Chapter 2
2.0 Introduction

This chapter provides us some unification and representation of Voigt function \( k(x,y) \) and \( L(x,y) \) which play an important role in several diverse field of physics such as astrophysical spectroscopy, emission, absorption and transfer of radiation in heated atmosphere and the theory of neutron reactions. We have derived several representations of these functions in terms of series and integrals which are specially useful in situations where the parameter and variables take on particular values.

Furthermore, the function

\[ k(x,y) + i L(x,y) \]

is, except for a numerical factor, identical to the so-called 'plasma dispersion function' which is tabulated by Fried and Conte [26] and Fetties et al. [25].

In many given physical problems, a numerical and analytical evaluation of Voigt function is required. For an
excellent review of various mathematical properties and computational method concerning the Voigt function see for example, Armstrong and Nicholls [5] and Houbold and John [31].

On the other hand, it is well known that Bessel function are closely associated with problems possessing circular or cylindrical symmetry for example, they arise in the theory of electromagnetism and in the study of free vibration of a circular membrane [47].

Srivastava and Miller [80] established a link of Bessel functions with the generalized voigt function

In section (2.1) some unification and presentation of voigt function is given in terms of series and integral which are specially useful in situations when the parameters and variables take on particular values.

Section (2.2) aims at presenting multiindices and multivariables study of the unified (or generalized) Voigt functions which play an important role in the several diverse field of physics. Some expressions of these functions are given in terms of familiar special functions of multivariable.
Further representation and series expansions involving multidimensional classical polynomials (Laguerre and Hermite) for mathematical physics are established in section (2.3).

2.1 Some known representation of Voigt functions

First of all, we recall here the following integral representations due to Reiche [64]

\[ k(x,y) = (\pi)^{1/2} \int_0^\infty \exp(-yt - \frac{1}{4}t^2) \cos(xt) \, dt \]  

\[ L(x,y) = (\pi)^{1/2} \int_0^\infty \exp(-yt - \frac{1}{4}t^2) \sin(xt) \, dt \]  

so that

\[ K(x,y) + iL(x,y) = (\pi)^{-1/2} \int_0^\infty \exp(y-ix)t - \frac{1}{4}t^2)dt \]  

\[ = \exp[(y - ix)^2] (1 - erf(y - ix)) \]

and

\[ K(x,y) - iL(x,y) = (\pi)^{-1/2} \int_0^\infty \exp(y + ix)t - \frac{1}{4}t^2)dt \]  

\[ = \exp[(y + ix)^2] (1 - erf(y - ix)) \]
where we have used an elementary integral given in Gradshteyn and Ryzhik [30, p.307; 3.322(2)] Since the error function (See Srivastava and Kashyap [77 p. 17(71)])

\[
\text{erf}(z) = \left( \frac{2z}{\sqrt{\pi}} \right) \mathbf{I}_1 \left[ \frac{1}{2}, \frac{3}{2}; -z^2 \right]
\]

\[
= \left( \frac{2z}{\sqrt{\pi}} \right) \exp(-z^2) \mathbf{I}_1 \left[ \frac{1}{2}, \frac{3}{2}; z^2 \right], \quad |z| < \infty,
\]

by Kummer's transformation for the confluent hypergeometric function \( \mathbf{I}_1 \) substitution in (2.1.3) and (2.1.4) followed by separation of real and imaginary part will readily yield the corrected version of \( \mathbf{I}_1 \) representation for \( K(x,y) \) and \( L(x,y) \) due to Exton [44] as already observed by Katriel [44] and Pettis [24]. It should be remarked in passing that, in view of the above relation [77, p. 17(71)], the corrected version of Exton's \( \mathbf{I}_1 \) representations for the Voigt function would follow directly from (2.1.1) and (2.1.2) by appealing to some known integral formula [19, p. 15(16); p. 74 (27)]

Srivastava and Miller [80; p.113(8)] introduced and studied systematically a unification (and generalization) of the Voigt functions \( K(x,y) \) and \( L(x,y) \) in the form:
\[ V_{\mu,\nu}(x,y) = \left(\frac{x}{2}\right)^{1/2} \int_{0}^{\infty} t^\mu \exp(-yt - \frac{1}{4}t^2) J_\nu(xt) \, dt \quad (2.1.5) \]

so that

\[ V_{1/2,-1/2}(x,y) = k(x,y) \quad \text{and} \quad V_{1/2,1/2}(x,y) = L(x,y) \quad (2.1.6) \]

where the Bessel function \( J_\nu(z) \), of order \( \nu \), is defined by (1.4.2) and (1.6.3).

