Chapter 1

INTRODUCTION

1.1 General introduction

A precise computation and prediction of Earth's satellite is a basic requirement for accurate mission planning, satellite geodesy, re-entry and orbital lifetime estimates. A satellite moving under the gravitational field of Earth experiences the effect of forces, known as the perturbing forces like the shape of the Earth, the Sun's radiation, the resistance of the atmosphere, gravitational attraction due to the Sun and the Moon, the Earth's magnetic field etc. The combined effects of these perturbations cause the satellite orbit to deviate from two-body elliptic orbit. For near-Earth orbits, the forces due to the Earth's flattening and air drag are responsible to bring the satellite back to the Earth. Thus inclusion of the effect of these perturbing forces becomes important for precise orbit computation of near-Earth orbits. For calculating the ephemeris of an artificial Earth's satellite, it has become necessary to use extremely complex force models in order to describe satellite motion with the accuracy that is consistent with the present day operational requirements and observational techniques. The effect of the atmosphere is difficult to determine because the atmospheric density and hence the drag undergoes large modeled fluctuations. To predict the motion precisely a mathematical representation for these forces must be selected for integrating the resulting differential equations of motion. The options for mathematical solutions can be classified as analytical, semi-analytical and numerical. Both analytical and semi-analytical methods use analytical transformations to produce mean, slowly varying differential equations. The major distinction lies in the fact that semi-analytical methods provide a totally analytical solution to these differential equations. However, the numerical integration methods can provide very accurate ephemeris of a satellite with respect to any type of perturbing forces. But these methods prove very costly in terms of computer time and at times are of questionable validity in high accuracy region.
In the literature, there are a number of analytical solutions, which describe the effects of atmospheric drag on the motion of an artificial satellite in the gravitational field of an oblate Earth. Newton [1] was the first scientist who studied the effect of drag on the orbit of a satellite. He showed that a body acted on by an inverse square gravitational attraction and moving in an atmosphere with density proportional to $1/r$ follows a contracting equiangular spiral path. This was the general estimation for the next 250 years. Singer [2] developed a semi-analytical method to evaluate first the lifetimes of circular orbits and then by assuming an impulsive deceleration at perigee, he estimated the lifetimes for elliptic orbits. His choice of the profile of the air density was quite close to that which was later established for times of low solar activity. Henry [3] discussed lifetimes of the elliptic orbits. A semi-numerical method based on the classical Newtonian equations was presented by Davis, Whipple and Zirker [4]. All these works were carried out before the launching of Sputnik in October 1957. The launching of Sputnik provoked several further attempts at methods of predicting satellite lifetimes. There were a number of papers on determining density from the changes in the orbital period of a satellite. Methods of increasing power were given by Groves [5], Sterne [6] and King-Hele [7]. Theories defining the variation of orbital period, perigee distance and eccentricity with time were first developed by Nonweiler [8] and then by King-Hele and Leslie [9]. The latter work is developed into a very important book [10]. The book was considerably improved later [11] by adding new researches in the area of satellite motion under the effect of air drag force and some other perturbing forces. Perkins [12] used perturbation solutions to obtain life time estimates and results were given for the cases of exponential and power-law density variation. Parkyn [13] integrated analytically Lagranges’ planetary Equations [14] and obtained the variations in $\omega$ and $e$ in terms of Bessel functions [15]. The effect of non-stationary atmosphere on the orbital inclination were considered by Bosanquet [16], Vinti [17] and also by Merson and Plimmer [18]. Using a power-law variation of density with height Michielsen [19] formulated the orbital variations during a satellite’s whole lifetime while per-orbit changes were investigated by Parkyn [20], Ewart [21] and many others.

The variational equations which describe the motion of an artificial satellite about the center of the Earth are usually expressed in terms of classical orbital
elements (i.e., semi major axis $a$, eccentricity $e$, inclination $i$, ascending node $\Omega$, argument of perigee $\omega$ and mean anomaly $M$). The dynamical system of a satellite motion perturbed by both atmospheric drag and gravitational attraction is nonlinear, non conservative in form and the integration of the system, in general, is analytically intractable. Some of the early studies and analytical difficulties for the coupled problem were addressed by de Nike [21]. Suitable perturbation methods in Celestial Mechanics to derive an approximate solution with desirable accuracy are (i) the von Ziepel method [22,23,24] (ii) the two-variable asymptotic expansions [25,26] (iii) the Lie series [27,28] and (iv) the general theory of the method of averaging [29,30]. Morrison [30,31] has showed that both the von Ziepel method and the two-variable asymptotic expansions are special cases of the general theory of the method of averaging. Hori [32], Kamel [33] and Choi and Tapley [34] have discussed the extension of the Lie series to non canonical systems. While, Shinad [35] discussed the equivalence of the Von Zeipel method and the Lie series. Watanabe [36] and Ahmed and Tapley [37] discussed the equivalence of the method of averaging and the Lie transform. Brouwer [38] used the Von Zeipel method to obtain a drag free solution for an artificial satellite perturbed by $J_2$ to $J_5$ zonal harmonics of the Earth's gravitational potential. Later Brouwer and Hori [39] extended the solution by including the atmospheric drag. They used a non rotating spherical density model. The exponential density function is approximated using Taylor series expansion. The solutions do not have sufficient convergence if the exponent is not small enough. The authors [40] discussed a similar approximation of the density function without modification of the solution form such that it would offer better convergence.

After Brouwer and Hori [39] published their work in 1961, Lane [41], Lane and Cranford [42], Zee [43], Barry and Rowe [44], Willey and Piscane [45], Chen [46], Watson et al [47], Santora [48], Muller et al [49], Hoots [50,51] and Vilhena de Moraes [52] have obtained analytic solutions. Lane [41] used a non rotating spherical power function density model [53]. The theory has been carried out to the same order as that of the Brouwer-Hori theory and has the same limitations. The use of the power function density model removed the convergence problem. Lyddane [54] and Davenport [55] examined the problems of small eccentricity and small inclination for Brouwer's [38] drag free solution and gave their modifications to remedy the singularities. Lane and
Cranford [42] improved Lane's [41] theory by reformulating the theory to eliminate the \( e \) and \( \sin i \) divisors. It is also an extension of Lyddane's [54] modification for a drag-free solution to remove the additional small divisors present in the drag terms. The complete explicit solutions for all six orbital elements were given. Though numerical results were not included in their theory, truncated version of their theory [56] has been used in the NORAD operational system for many years. Zee [43] used a set of dimensionless variables derived from the spherical coordinates and an averaging method [57] to obtain a first-order singly averaged dynamical system. In his theory, only the second zonal harmonics and a non rotating spherical exponential density model are taken into consideration and are restricted to small values of eccentricity.

Barry and Rowe [44] used a Fourier series expression for the density model in which the coefficients were determined by the Jacchia 1970 model [58]. The solutions are obtained through first-order periodic and second-order secular effects for the Earth's oblateness \( J_2, J_3 \) and \( J_4 \), and first-order drag effects. Lorell and Liu [59], Liu and Alford [60], Liu [61] and Slutsky and McClain [62] propose the drag-free theories in an analytical form, by the averaging of conservative perturbations. Willey and Pisacane [45] extended the Lane theory [41] by introducing a power function density model with a quadratic instead of a linear density scale height. Their complete solutions and detailed analysis are given in [63]. Chen [46] introduced a modified exponential density function to take the atmospheric oblateness and the diurnal variation of the density into consideration. The values of the parameters, density \( \rho_0 \), density scale height \( H \) and the measure of the amplitude of the day to night variation in density \( F \) are computed using Jacchia 1971 density model [64]. Chen assumed that the drag perturbing force is a second-order quantity and the associated perturbing terms are expanded in power series of \( e \) and retaining terms up to \( e^2 \). He then extended his drag free analysis [65] with \( J_2, J_3 \) and \( J_4 \) and obtained a second-order solution for the dynamical system with the combined effect of the Earth's oblateness and a rotating atmosphere using two-variable asymptotic expansions. Watson et al [47] introduced an analytic iterative method to avoid the complexity of the oblateness-drag effects illustrated by Brouwer and Hori [39] and Sherril [66]. In their method, the effect of Earth's oblateness is accounted for by the Vinti Spheroidal theory [67]. Two test cases were given for the long-term decay predictions. In both cases the predicted lifetimes...
were within 4% of the true values without the use of the Vinti differential correction algorithm [68].

Santora [48] extended King-Hele's drag only solution [17] to include both the oblate Earth figure and diurnal density effects for orbits of small eccentricity. The average changes in $a$ and $\chi = a e$ due to drag in one revolution were obtained in terms of the modified Bessel functions. He used Kozai's [69] drag free solution to improve the determination of decay rates. Muller et al. [49] presented an analytic theory to include the short and long-periodic, secular effects of $J_2$ and higher-order zonal harmonics, secular and quadratic drag effects. Hoots [50] used the gravitational and atmospheric models as used by Lane [41] and arrived at an improved analytical solution. A numerical comparison done with a slightly modified Lane and Cranford theory [42] using the same reference orbits showed a noticeable improvement in accuracy. For higher altitude satellites and smaller drag coefficients, the two solutions are generally comparable. Vilhena de Moraes [52] extended Ferraz Mello's [70] drag free solution to include the atmospheric drag modeled by the same exponential density function used in Brouwer and Hori theory [39]. He adopted the transformation suggested by Ferraz Mello [71] and the Delauny angular variables are modified to avoid the appearance of Poisson's terms. He then applied the method of variations of arbitrary constants and successive approximations to obtain his coupled solution. Hoots [51] applied his assumed solutions for the motion and eccentricity and Liu's singly averaged variational equations [72] to arrive at an analytical solution for the same dynamical system (with $J_2$, $J_3$ and $J_4$ and drag) by choosing a rotating empirical density model. The constant parameters and the assumed solutions were determined as in [50].

