in F \) or \( a' \not\in F \).
ii) implies iii): Let \( a, b \in L(X) \), be such that \( a \lor b \) is in \( F \). Suppose \( a \notin F \). Then by ii) \( a' \) belongs to \( F \). Thus \( a' \) and \( a \lor b \) are in \( F \). Hence, \( a' \land (a \lor b) = a' \land b \in F \). Thus \( b \in F \), since \( b \geq b \land a' \). Therefore, \( a \lor b \in F \) implies either \( a \) or \( b \), belongs to \( F \).

iii) implies i): Let \( F \) be a fuzzy filter on \( X \) satisfying iii) Suppose \( G \) is a ultrafuzzy filter containing \( F \). Then, for \( a \) in \( G \), since \( a \lor a' = 1 \), and \( 1 \in F \), by iii), either \( a \in F \) or \( a' \in F \). But \( a' \) cannot belong to \( F \), since \( a \in G \), and \( F \) is a subset of \( G \). Thus \( a \not\in F \). Therefore, \( G \) is subset of \( F \). Hence \( F \) is an ultrafuzzy filter.