The analytic expressions of the density models not only lack the dynamic representation of the atmosphere but also may not provide accurate values for the density. An alternative approach of the coupled problem is to adopt a combination of general and special perturbation technique, referred to as a semi-analytic method. This method enhances efficiency through the use of analytical techniques whenever possible and makes sufficient numerical methods to permit the inclusion of an empirical density model without using the series expansions. Well known and commonly used models are Jacchia 1964 [73], Jacchia 1970 [58], Jacchia 1971 [72], Jacchia 1977 [74] and MSIS 78.
The empirical density models [58,72,75] treat the density as a function of altitude above the surface of an oblate Earth, longitude, latitude, solar flux, geomagnetic index and time. In an analytic version of Jacchia 1977 model [74], de Lafontaine and Hughes [79] known as the Global Analytical Model (GAM) avoids the large memory space requirements of the tabular models and the extensive computer time needed by the numerical models. In addition to static variations (including flattening of the atmosphere), the GAM accounts for the solar activity, geomagnetic activity, diurnal, semi-annual and seasonal-latitudinal cycles of density variations. These variations are graphically illustrated in de Lafontaine [80] and de Lafontaine and Mammen [81]. A comparison of the Jacchia 1977 model [74] with accelerometer density data [82] concludes that the root-mean-squared error in the Jacchia density model is around 10%. MSIS-90 uses analytic models to model the lower altitudes to account for disturbances such as solar activity, magnetic storms, and daily variations, as well as latitude, longitude and monthly variations. The application of these and some other empirical atmospheric density models in orbital mechanics in the real world environment has been discussed by Liu et al [82].

Representative works in semi-analytic approach are by Pimm [83], Kaufman and Dasenbrock [84], Barry, Pim and Rowe [85], Dallas and Khan [86], Wu et al [87], Lidov and Solov'ev [88], Green and Cefola [89] and Liu and Alford [90]. Pimm [83] developed a semi-analytic, long-term orbit theory that used Simpson's method to evaluate the averaged drag effects for a rotating atmosphere. Different analytic solutions [69,91] of the Earth's gravitational perturbation due to J₂ to J₆ zonal harmonics were adopted to obtain a combined solution by means of a fourth-order Runge-Kutta (R-K) method. The Jacchia 1964 atmospheric density model [73] was used for the analysis. The historical Smithsonian Astrophysical Observatory (SAO) mean orbital elements, determined by the SAO Differential Orbit Improvement Programme published by the SAO special report [80] were used as the data base. Kaufman and Dasenbrock [83] formulated a lengthy semi-analytic solution for analysing both lunar and terrestrial orbiters. Dallas and Khan [86] developed a semi-analytic theory using the singly averaged differential equations in terms of parameters valid for all eccentricities < 1. Wu et al [87] used an averaging technique similar to that of Kozai [69] and developed a second-order semi-analytic theory to include the perturbations.
due to the non spherical Earth, atmospheric drag, third-body effects and the solar radiation pressure. Two rotating empirical density models, CIRA 1972 [92] and Jacchia 1977 [74] are considered in their theory. In their analysis, all the perturbing forces are treated as second-order quantities except $J_2$. Solov'ev [93] described a semi-analytic drag free theory (including non spherical Earth and third-body perturbations) using singly averaged Delauny variables and the von Zeipel method. Lidov and Solov'ev [88] extended the theory to include the atmospheric drag for a high eccentricity resonant orbit with an exponential density function. Cefola et al [94] discussed a semi-analytic approach to include the Earth's oblateness, third-body perturbations, atmospheric drag and solar radiation pressure using the generalised method of averaging. Green and Cefola [89] assumed Fourier series expansions for the short-periodic variations and developed a semi-analytic solution. The coefficients for the drag variations are determined using a method similar to that of Lutsky and Uphoff [95] by numerical quadrature technique. Using Liu's singly averaged drag free equations [96] which include $J_1$, $J_2$, and $J_3$, Alford and Liu [90] developed a semi-analytic long-term orbit theory using the generalised method of averaging with the assumption that the drag force is a second-order quantity. The procedure is to extend a system of first-order differential equations for a set of well-defined mean orbital elements to include the drag effect due to a rotating atmosphere. Numerical results [97] for the long-term decay predictions versus satellite orbital data obtained from SAO Special Reports demonstrated that the semi-analytic theory can provide a means to estimate the orbital decay history and lifetime with good accuracy and efficiency. Liu and Alford [90] extended their long-term solutions to include the computation of the fast variable, mean anomaly, and the use of an initial orbit determination algorithm [98] so that accurate short-term ephemeris can be generated.

The Stroboscopic method developed by Roth and used in different applications [99,100,101,102] expresses the Variation-of-Parameter (VOP) equations with the true anomaly as the independent variable and time as a dependent variable. In fact most semi-analytic theories [83], [195], [103], Wagner, Douglas and Williamson [104] and Alford and Liu [105] carry out their numerical averaging with respect to the true anomaly although the propagation equations of the mean orbital state always rely on the mean anomaly. The computation of the mean anomaly from the true anomaly is
an explicit operation. de Lafontaine [80] discusses this method to be an extension of
Stroboscopic method. Theories, which discuss the complete transformation of the first-
order in \( J_2 \), will exhibit a second-order secular mean anomaly error due to the
initialization procedure. Lyddane and Cohen [107] have demonstrated this fact by
recovering the second-order quantities due to \( J_2 \) in semi-major axis. Later Breakwell
and Vagners [108] also investigated the problem and concluded that accuracy may be
kept to the second-order by either including the mean motion with the aid of the
energy integral or fitting an orbit theory to data over many revolutions. A theoretical
determination of the projected area, \( A \), and of the drag coefficient, \( C_D \), is also a very
involved field. The most relevant papers on the aerodynamic properties of the
satellites are those by Cook [109,110], Nocilla [111], and Jastrow and Pearse [112].
Drag coefficients were investigated in King-Hele [10], Cook [109,110], Williams
[113,114] and Nocilla [111]. The effects of uncertainty and variation in \( C_D \) are treated
in Hunziker [115] and a discussion of the thermal accommodation coefficient and other
related parameters is found in King-Hele [10], Ladner and Ragsdale [116] and Wachman
[117]. The works in the area of orbital mechanics and the numerous related fields are
provided in Szebehely [118] and methods of orbital determination is explained in
Escobal [119,120].

Sterne [121] and Ewart [122] evaluated the changes in the orbital elements
at the end of one revolution in a general form. Davies [123] evaluated the effects of
atmospheric oblateness. Santora [124,125] studied the combined effect of atmospheric
oblateness and the day-to-night variation with low eccentricity orbits. The effects of
atmospheric rotation and of geomagnetic and solar activity on the accuracy of
prediction of satellite position are studied in [126]. Swinerd and Boulton [127]
presented a more comprehensive atmospheric model that combines atmospheric
rotation, oblateness and the daytime bulge. They determined the perturbations over
one revolution with third-order accuracy and are an improvement over Santora's. King-
Hele [128] used graphical approximation to consider the dynamic density variations of
the atmosphere. The effect of atmospheric rotation on both satellite's orbital
inclination and \( \Omega \) were evaluated fully by Sterne [121] and by Cook and Plimmer [129].
Subsequent works for an oblate atmosphere [130], an atmosphere with \( H \) varying with
height King-Hele and Scott [131] and an atmosphere with day to night variation King-
Hele and Walker [132] were carried out. Using the same atmospheric model Cook and King-Hele [133] evaluated the effects on near-circular objects. Results for an oblate atmosphere with diurnal variation for near-circular orbits were derived by Swinerd and Boulton [134]. They go on to consider the effects of the variation of density scale height with altitude [127]. In another paper [135] the change of $\omega$ during one revolution to near circular orbits with an oblate diurnally varying atmosphere is devoted. A theory for high eccentricity orbits in a spherically symmetrical atmosphere was developed by King-Hele [136]. King-Hele [137] evaluated the effects of meridional winds on orbits of small eccentricity for a spherical atmosphere. Results were derived for orbits of $e > 0.1$ by King-Hele and Walker [138] in an oblate atmosphere. An improved method applicable for all $e$ has developed by King-Hele and Walker [139]. Works during the 1970s was reviewed in the thesis of de Lafontaine which also includes discussion of the sources of error. There was a more general review by de Lafontaine and Garg [140].

The KS total-energy elements equations [141] is a very powerful method for numerical solution with respect to any type of perturbing forces as the equations are less sensitive to round-off and truncation errors in the numerical integration algorithm - Merson [142] Graf et al [143], Sharma [144,145], Sharma and Mani [146]. An orbital frequency based on the total energy gives more accuracy to in-orbit position calculations; the equations are everywhere regular in contrast with the classical Newtonian equations, which are singular at the collision of the two bodies. The equations are smoothed for eccentric orbits because eccentric anomaly is the independent variable. Due to symmetry in these equations, only two of the nine equations were solved analytically to get the complete solution. Sharma [147] generated analytical expressions for short-term orbit predictions with zonal harmonics $J_2$ by the method of series expansions. The study was continued with $J_3$ and $J_4$ by Sharma [148] and Sharma and Xavier James Raj [149] with $J_2$ to $J_8$ terms using KS element equations. In an attempt to compute very accurate short-periodic terms due to $J_2$, even for very high eccentricity orbits, Sharma [150] led to integration of the KS element equations analytically after a close examination of the equations.

Sharma [151] made an attempt to get an analytical solution using an analytical oblate exponential atmospheric density model by series expansion which
include up to quadratic terms in e and c, a small parameter depending on the ellipticity of the atmosphere. Sharma [152] extended the work in a wide range of e with perigee heights near to 300 km at three different inclinations, which include third order terms. Sharma [153] developed a third order non-singular solution with an oblate atmospheric model by including the effect of diurnal bulge. Another attempt by Sharma [154] was carried out with a spherical symmetrical atmosphere for high eccentricity satellite orbits.

In this thesis we have generated analytical solutions for orbit predictions with the KS element equations of motion with respect to the perturbations due to the Earth's oblateness and atmospheric drag. The thesis carries eight chapters.

Chapter 1 contains two sections. Section one deals with the historical developments in predicting satellites motion using analytic as well as semi-analytic method. It is not easy to bring out the complete work in the literature. Some of the works in this field, which helped to bring the theory in this stage, has been explained. Section two describes the different steps used to obtain the KS element equations in terms of the eccentric anomaly. The Keplerian planetary equations are touched upon. The perturbing aerodynamic drag force, which is likely to act on an artificial satellite, is also discussed. In our studies, we come across lot of integrals of the form of the modified Bessel's function. The integral representation of the Bessel function of the first kind and of imaginary arguments is quoted with some properties. The chapter ends with the explanation of the advantages of the KS element equations over the Newtonian equations of motion.

Chapter 2 provides a non-singular analytical theory of short-term orbital motion of satellites in terms of the KS elements, in closed form in eccentricity for the Earth's zonal harmonics \( J_6 \) to \( J_8 \). Following the approach of Sharma [150], the generalized form of the KS element equations to compute terms due to the Earth's higher zonal harmonics \( J_n \) (\( n > 8 \)) is explained. The chapter ends with the numerical results giving the changes in the orbital elements due to the Earth's zonal harmonics \( J_6 \) to \( J_8 \).
Chapter 3 explains the orbital theory for an oblate exponential atmosphere in terms of KS elements. The model for atmospheric density is expressed in terms of the angular position of the satellite in its orbit. The analytical model is developed retaining terms up to fourth-order terms in \( e \) and \( c \) and is provided in terms of the eccentric anomaly. The change in KS elements using this analytical oblate exponential atmospheric density model is calculated. A comparison of the analytically integrated values with fourth-order theory is done with the numerically integrated values of the KS element equations.

Chapter 4 discusses the orbital theory for an atmosphere with day-to-night density variation which is caused due to the day-to-night temperature variation. The day-to-night density variation is defined as a daytime bulge. A fourth order nonsingular solution is developed in terms of KS elements with terms up to fourth-order in \( e \) to study the effect of the perturbation caused by a diurnally varying atmosphere. The decrease in \( a \) and \( e \) of the perturbed orbit are calculated in terms of KS elements. Comparisons are made with numerically and analytically integrated solutions as well as with the third-order and fourth-order theory.

Chapter 5 provides the orbital theory in terms of KS elements to find the combined effect of the perturbation due to the diurnally varying and oblate atmospheric density in satellite’s orbit prediction. The analytical model for the density is developed up to fourth-order terms in \( e \) and \( c \). The KS element equations are formulated using the analytically developed atmospheric density model. Comparison is made between the analytically integrated values with third-order and fourth-order theories after 100 revolutions to emphasize the importance of extending the theory to fourth-order terms.

Chapter 6 introduces the orbital theory in terms of the KS elements when atmospheric scale height varies with altitude. An analytical density model which provides the scale height variation with altitude is developed retaining fourth-order terms in \( e \). The KS element equations are expanded using the density function up to fourth-order terms. The numerical results from the analytical solution are compared with numerically integrated values as well as with the third-order theory.
Chapter 7 describes the orbital theory in terms of KS elements for an oblate diurnally varying atmosphere with variation of the scale height depending on altitude. An analytical density model with fourth-order terms in c and e as well as the second order terms in $\mu$, gradient of the scale height altitude as in [127] is developed. The KS element equations are analytically integrated to obtain the changes in the KS elements due to the new terms added to the equations which are contributed by the new density function. Comparison of the analytical solution is made with the numerically integrated values.

In Chapter 8 the theory discussed in Swinerd and Boulton [127] is extended by expanding the integrand to fourth-order terms. The changes in $\alpha$ and $e$ obtained by the extended theory are compared with the values of the original theory. The results in the analytical method of the orbital theory in terms of KS elements for a diurnally varying oblate atmosphere with scale height variations discussed in chapter 7 are also compared with the values obtained by the analytically integrated values of the present theory after 500 revolutions for orbits having different eccentricities and inclinations. In the KS theory only two equations are handled analytically against three by Swinerd and Boulton. Also the present solution is nonsingular whereas in Boulton's expression for variation in argument of perigee, $e$ is in the denominator.

1.2 KS element equations

In this section we describe the KS element equations of motion derived from the Newtonian theory of gravitational motion. We divide this section into 6 parts. In part 1 we discuss Levi-Civita's transformation that helps to remove the singularity of the Newtonian equations of motion in a plane at collision. The equations of motion in space are explained by introducing the KS transformation in part 2. The equations of motion are also transformed by introducing the generalized eccentric anomaly. Part 3 discusses Lagrange's planetary equations which give the changes in the orbital elements of a satellite caused by perturbing forces. In part 4 the aerodynamic drag force is explained in terms of the KS elements and the equations of motion are brought into KS element equations. The integral representation of the first kind and of imaginary arguments is introduced in part 5. Some properties of the Bessel functions, which help.
us for integration, are also stated in this part. This chapter ends in part 6 with a conclusion describing the advantages of the KS element equations.

1.2.1 Equations of motion in a plane

The Newtonian equations of motion of a particle of mass \( m \) attracted by a central body of mass \( M \) at a distance \( r \) with respect to a co-ordinate system centered at \( M \) is given by

\[
\ddot{x} + \frac{k^2}{r^3} \dot{x} = 0 \quad \text{or} \quad K^2 = k^2 (M + m);
\]  

(1.2.1.1)

\( k^2 \) is the universal gravitational constant. Forces other than central attraction are produced by drag, light pressure or by asphericity of the central body. All such additional forces, called the perturbing force, are represented by single force \( \vec{P} \) acting per unit mass of the particle. The existence of a scalar function \( V(t, \vec{x}) \) depending on time and position of the particle generates a perturbing force given by the gradient of the perturbing potential \( V(t, \vec{x}) \). Thus the equations of motion are modified to the form

\[
\ddot{x} + \frac{k^2}{r^3} \dot{x} = \vec{P} - \frac{\partial V}{\partial \vec{x}}. 
\]  

(1.2.1.2)

Obviously at the point of collision of the two bodies there will be an infinite increase of the velocity so that Eq. (1.2.1.2) is singular at \( r = 0 \). Hence the equations are not suitable for numerical integration at small values of \( r \). To avoid these singularities we regularize the equations of motion by multiplying the velocity vector \( \dot{\vec{x}} \) by an appropriate factor which compensates the growth of velocity at collision. Such a scaling factor that vanishes at collision is the distance \( r \) from the centre of the mass regardless of the direction of the incoming particle. A new independent variable \( s \) called the fictitious time is adopted such that the velocity with respect to \( s \) is defined by

\[
\frac{d}{ds} = r \frac{d}{dt}. 
\]  

(1.2.1.3)
Thus

\[
\frac{dt}{ds} = r
\]

(1.2.1.4)

Thus

\[
\dot{\ddot{x}} = \frac{1}{r} \ddot{x},
\]

\[
\ddot{x} = \frac{d}{dt} \left( \dot{x} \right)
\]

\[
= \frac{1}{r} \frac{d}{dt} (\ddot{x}) + \ddot{x} \frac{d}{dt} \left( \frac{1}{r} \right).
\]

\[
\ddot{x} = \frac{1}{r} \frac{d}{ds} (\ddot{x}) \frac{ds}{dt} + \ddot{x} \left( \frac{1}{r^2} \right) \frac{ds}{dt},
\]

\[
\ddot{x} = \frac{r \dddot{x} - r^2 \dddot{x}}{r^3}.
\]

Hence Eq. (1.2.1.2) reduces to

\[
\frac{r \dddot{x} - r^2 \dddot{x}}{r^3} \frac{K^2}{r^3} \dot{x} = \beta - \frac{\partial V}{\partial \dot{y}}.
\]

The kinetic energy of the particle taken per unit mass is given by

\[
\mathcal{T} = \frac{1}{2} m \dot{\mathbf{v}}^2 = \frac{1}{2} \frac{m}{r^2} \dot{\mathbf{v}}^2 = \frac{1}{2} \left( \dot{x}, \dot{\ddot{x}} \right),
\]

(1.2.1.5)

where \( \mathbf{v} \) is the velocity of the particle and \( (,) \) denotes the scalar product of the two inserted vectors. On differentiation of Eq. (1.2.1.5), we get

\[
\frac{d}{dt} \left( \frac{\dot{v}^2}{2} - \frac{K^2}{r} \right) = \left( \dot{x}, \dot{P} \right) - \left( \dot{x}, \frac{\partial F}{\partial \dot{x}} \right).
\]

The terms inside the parenthesis of the left-hand side are called the Kepler energy of the motion and are denoted by \( -h_k \). Thus

\[
h_k = \frac{K^2}{r} \frac{\ddot{x}}{2}
\]

(1.2.1.6)
The total negative energy \( h \) is given by the relation \( h = h_0 - V \) which leads to the following law of energy

\[
h = - (\dot{x}, \dot{\hat{P}}) - \frac{\partial V}{\partial \dot{t}}.
\]  

(1.2.1.7)

Thus equations of motion and energy relations are given by

\[
r = -r \ddot{\hat{x}} + K \dot{\hat{x}} = r^0 (\dot{\hat{P}} - \frac{\partial V}{\partial \hat{x}}), \quad \dot{t} = r.
\]  

(1.2.1.8)

\[
h = \frac{K^0}{r} - \frac{1}{2} \frac{1}{r^2} \dot{\hat{x}}^2 - V.
\]  

(1.2.1.9)

\[
h = - (\dot{x} \dot{\hat{P}}) - r \frac{\partial V}{\partial \dot{t}}.
\]  

(1.2.1.10)

Since \( V \) is assumed to be given as a function of the independent variables \( t, x_1, x_2, x_3 \) it's partial derivatives with respect to \( t \) in latter equation can not be replaced by with respect to \( s \). Eq. (1.2.1.8) is still singular since solving for \( \ddot{\hat{x}} \) we get,

\[
\ddot{\hat{x}} = \frac{r}{r} \frac{\dot{K}^0}{r} \ddot{x} + r^0 (\dot{\hat{P}} - \frac{\partial V}{\partial \hat{x}}),
\]  

(1.2.1.11)

and thus \( r \) appears in the denominator.

The transformation from \( \hat{x} \) to a new co-ordinate \( \hat{u} \) defined by

\[
\hat{x} = \hat{u}.
\]  

(1.2.1.12)

is given in terms of complex variables by the mapping

\[
x_1 + i x_2 = (u_1 + i u_2)^0.
\]  

(1.2.1.13)

of a \( u_1, u_2 \) plane on to the physical plane known as Levi-Civita's transformation [155]. Eq. (1.2.1.13) is equivalent to

\[
x_1 = u_1, \quad x_2 = 2 u_1 u_2.
\]  

(1.2.1.14)

and by differentiation, the matrix relation


\[
\begin{pmatrix}
v_1' \\
v_2'
\end{pmatrix} = 2 \begin{pmatrix}
u_1 & -u_3 \\
u_3 & u_1
\end{pmatrix} \begin{pmatrix}
u \\
u
\end{pmatrix}
\]

(1.2.1.15)

is obtained. The above formula appears in more compact form by introducing the Levi-Civita matrix

\[
L(\vec{u}) = \begin{pmatrix}
u & -u_3 \\
u_3 & u_1
\end{pmatrix}
\]

(1.2.1.16)

Eqs. (1.2.1.14) and (1.2.1.15) with Levi-Civita Matrix (L-Matrix) (1.2.1.16) can be written as

\[
\begin{align*}
\vec{x} &= L(\vec{u}) \vec{u}, \\
\vec{x}' &= 2L(\vec{u}) \vec{u}'.
\end{align*}
\]

(1.2.1.17)

(1.2.1.18)

The L - Matrix satisfies the following three properties,

(i) \(L(\vec{u})\) is orthogonal.

\[\therefore \quad L(\vec{u}) L^T(\vec{u}) = (\vec{u}, \vec{u}) \Rightarrow L^{-1}(\vec{u}) = \frac{1}{(\vec{u}, \vec{u})} L^T(\vec{u}).\]

(ii) The elements of \(L(\vec{u})\) are linear and homogeneous functions of \(u_j\), i.e. \(L(\vec{u}) = L(\vec{u}').\)

(iii) The first column of \(L(\vec{u})\) is the position vector \(\vec{u}.'\)

Also the following two rules are valid for any two vectors \(\vec{u}\) and \(\vec{v}\).

\[
L(\vec{u}) \vec{v} = L(\vec{v}) \vec{u},
\]

(1.2.1.19)

\[
(\vec{u}, \vec{u}) L(T\vec{v}) - 2(\vec{u}, \vec{v})L(\vec{u})\vec{v} + (\vec{v}, \vec{v})L(\vec{u})\vec{u} = 0.
\]

(1.2.1.20)

1.2.2 Equations of motion in space

The generalisation of Levi-Civita matrix defined as

\[
L(\vec{u}) = \begin{pmatrix}
u & -u_3 & -u_3 & u_4 \\
u_3 & u & -u_3 & -u_3 \\
u_3 & u_4 & u_2 & u_2 \\
u_3 & u_4 & u_2 & -u_3
\end{pmatrix}
\]

(1.2.2.1)
is called KS (Kustanheimo-Stiefel) matrix \([157]\). In this matrix \(\vec{u}\) is the 4-vector composed of the components \(u_j\) \((j = 1, 2, 3, 4)\) and in the upper left hand corner the Levi-Civita matrix appears. Thus a vector \((x_1, x_2, x_3)\) in the physical space is transformed to a four vector \(\ddot{x}\) by adding a fourth component of value zero. Hence a similar transformation to (1.2.1.17)

\[
\ddot{x} = L(\vec{u}) \dot{u}.
\]

which is explicitly written as

\[
\begin{align*}
\dddot{x}_1 &= u_2 \dddot{x}_4 - u_3 \dddot{x}_4 + u_4 \dddot{x}_4, \\
\dddot{x}_2 &= 2(u_0 u_2 - u_1 u_4), \\
\dddot{x}_3 &= 2(u_0 u_3 + u_1 u_4),
\end{align*}
\]

is defined. This generalisation of Levi-Civita transformation is called KS transformation and the matrix satisfies the three properties satisfied by the L-Matrix. Also it satisfies the two rules given by Eqs. (1.2.1.19) and (1.2.1.20). Equation (1.2.1.20) results in the relation

\[
\dddot{u}_4 v_1 - \dddot{u}_1 v_2 + \dddot{u}_2 v_3 - \dddot{u}_3 v_4 = 0,
\]

known as bilinear relation and is of fundamental importance.

We postulate Eq. (1.2.2.2) to verify that the Eq. (1.2.1.8) is satisfied. For the initial condition of the differential equations assume the initial values \(\dddot{x}(0)\) and \(\dddot{\dot{x}}(0)\) of the functions \(\dddot{x}(s)\) and \(\dddot{\dot{x}}(s)\) at instant \(s = 0\). The initial value \(\dddot{u}(0)\) is chosen among the vectors corresponding to \(\dddot{x}(0)\) by the transformation given by equation (1.2.2.3). We define \(\dddot{u}(0)\), from Eq. (1.2.1.18) by multiplying \(L^{-1}(\vec{u})\) and using the property (ii) as

\[
\dddot{u}(0) = \frac{1}{2 |\dddot{u}(0)|} L^T(\vec{u}(0)) \dddot{\dot{x}}(0).
\]

Thus \(\dddot{u}(0)\) is uniquely determined after having chosen \(\dddot{u}(0)\).

Certain theorems \([141]\) show that \(\dddot{u}(s)\) and \(\dddot{u}(s)\) are two vectorial functions that satisfy
Therefore it follows from the KS transformation $\tilde{x} = L(\tilde{u}) \tilde{u}$ and from

$$\tilde{x}'' = 2 L(\tilde{u}) \tilde{u}$$  \hspace{1cm} (1.2.2.8)

that the Levi-Civita relation (1.2.1.18) holds true in space. Also the KS transformation (1.2.2.2) satisfies the Eqs. (1.2.1.8) as well as the given initial conditions $\tilde{x}(0)$ and $\tilde{x}''(0)$.

With this we finalise the KS equations of motion as

$$\tilde{x}'' + \frac{K}{\tilde{u} \tilde{u}'} \tilde{u}'' = \frac{1}{2} (\tilde{u}, \tilde{u}') L(\tilde{u}) \left( \frac{\partial V}{\partial \tilde{x}} + \tilde{p} \right).$$  \hspace{1cm} (1.2.2.9)

The perturbing potential $V$ becomes a function of the independent variable $t$, $u_1$, $u_2$, $u_3$, $u_4$ by the expressions given by Eq. (1.2.2.3) and

$$\frac{\partial V}{\partial u_j} = \sum_{i=1}^{4} \frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial u_j} \left. \right|_{j=1,2,3,4}.$$  \hspace{1cm} (1.2.2.10)

The vector $\frac{\partial V}{\partial \tilde{x}}$ appearing in Eq. (1.2.2.9) has a vanishing fourth component, as has any vector belonging to the physical space and hence

$$L(\tilde{u}) \left. \frac{\partial V}{\partial \tilde{x}} \right| \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ u_2 & u_3 & u_4 & u_1 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \\ \frac{\partial V}{\partial x_3} \\ \frac{\partial V}{\partial x_4} \end{pmatrix}.$$
Writing the energy relations (1.2.1.9) and (1.2.1.10) in u language,

\[
L' (\ddot{u}) \frac{dU'}{dx} = \frac{1}{2} \left( \frac{\partial V}{\partial \dot{u}} \right).
\]  

\[ (1.2.2.11) \]

Elements are introduced into differential equations to produce stabilization. An element is any quantity, which is a linear function of the independent variable during a pure Kepler motion. If the physical time is the independent variable, the mean anomaly given by \( M = (KL/\ell^{3/2}) \) \( t \) is an element since it varies linearly with \( t \).

Here we introduce an element with respect to the physical time called time-element denoted as

\[
\tau = t + \frac{1}{h} (\ddot{u}, \dddot{u}),
\]

\[ (1.2.2.15) \]

The differential equation describing the variation of \( t \) caused by the perturbing forces is

\[
\tau' = \frac{1}{2h} \left( K' - 2r' V' \right) - \frac{r}{4h} \left( \ddot{u} \frac{\partial V}{\partial \dot{u}} - 2L' (\ddot{u}) \dddot{P} \right) - \frac{h'}{h^2} (\dddot{u}, \dddot{u}').
\]

\[ (1.2.2.16) \]

where \( h' \) is given by

\[
h' = -r \frac{\partial V}{\partial \ell} - 2 (\dddot{u}, L' (\dddot{u}) \dddot{P}).
\]

\[ (1.2.2.17) \]

The physical time \( t' \) is computed from Eq. (1.2.2.15) as

\[
t = \tau - \frac{1}{h} (\dddot{u}, \dddot{u}').
\]

\[ (1.2.2.18) \]
By assuming \( h > 0 \) the elements attached to the parametric co-ordinates \( u_j \) can be constructed by introducing the generalized eccentric anomaly of a pure Kepler motion as

\[
E' = 2w, \quad w = \sqrt{\frac{h}{2}}.
\]

and hence \( E' = 2w \). In the perturbed case the relation can define the new independent variable \( E \) as

\[
\frac{d}{ds} = 2w \frac{d}{dE}, \quad (1.2.19)
\]

Thus equations of motion are given by

\[
\frac{d^2u}{dE^2} + \frac{1}{4} \frac{d}{dE} = -\frac{1}{4w} \left[ \frac{V'}{2} \frac{d}{dE} + r \left( \frac{\partial V'}{\partial u} \frac{d}{dE} - 2L' (u) \right) \right] = \frac{1}{w} \frac{d}{dE} \frac{d}{dE}, \quad (1.2.20)
\]

and Eq. (1.2.16) becomes

\[
\frac{d\tau}{dE} = \frac{1}{8w} \left[ K - 2rV' \right] = \frac{1}{16w} \left( \frac{\partial V'}{\partial u} \frac{d}{dE} - 2L' (u) \right) = \frac{2}{w^2} \frac{d}{dE} \frac{d}{dE}, \quad (1.2.21)
\]

Equation of the angular frequency (1.2.17) is

\[
\frac{d}{dE} \frac{d}{dE} = -\frac{r}{8w} \frac{\partial V'}{\partial \tau} - \frac{1}{2w} \left( \frac{\partial}{\partial E} \frac{\partial}{\partial E} - L' (u) \right). \quad (1.2.22)
\]

The energy Eq. (1.2.12) is

\[
\frac{r}{K} - \frac{2}{rV'} - \frac{1}{2w} = 0. \quad (1.2.23)
\]

The equations of motion (1.2.20) are now solved by the method of variation of constants. Let

\[
\tilde{u} = (\alpha (E) \cos \frac{E}{2} + \beta (E) \sin \frac{E}{2}, \quad (1.2.24)
\]

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be a solution where vectors $\alpha$ and $\beta$ are to be determined. Solving for $\frac{d\alpha}{dE}$ and $\frac{d\beta}{dE}$ the equations of motion are

$$
\frac{d\alpha}{dE} = \frac{1}{2w^2} \left[ \frac{1}{2} \ddot{u} + \frac{r}{4} \left( \frac{\partial V}{\partial u} - 2 L \left( \dot{u} \dot{P} \right) \right) + \frac{2}{w} \frac{d w}{d E} \frac{d \ddot{u}}{d E} \right] \sin \frac{E}{2},
$$

$$
\frac{d\beta}{dE} = -\frac{1}{2w^2} \left[ \frac{1}{2} \ddot{u} + \frac{r}{4} \left( \frac{\partial \dot{u}'}{\partial u} - 2 L' \left( \dot{u} \dot{P} \right) \right) + \frac{2}{w} \frac{d w}{d E} \frac{d \ddot{u}}{d E} \right] \cos \frac{E}{2}. \tag{1.2.2.25}
$$

When no perturbation occurs, i.e. when $\bar{P} = 0$ and $V = 0$, it can be seen from Eqs. (1.2.2.25) and (1.2.2.22) that $\alpha$ and $\beta$ are constants. Since the state vector $\ddot{u}$ is explicitly represented in terms of these elements, the vectors $\ddot{\alpha}$ and $\ddot{\beta}$ are the vector elements corresponding to $\ddot{u}$. Together with the elements $w$, $x$, they form a set of ten scalar elements and thus there is the total order ten of the differential system. Thus they become regular elements. The differential equations describing the variation of the elements caused by the perturbations are the Eqs. (1.2.2.21), (1.2.2.22) and (1.2.2.25). These are referred to as the element equations. From the bilinear relation

$$
u_1 
u_1 - u_2 \nu_2 + u_3 \nu_3 - u_4 \nu_4 = 0,
$$

and from Eq. (1.2.2.24), it is easily seen that

$$
u_1 \frac{d u_1}{d E} - u_2 \frac{d u_2}{d E} + u_3 \frac{d u_3}{d E} - u_4 \frac{d u_4}{d E} = 0,
$$

and consequently,

$$\alpha \beta - \alpha \beta + \alpha \beta - \alpha \beta = 0.
$$

Thus the vector elements satisfy a bilinear relation. For convenience we use the notation $\frac{d \ddot{u}}{d E} = \ddot{u}^\ast$. To get the initial conditions we have the following:

$$r^\ast = (\dddot{x}, \dddot{\dot{x}}) = |\dddot{x}|^2 = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}},
$$

$$\nu = \left[ \frac{1}{2} \left( \frac{K}{r^\ast} - \frac{1}{2} |\dddot{x}|^2 - V \right) \right]^{\frac{1}{2}}.
$$

In terms of the parameter $u_1$,}
\[ v_1 = u_1 - u_7^2 - u_8^2 + u_4^2; \quad x_2 = 2 (u_1 u_2 - u_1 u_4); \quad x_3 = 2 (u_1 u_3 + u_1 u_4), \]

and

\[ \hat{v} = \hat{u} = u_7^2 + u_8^2 + u_4^2. \]

Thus given the position and velocity vectors \( \hat{x} \) and \( \hat{x} \) at the instant \( t = 0 \), the initial state vector is found either from,

\[ u_1 + u_4^2 = \frac{1}{2} \left( r - x_1 \right); \]

\[ u = \frac{x_1 u_1 + x_3 u_4}{r - x_1}; \]

\[ u_1 = \frac{x_1 u_1 - x_3 u_4}{r - x_1}, \quad \text{if} \quad x_1 \geq 0 \quad (1.2.2.26) \]

or from

\[ u_1 + u_4^2 = \frac{1}{2} \left( r + x_1 \right); \]

\[ u = \frac{x_1 u_1 + x_3 u_4}{r + x_1}; \]

\[ u_1 = \frac{x_1 u_1 - x_3 u_4}{r + x_1}. \quad (1.2.2.27) \]

Also

\[ \hat{u} = \frac{1}{2r} L^T (\hat{u}) \hat{x} \Rightarrow \hat{u} = \frac{1}{2} L^T (\hat{u}) \hat{x}, \]

\[ \hat{u} = \frac{1}{4w} L^T (\hat{u}) \hat{x}. \]

\[ \hat{u} = \frac{1}{4w} (u_1 \hat{x}_1 + u_2 \hat{x}_2 + u_3 \hat{x}_3), \]

\[ \hat{u} = \frac{1}{4w} (u_1 \hat{x}_1 + u_2 \hat{x}_2 + u_3 \hat{x}_3), \]

\[ \hat{u}_1 = \frac{1}{4w} (u_2 \hat{x}_2 + u_3 \hat{x}_3), \]

\[ \hat{u}_1 = \frac{1}{4w} (u_2 \hat{x}_2 + u_3 \hat{x}_3), \]

\[ \hat{u}_1 = \frac{1}{4w} (u_3 \hat{x}_3). \quad (1.2.2.28) \]
And by adopting $E = 0$ as initial value of the eccentric anomaly, we obtain

$$\dot{\alpha} = \dot{u} \quad \text{and} \quad \dot{\beta} = 2u^* \quad (1.2.2.29)$$

If the perturbing force $\mathbf{P}$ only is considered, (the perturbing potential $V$ is assumed to be zero) then the equations of motion (1.2.2.21), (1.2.2.22) and (1.2.2.25) become

$$\frac{d}{dt} \mathbf{F} = -\frac{1}{2w} \left[ \mathbf{E}^* \cdot \mathbf{L}'(\tilde{u}) \mathbf{P} \right], \quad (1.2.2.30)$$

$$\frac{d}{dE} \mathbf{E} = \frac{1}{8w^2} \left[ K^2 + r(\tilde{u}, \mathbf{E}) \mathbf{P} - 16w \frac{d}{dE} (\tilde{u}, \mathbf{E}) \mathbf{u}^* \right], \quad (1.2.2.31)$$

$$\frac{d}{dE} \alpha = -\frac{1}{w^2} \left[ -\frac{r}{4} \mathbf{L}'(\tilde{u}) \mathbf{P} + 2w \frac{d}{dE} \mathbf{u}^* \right] \sin \frac{E}{2}, \quad (1.2.2.32)$$

$$\frac{d}{dE} \beta = -\frac{1}{w^2} \left[ -\frac{r}{4} \mathbf{L}'(\tilde{u}) \mathbf{P} + 2w \frac{d}{dE} \mathbf{u}^* \right] \cos \frac{E}{2} \quad (1.2.2.32)$$

Considering the elliptic motion, when the position and velocity vectors $\mathbf{x}$ and $\dot{\mathbf{x}}$ are given, then $\alpha$ and $\epsilon$ are given by

$$\frac{1}{a} = \frac{2}{r} - \frac{\mathbf{x}^2}{K^2}, \quad (1.2.2.33)$$

$$\epsilon = \left[ (1 - \frac{c}{a})^2 + \frac{(x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3)^2}{K^2 a} \right]^{1/2}. \quad (1.2.2.34)$$
in terms of the eccentric anomaly $E$, from (fig. 1.2.1)

\[ r \cos \theta = a (\cos E - e), \]
\[ r \sin \theta = a(1-e^2)^{\frac{1}{2}} \sin E, \]
\[ r = a (1 - e \cos E). \]  

1.2.3 Lagrange's planetary equations

Lagrange's planetary equations give the changes in the orbital elements of a satellite caused by perturbing forces. If a force $\vec{f}$ per unit mass perturbs an orbit, we first resolve $\vec{f}$ into components $\vec{f}_r$, radially (parallel to $\vec{r}$), $\vec{f}_t$, transverse (perpendicular to $\vec{r}$ in the osculating plane) and $\vec{f}_n$, normal to the osculating plane, parallel to $\vec{n}$ as in Figure 1.2.

Then Lagrange's planetary equations are:

\[ \dot{a} = \frac{2a^2}{\dot{u} (1-e^2)^{\frac{1}{2}}} \left\{ \vec{f}_r \sin \theta + \vec{f}_n (1+e \cos \theta) \right\}, \]
\[ \Omega \sin i = \frac{r f_n \sin u}{K a^2 (1-e^2)^{\frac{1}{2}}} \quad \text{where} \quad u = \omega + \theta, \]
\[ \frac{d \Omega}{dt} = \frac{1}{K a^2 (1-e^2)^{\frac{1}{2}}} r f_n \cos u, \]
\[ e = \frac{a - (1-e^2)^{\frac{1}{2}}}{K} \left\{ \vec{f}_r \sin \theta + \vec{f}_t (\cos \theta + \cos E) \right\}, \]
and
\[ \dot{\omega} + \dot{\Omega} \cos i = \frac{1}{e} \left( \frac{a - (1-e^2)^{\frac{1}{2}}}{K} \right) \left\{ -\vec{f}_r \cos \theta + \vec{f}_n \left( \frac{1+\frac{r}{a(1-e^2)}}{\cos \theta} \right) \right\}. \]

Let $\vec{f}_r$ and $\vec{f}_n$ denote the components of $\vec{f}$ along the tangent to the orbit and along the inward normal to the orbit in the orbital plane. If the tangent to the orbit makes an angle $\psi$ with the transverse direction along which $\vec{f}_t$ acts, then
\[ \mathbf{f}_r = f_r \cos \psi + f_n \sin \psi = \frac{1}{|\mathbf{v}|} \frac{K}{a^2 \left(1 - e^2\right)^2} \left\{ \frac{\mathbf{v}_r}{|\mathbf{v}|} (1 + e \cos \theta) + \mathbf{f}_n \right\}, \]

\[ \mathbf{f}_z = f_r \cos \psi - f_n \sin \psi = \frac{1}{|\mathbf{v}|} \frac{K}{a^2 \left(1 - e^2\right)^2} \left\{ \frac{\mathbf{v}_z}{|\mathbf{v}|} e \cos \theta - \mathbf{f}_n (1 + e \cos \theta) \right\}. \]

Thus

\[ u = \frac{2 a^2 |\mathbf{v}|}{K} f_r, \quad (1.2.3.1) \]

\[ \mathbf{v} = \frac{1}{|\mathbf{v}|} \left\{ 2 f_r \left(1 + e \cos \theta\right) - f_n \frac{r}{a} \sin \theta \right\}. \quad (1.2.3.2) \]

If \( \mathbf{P} \) is the aerodynamic drag force per unit mass acting on a satellite of mass \( m \), then

\[ \mathbf{P} = -\frac{1}{2} \rho \frac{S C_o}{m} \mathbf{v}_r |\mathbf{v}| \mathbf{v}. \quad (1.2.3.3) \]

where \( S \), the effective area of the satellite, \( C_o \), the drag coefficient, \( \rho \) is the atmospheric density at the point of calculating atmospheric drag force and \( \mathbf{v}_r \) is the velocity of the satellite relative to the ambient air. If \( \mathbf{v} \) is the velocity of the satellite relative to the Earth's centre, then \( \mathbf{v}_r = \mathbf{v} - \mathbf{v}_a \) where \( \mathbf{v}_a \) is the velocity of the air relative to the Earth's centre. \( \mathbf{v}_a \) is assumed to be west to east. Further,

\[ \mathbf{v}_r = \mathbf{v}_r + \mathbf{v}_o - 2 \mathbf{v} \mathbf{v}_o \cos \gamma, \]

where \( \gamma \) is the angle between \( \mathbf{v}_o \) and \( \mathbf{v} \). As in King-Hele [10], \( \mathbf{v}_o \) can be approximately written as

\[ \mathbf{v}_o = \mathbf{v} \left(1 - \frac{r_o}{r} \Lambda \cos i_o\right). \]

\( \Lambda \), the rotational rate of the atmosphere about the Earth's axis and \( i_o \), the initial inclination. \( r_o = a_o (1 - e_{oi}) \) is the initial perigee radius, \( \mathbf{v}_{oi} \), the velocity at the initial perigee. Then the drag force per unit mass tangential to the orbit given by equation (1.2.3.3) can be written as

\[ \mathbf{f}_\theta = P = -\frac{1}{2} \rho \left| \mathbf{v} \right| \left( \mathbf{v} \right). \quad (1.2.3.4) \]
where \( \delta = \frac{FS\delta_{0}}{m} \), and \( F = (1 - \frac{r_{pl}^{2}}{v_{pl}^{2}} \Lambda \cos \theta_{0})^{3} \).

The small component \( \vec{f}_{p} \), normal to the orbit in the orbital plane is ignored here. In equation (1.2,3.4)

\[
\vec{v} = \vec{v}_{r} + \vec{v}_{p} \equiv \vec{v}_{l} \\
\text{where} \quad \vec{v}_{l} = \frac{4\pi r}{v} \left[ u_{1}u_{1}' + u_{2}u_{2}' + u_{3}u_{3}' \right], \\
\vec{w}_{l} = -\frac{4\pi r}{v} \left[ u_{1}u_{1}' - u_{2}u_{2}' - u_{3}u_{3}' \right], \\
\vec{c}_{l} = \frac{4\pi r}{v} \left[ u_{1}u_{1}' + u_{2}u_{2}' + u_{3}u_{3}' \right],
\]

### 1.2.4 Drag Force in terms of KS elements

Expressing \( \vec{x}_{l} \) in terms of the vectors \( \vec{a} \) and \( \vec{b} \)

\[
\vec{x}_{l} = \frac{4\pi r_{pl}^{2}}{v} \left[ \frac{1}{2} \left( \alpha_{1} \cos \frac{E}{2} + \beta_{1} \sin \frac{E}{2} \right) \left( -\alpha_{1} \sin \frac{E}{2} + \beta_{1} \cos \frac{E}{2} \right) \\
+ \frac{1}{4} \left[ -\alpha_{1}^{2} + \alpha_{1}^{2} + \alpha_{1}^{2} - \alpha_{1}^{2} \right] \sin E \right].
\]

Hence

\[
\vec{x}_{l} = \frac{\gamma_{l}}{r} \left[ \cos E + \sin E \right], \\
\vec{x}_{l} = \frac{\gamma_{l}}{r} \left[ \sin E + \cos E \right],
\]

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\[
\dot{x}_i = \frac{\omega}{r} \left( d_{13} \cos E + d_{12} \sin E \right),
\]

where

\[
d_{14} = (\alpha_1 \beta - \alpha_2 \beta_2 + \alpha_3 \beta_3 + \alpha_4 \beta_4).
\]
\[
d_{24} = \frac{1}{2} \left[ -\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 \right] - \left( \beta_1^2 + \beta_2^2 + \beta_3^2 - \beta_4^2 \right),
\]
\[
d_{34} = (\alpha_1 \beta_1 + \alpha_2 \beta_2 - \alpha_3 \beta_3 - \alpha_4 \beta_4),
\]
\[
d_{44} = (\alpha_1 \alpha_2 + \beta_1 \beta_2 + \alpha_3 \alpha_4 - \beta_3 \beta_4),
\]
\[
d_{54} = (\alpha_1 \beta - \alpha_2 \beta_2 + \alpha_3 \beta_3 + \alpha_4 \beta_4),
\]
\[
d_{64} = (-\alpha_1, -\alpha_2, \alpha_3 + \beta_1 \beta_3 + \beta_2 \beta_4).
\]

then,

\[
\ddot{p} = p_1 \ddot{i} + p_2 \ddot{j} + p_3 \ddot{k},
\]

where

\[
p_1 = -\frac{\rho \delta \omega |\dot{p}|}{2r} \left( d_{11} \cos E + d_{12} \sin E \right),
\]
\[
p_2 = -\frac{\rho \delta \omega |\dot{p}|}{2r} \left( d_{21} \cos E + d_{22} \sin E \right),
\]
\[
p_3 = -\frac{\rho \delta \omega |\dot{p}|}{2r} \left( d_{31} \cos E + d_{32} \sin E \right),
\]

\[
: \because (\ddot{u} \cdot L \cdot (\ddot{u}) \dot{p}) = -\frac{\rho \delta \omega |\dot{p}| \sin E}{8r} \left( -\alpha_1 \sin \frac{E}{2} + \beta_1 \cos \frac{E}{2} \right) \left( \alpha_1 \cos \frac{E}{2} + \beta_1 \sin \frac{E}{2} \right)
\]
\[
\cdot (d_{11} \cos E + d_{12} \sin E) + (\alpha_2 \cos \frac{E}{2} + \beta_2 \sin \frac{E}{2})
\]
\[
\cdot (d_{11} \cos E + d_{12} \sin E) + (\alpha_3 \cos \frac{E}{2} + \beta_3 \sin \frac{E}{2})
\]
\[
\cdot (d_{11} \cos E + d_{12} \sin E) + (\alpha_3 \cos \frac{E}{2} + \beta_3 \cos \frac{E}{2})
\]
\[
\cdot (\alpha_2 \cos \frac{E}{2} + \beta_2 \sin \frac{E}{2}) (d_{11} \cos E + d_{12} \sin E)
\]

27
\[
\begin{align*}
+ (\alpha_1 \cos \frac{E}{2} + \beta_1 \sin \frac{E}{2}) (d_{11} \cos E + d_{12} \sin E) \\
+ (\alpha_2 \cos \frac{E}{2} + \beta_2 \sin \frac{E}{2}) (d_{21} \cos E + d_{22} \sin E) \\
+ (\alpha_3 \sin \frac{E}{2} + \beta_3 \cos \frac{E}{2}) (-d_{11} \cos E + d_{12} \sin E) \\
+ (\alpha_4 \cos \frac{E}{2} + \beta_4 \sin \frac{E}{2}) (d_{21} \cos E + d_{22} \sin E) \\
\end{align*}
\]

\[
(d_{11} \cos E + d_{12} \sin E) - (\alpha_1 \cos \frac{E}{2} + \beta_1 \sin \frac{E}{2}) (d_{11} \cos E + d_{12} \sin E) \\
(d_{21} \cos E + d_{22} \sin E) + (\alpha_2 \cos \frac{E}{2} + \beta_2 \sin \frac{E}{2}) (d_{21} \cos E + d_{22} \sin E) \\
(d_{11} \cos E + d_{12} \sin E) + (\alpha_3 \sin \frac{E}{2} + \beta_3 \cos \frac{E}{2}) (-d_{11} \cos E + d_{12} \sin E) \\
(d_{21} \cos E + d_{22} \sin E) + (\alpha_4 \cos \frac{E}{2} + \beta_4 \sin \frac{E}{2}) (d_{21} \cos E + d_{22} \sin E) \\
\}
\]

\[\text{i.e. } (\ddot{u} \cdot L) (\ddot{u} \cdot P) = \frac{\rho v^2}{8r} \{ f_0 + f_1 \cos 2E + f_2 \sin 2E \}.
\]

\[
\begin{align*}
\begin{align*}
\begin{align*}
\begin{align*}
\end{align*}
\end{align*}
\end{align*}
\end{align*}
\]

where

\[
\begin{align*}
t_{11} &= d_{11} (2 \alpha_1 \beta_1 - 2 \alpha_1 \beta_2 - 2 \alpha_2 \beta_1 + 2 \alpha_4 \beta_4) \\
+ d_{12} (-\alpha_1 \beta_2 + \alpha_2 \beta_1 - \alpha_3 \beta_2 - \alpha_4 \beta_3 - \beta_1) \\
+ d_{21} (2 \alpha_2 \beta_1 - 2 \alpha_2 \beta_3 + 2 \alpha_3 \beta_1 - 2 \alpha_4 \beta_4) \\
+ d_{22} (-2 \alpha_1 \beta_2 + 2 \alpha_2 \beta_1 + 2 \alpha_3 \beta_1 - 2 \beta_1) \\
+ d_{31} (2 \alpha_3 \beta_2 + 2 \alpha_4 \beta_3 + 2 \alpha_4 \beta_1 + 2 \alpha_4 \beta_4) \\
+ d_{32} (-2 \alpha_1 \beta_3 - 2 \alpha_2 \beta_3 + 2 \beta_2) \\
\}
\end{align*}
\]

\[
\begin{align*}
t_{12} &= 2 \{ d_{11}^2 + d_{12}^2 \} \\
+ \{ d_{21}^2 + d_{22}^2 + d_{31}^2 + d_{32}^2 \} \\
\}
\end{align*}
\]

\[
\begin{align*}
t_{13} &= 2 d_{11} (\alpha_1 \beta_1 - \alpha_3 \beta_2 - \alpha_4 \beta_3) \\
+ d_{12} (\alpha_2 \beta_1 + \alpha_2 \beta_3 + \alpha_3 \beta_1 + \alpha_4 \beta_4) \\
+ d_{22} (-2 \alpha_1 \beta_2 + 2 \alpha_2 \beta_1 + 2 \alpha_3 \beta_1 - 2 \alpha_4 \beta_4) \\
+ d_{31} (2 \alpha_3 \beta_2 + 2 \alpha_4 \beta_3 + 2 \alpha_4 \beta_1 + 2 \alpha_4 \beta_4) \\
\}
\end{align*}
\]

\[28\]
Coefficients of \( \sin E \) and \( \cos E \) are

\[
2 d_{11} (\alpha_1 \beta_1 - \alpha_1 \beta_2 + \alpha_1 \beta_3 - \alpha_1 \beta_4 + \alpha_2 \beta_2 \beta_4 - \alpha_2 \beta_3 \beta_4 + \alpha_2 \beta_4 \beta_3 - \alpha_2 \beta_4 \beta_1) + 2 d_{12} (\alpha_1 \beta_1 \beta_2 - \alpha_2 \beta_2 \beta_3 + \alpha_4 \beta_4) + 2 d_{21} (\alpha_1 \beta_2 \beta_3 + \alpha_2 \beta_2 \beta_4 + \alpha_4 \beta_4) + 2 d_{22} (\alpha_1 \beta_2 \beta_4 + \alpha_2 \beta_3 \beta_4 + \alpha_4 \beta_4) + 2 d_{31} (\alpha_1 \beta_3 \beta_1 + \alpha_2 \beta_3 \beta_4 + \alpha_4 \beta_4) + 2 d_{32} (\alpha_1 \beta_3 \beta_4 + \alpha_2 \beta_4 \beta_3 + \alpha_4 \beta_4) = 4 \{ d_{11} d_{22} + d_{12} d_{21} + d_{31} d_{32} \},
\]

respectively which shows coefficients of \( \sin E \) and \( \cos E \) are zero. Hence equation (1.2.2.26) is

\[
\frac{d w}{d E} = \frac{\rho \delta [v]}{16 r} \left[ f_0 + f_1 \cos 2E + f_2 \sin 2E \right]. \tag{1.2.4.1}
\]

To get equation (1.2.2.28) in terms of the vectors \( \hat{\alpha} \) and \( \hat{\beta} \), first let us express \( u_1 \sin \frac{E}{2} \) and \( u_2 \sin \frac{E}{2} \) in the following way;

\[
u_1 \sin \frac{E}{2} = (\alpha \cos \frac{E}{2} + \beta \sin \frac{E}{2}) \sin \frac{E}{2} = \frac{1}{2} \left[ q_{11} \sin E + q_{11} \cos E \right],
\]

\[
u_2 \cos \frac{E}{2} = \left( -\frac{1}{2} \alpha \sin \frac{E}{2} + \frac{1}{2} \beta \cos \frac{E}{2} \right) \sin \frac{E}{2}.
\]
\[ \frac{d\mathbf{\alpha}_1}{dE} = \frac{1}{w} \left\{ \frac{r}{4} \times \left( -\frac{\rho \delta [v]}{2} \right) \mathbf{w} \right\} \left( d_{11} \cos E + d_{12} \sin E \right) \]

\[ + w \times \frac{\rho \delta [v]}{16 \rho} \left( f_0 + f_1 \cos 2E + f_2 \sin 2E \right) \]

\[ + \left( q_{00}^{(1)} + q_{12}^{(1)} \cos E + q_{22}^{(1)} \sin E \right) \]

Thus equation (1.2.2.28) can be written as

\[ \frac{d\mathbf{\alpha}_1}{dE} = \frac{\rho \delta [v]}{16w} \left\{ \left( d_{11} q_{11}^{(1)} + d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} + d_{22} q_{22}^{(1)} + d_{31} q_{31}^{(1)} + d_{32} q_{32}^{(1)} \right) \right. \]

\[ + 2(d_{11} q_{00}^{(1)} + d_{12} q_{02}^{(1)} + d_{12} q_{02}^{(1)} \cos E \right. \]

\[ + 2(d_{12} q_{01}^{(1)} + d_{22} q_{02}^{(1)} + d_{22} q_{02}^{(1)} \sin E \right. \]

\[ + t_0 q_{11}^{(2)} - t_1 q_{11}^{(2)} + t_2 q_{22}^{(2)} + t_3 q_{22}^{(2)} \cos 2E \right. \]

\[ + (d_{11} q_{11}^{(1)} + d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} + d_{22} q_{22}^{(1)} + d_{31} q_{31}^{(1)} + d_{32} q_{32}^{(1)} \sin 2E) \right. \]

\[ \left. - \frac{1}{r} f_0 q_{10}^{(1)} + f_0 q_{11}^{(1)} + \frac{1}{2} f_1 q_{11}^{(1)} + \frac{1}{2} f_2 q_{12}^{(1)} \cos E \right. \]

\[ + (f_0 q_{12}^{(1)} - \frac{1}{2} f_1 q_{12}^{(1)} + \frac{1}{2} f_2 q_{12}^{(1)} \sin E \right. \]

\[ + t_1 q_{11}^{(1)} \cos 2E - t_2 q_{12}^{(1)} \sin 2E + \left. \frac{1}{2} (f_1 q_{12}^{(1)} - f_2 q_{22}^{(1)} \cos 3E \right) \]

\[ + \left. \frac{1}{2} (f_1 q_{12}^{(1)} + f_2 q_{12}^{(1)} \sin 3E \right) \right\}, \quad (1.2.4.2) \]

From King-Hele [10] and using equation (1.2.2.37)
After using binomial expansion up to fourth-order terms in $e$ we write

$$|\psi| = \frac{K}{\alpha^2} \left[ 1 + e \left( 1 + \frac{9e^4}{64} + (e + \frac{3e^3}{8}) \cos E + \left( \frac{e^2}{4} + \frac{3e^4}{16} \right) \cos 2E \right) \right. \left. + \frac{e^4}{8} \cos 3E + \frac{3e^4}{64} \cos 4E \right],$$

(1.2.4.3)

and

$$|\psi| = \frac{K}{\alpha^2} \left[ 1 + e \left( 1 + \left( \frac{27}{64} e^4 \right) \cos E + \left( \frac{e^2}{4} + \frac{3e^4}{16} \right) \cos 2E \right) \right. \left. + \frac{3e^4}{64} \cos 3E + \frac{27 e^4}{64} \cos 4E \right].$$

(1.2.4.4)

Substituting equations (1.2.4.3) and (1.2.4.4) in equations (1.2.4.1) and (1.2.4.2) we get

$$\frac{dw}{dE} = \frac{\rho \delta K}{10 \alpha^2} \left[ \frac{1}{2} f_0 + \frac{5}{8} e^2 (2 f_0 + f_1) + \frac{27}{64} e^4 (3 f_0 + 2 f_1) \right] + \{ e (2 f_0 + f_1) \}

+ \frac{3}{4} e^2 (3 f_0 + 2 f_1) \cos E + \left( e + \frac{3}{4} e^3 \right) f_2 \sin E

+ \frac{1}{4} f_1 + \frac{5}{4} e^2 (f_0 + f_1) + \frac{27}{128} e^4 (8 f_0 + 7 f_1) \cos 2E

+ (1 + \frac{5}{4} e^2 + \frac{135}{128} e^4 ) f_2 \sin 2E

+ \frac{1}{8} e f_1 + \frac{3}{8} e^2 (3 f_1 + 2 f_0) \cos 3E + \left( e + \frac{9}{8} e^3 \right) f_2 \sin 3E

+ \frac{1}{8} e^2 f + \frac{27}{64} e^4 (f_0 + 2 f_1) \cos 4E

+ \frac{5}{8} e^2 f + \frac{27}{32} e^4 f_2 \sin 4E + \frac{3}{8} e^3 f_1 \cos 5E.$$
\[
\frac{d\alpha_t}{dE} = \frac{\rho \delta K}{16 \omega a^3} \left[ g_{01} + h_{11} + \frac{e}{2} \left( g_{11} + 2h_{11} \right) \right] \\
+ \frac{e^2}{8} \left( 2g_{01} + 10h_{11} + g_{31} + 5h_{31} \right) \\
+ \frac{e^3}{16} \left( 3g_{11} + 18h_{11} + 6h_{31} \right) \\
+ \frac{e^4}{64} \left( 9g_{01} + 81h_{11} + 6g_{31} + 54h_{31} \right) \\
+ \left\{ g_{11} + h_{11} + \frac{e}{2} \left( 2g_{01} + 4h_{01} + g_{31} + 2h_{31} \right) \right\} \cos E \\
+ \left\{ g_{21} + h_{21} + \frac{e}{2} \left( 2g_{11} + h_{41} \right) \right\} \cos E \\
+ \frac{e^2}{8} \left( g_{21} + 5h_{11} + 5h_{61} \right) \\
+ \frac{e^3}{8} \left( g_{41} + 6h_{41} \right) \\
+ \frac{e^4}{128} \left( 6g_{21} + 54h_{11} + 81h_{61} \right) \sin E \\
+ \left\{ g_{31} + h_{31} + \frac{e}{2} \left( g_{11} + 2h_{11} + 2h_{31} \right) \right\} \sin E \\
+ \frac{e^2}{4} \left( g_{01} + 5h_{01} + g_{31} + 5h_{51} \right) \\
+ \frac{e^3}{8} \left( 2g_{11} + 12h_{11} + 9h_{51} \right)
\]
\[ \begin{align*}
&+ \frac{e^1}{16} (g_{41} + 5h_{61}) \\
&+ \frac{e^2}{8} (g_{21} + 5h_{31}) + \frac{e^3}{16} (g_{41} + 18h_{31}) \rangle \sin 3E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^2}{16} (g_{41} + 18h_{31}) + \frac{e^3}{128} (g_{41} + 135h_{31} + 162h_{31}) \rangle \sin 3E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^3}{64} (3g_{41} + 108h_{31} + 54h_{31}) \rangle \cos 4E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^3}{32} (3g_{41} + 27h_{31} + 54h_{31}) \rangle \sin 4E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^3}{5h_{31} + \frac{e^3}{16} (g_{41} + 6h_{31}) \rangle \cos 5E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^3}{128} (3g_{41} + 108h_{31} + 108h_{31}) \rangle \sin 5E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^3}{128} (3g_{41} + 27h_{31} + 108h_{31}) \rangle \cos 6E
\end{align*} \]

\[ \begin{align*}
&+ \frac{e^3}{128} (3g_{41} + 27h_{31} + 108h_{31}) \rangle \sin 6E
\end{align*} \]
\[ + \frac{3 \epsilon^4}{8} h_{63} + \frac{\epsilon^4}{128} (3q_{34} + 27 h_{44}) \sin 6E \]
\[ + \frac{27 \epsilon^4}{128} h_{63} \sin 7E. \]  

(1.2.4.6)

where

\[ g_{01} = \frac{d_{11} q_{11}^{(1)} + d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} + d_{22} q_{22}^{(1)}}{d_1} \]
\[ g_1 = 2 (d_{11} q_{11}^{(1)} + d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} + d_{22} q_{22}^{(1)}) \]
\[ g_2 = 2 (d_{11} q_{11}^{(1)} + d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} + d_{22} q_{22}^{(1)}) \]
\[ g_3 = d_{11} q_{11}^{(1)} - d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} - d_{22} q_{22}^{(1)} + d_{31} q_{31}^{(1)} - d_{32} q_{32}^{(1)} \]
\[ g_4 = d_{11} q_{11}^{(1)} + d_{12} q_{12}^{(1)} + d_{21} q_{21}^{(1)} + d_{22} q_{22}^{(1)} + d_{31} q_{31}^{(1)} + d_{32} q_{32}^{(1)} \]
\[ h_{01} = d_0 q_{01}^{(1)} / \alpha \]
\[ h_1 = \frac{(2 f_0 q_{12}^{(1)} + f_1 q_{11}^{(1)} + f_2 q_{22}^{(1)})}{\alpha} \]
\[ h_2 = \frac{(2 f_0 q_{22}^{(1)} - f_1 q_{11}^{(1)} + f_2 q_{22}^{(1)})}{\alpha} \]
\[ h_3 = \frac{2 f_1 q_{11}^{(1)}}{\alpha} \]
\[ h_4 = \frac{2 f_2 q_{22}^{(1)}}{\alpha} \]
\[ h_5 = \frac{(f_1 q_{11}^{(1)} - f_2 q_{22}^{(1)})}{\alpha} \]
\[ h_6 = \frac{(f_2 q_{22}^{(1)} + f_1 q_{11}^{(1)})}{\alpha} \]

Similarly we can write \( \frac{d \alpha}{d E}, \frac{d \alpha}{d E} \) and \( \frac{d \alpha}{d E} \). For \( \frac{d \beta}{d E} \) we write

\[ u, \cos \frac{E}{2} = (\alpha, \cos \frac{E}{2} + \beta, \sin \frac{E}{2}) \cos \frac{E}{2} \]
\[ = \frac{1}{2} \left[ \sum_{0}^{(1)} + \sum_{1}^{(1)} \cos E + q_{01}^{(1)} \sin E \right]. \]

and

\[ u, \cos \frac{E}{2} = \left( - \frac{1}{2} \alpha, \sin \frac{E}{2} + \frac{1}{2} \beta, \cos \frac{E}{2} \cos \frac{E}{2} \right) \cos \frac{E}{2} \]
\[ = \frac{1}{2} \left[ \sum_{0}^{(1)} + \sum_{1}^{(1)} \cos E + q_{01}^{(1)} \sin E \right]. \]

with \( q_{01}^{(1)} = \alpha, q_{11}^{(1)} = \alpha, q_{21}^{(1)} = \beta, \) and \( q_{01}^{(1)} = \alpha, q_{01}^{(1)} = \alpha, \) and \( q_{21}^{(1)} = \beta. \)
To find the increments $\Delta w$ and $\Delta \alpha$, by choosing an appropriate density model, as the case may be, we evaluate the integrands in equations (1.2.4.5) and (1.2.4.6).

### 1.2.5 Bessel functions of imaginary argument $I_n(z)$

The integrals are evaluated by using the integral representation of the Bessel function of the first kind and of imaginary argument

$$ I_n(z) = \frac{1}{2\pi} \int_0^\infty \exp(z \cos \theta) \cos n \theta \, d\theta $$

$$ = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (n+m)!} z^{n+m} \quad (1.2.5.1) $$

The function follows some properties known as Recurrence relations. They are,

$$ I_{n+1}(z) - I_{n-1}(z) = 2n I_n(z), $$

$$ I_{n+1}(z) + I_{n-1}(z) = 2I_n(z), $$

$$ z I_{n+1}(z) - I_n(z) = z I_{n-1}(z), $$

$$ z I_{n+1}(z) - I_{n-1}(z) = z I_{n-1}(z), $$

where $I_n$ denotes $\frac{d^{N-1}I_n}{dz^N}$.

### 1.2.6 Conclusion

We list the advantages of the KS element equations as follows:

(i) Instabilities associated with solving the two-body (conic) equations are eliminated.

(ii) An orbital frequency based on the total energy gives more accuracy to calculations of in orbit positions.

(iii) The differential equations are smoothed for eccentric orbits because eccentric anomaly is the independent variable.
(iv) The equations are less sensitive to round off and truncation errors in the numerical integration algorithm.

(v) The pure two-body non-linear Keplerian motion is described by harmonic oscillations governed by linear differential equations with constant coefficients. The linear differential equations are everywhere regular.

(vi) Only two of the equations are need to be solved analytically to compute the state vector at the end of each revolution due to the symmetry in KS element equations.

(vii) The accuracies of the numerical computations can be examined with the help of the bilinear relation in the KS elements.

Thus we conclude that the KS total energy orbital elements are a very powerful method for the numerical solutions of the differential equations of satellite motion.