Further representation (Series) of the unified Voigt function \( V_{\mu,\nu}(x,y) \) is given by Srivastava and Miller [80; p.113 (11)]

\[
V_{\mu,\nu}(x,y) = \frac{2^{\mu-1/2} x^{\nu+1/2}}{\Gamma(\nu+1)} \left\{ \Gamma\left[\frac{1}{2}(\mu+\nu+1)\right] \psi_2\left[\frac{1}{2}(\mu+\nu+1); \nu+1, \frac{1}{2}; -x^2, y^2\right] 
- 2y\Gamma\left[\frac{1}{2}(\mu+\nu+2)\right] \psi_2\left[\frac{1}{2}(\mu+\nu+2); \nu+1, \frac{3}{2}; -x^2, y^2\right] \right\} \quad (2.1.7)
\]

\((\mu, \nu \in \mathbb{R}^+, x \in \mathbb{R} \text{ and } \Re(\mu+\nu) > -1)\)

where \( \psi_2 \) denotes the Humbert's confluent hypergeometric function of two variables, defined by (1.3.8)

Subsequently following the work of Srivastava and Miller [80] closely, Klusch [45] proposed a unification of the
Voigt functions $K_\mu[x,y,z]$ and $L_\mu[x,y,z]$ in a slightly modified form (see [45; p. 233 (22)].

\[
\Omega_{\mu,\nu}(x,y,z) = \left(\frac{x}{2}\right)^{1/2} \int_0^\infty t^\mu \exp(-yt - zt^2)J_\nu(xt) \, dt,
\]

\[(x,y,z \in \mathbb{R}^+; \Re(\mu+\nu)>-1).\]

so that

\[
\begin{align*}
\Omega_{\mu,-1/2}(x,y,z) &= k_{\mu+1/2}(x,y,z), \\
\Omega_{\nu,1/2}(x,y,z) &= L_{\nu+1/2}(x,y,z)
\end{align*}
\]

\[
\begin{align*}
\Omega_{1/2,-1/2}(x,y,\frac{1}{4}) &= k(x,y), \\
\Omega_{1/2,1/2}(x,y,\frac{1}{4}) &= L(x,y)
\end{align*}
\]

By comparing (2.1.5) and (2.1.8) we find that

\[
\Omega_{\mu,\nu}(x,y,z) = (2\sqrt{z})^{-\mu-1/2}V_{\mu,\nu}\left(\frac{x}{2\sqrt{z}}, \frac{y}{2\sqrt{z}}\right)
\]

and

The relations (2.1.5) and (2.1.8) are, in fact unification (and generalization) of the Voigt function $k(x,y)$ and $L(x,y)$.

In an attempt to generalize the work of Srivastava and Millar [80], Siddiqui [67], Srivastava and Chen [74], Siddiqui and Uppal [68] and Yang [94] studied the following unification (and generalization) of the Voigt function $k(x,y)$ and $L(x,y)$ in the form
\[ \Omega_{\eta,\nu,\lambda}(x,y,z) = \left( \frac{x}{2} \right)^{1/2} \int_0^\infty t^\mu \exp(-yt - \frac{1}{4} t^2) J_{\nu,\lambda}^\mu(xt) \, dt \quad (2.1.11) \]

\[(x,y,\mu \in \mathbb{R}^+; \, R(\eta+\nu+2\lambda)>-1).\]

where

\[ J_{\nu,\lambda}^\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z^2)^{\nu+2\lambda+2m}}{\Gamma(\lambda+m+1) \Gamma(\nu+\lambda+m\mu+1)} \quad (2.1.12) \]

Srivastava, Pathan and Kamarujjama [85] have studied and investigated a slightly modified form of formula (2.1.11) in the form given below:

\[ \Omega_{\eta,\nu,\lambda}(x,y,z) = \left( \frac{x}{2} \right)^{1/2} \int_0^\infty t^\mu \exp(-yt - zt^2) J_{\nu,\lambda}^\mu(xt) \, dt \quad (2.1.13) \]

\[(x,y,z,\mu \in \mathbb{R}^+; \, R(\eta+\nu+2\lambda)>-1).\]

The following explicit expression of the generalized Voigt function \( \Omega_{\mu,v}[x,y,z] \) of Klusch is given by Srivastava, Pathan and Kamarujjama [85]

\[ \Omega_{\mu,v}[x,y,z] = \frac{Z^{-1/2(\mu+v+1)} X^{v+1/2}}{2^{v+1/2}} \sum_{m,n=0}^{\infty} \frac{\left(\frac{-x^2}{4z}\right)^m \left(\frac{y^2}{4z}\right)^n}{m! n! \Gamma v + m + 1} \]
\[ \Gamma \left( \frac{1}{2} \left( \mu + \nu + 2m + n + l \right) \right) \quad (2.1.14) \]

and

\[ \Omega_{\mu, \nu}[x, y, z] = \frac{Z^{1/2(\mu+\nu+1)} x^{\nu + 1/2}}{2^{\nu + 1/2} \Gamma \nu + 1} \]

\[ \left\{ \Gamma \left[ \frac{1}{2} (\mu + \nu + 1) \right] \Psi_2 \left[ \frac{1}{2} (\mu + \nu + 1); \nu + 1, \frac{1}{2}, -\frac{x^2}{4z}, \frac{y^2}{4z} \right] \right\} \]

\[ \frac{-y}{\sqrt{z}} \Gamma \left[ \frac{1}{2} (\mu + \nu + 2) \right] \Psi_2 \left[ \frac{1}{2} (\mu + \nu + 2); \nu + 1, \frac{3}{2}, -\frac{x^2}{4z}, \frac{y^2}{4z} \right] \right\} \quad (2.1.15) \]

\( (x, y, z \in \mathbb{R}^+; \text{Re}(\mu+\nu) > -1) \)

For \( \nu = \frac{1}{2} \), equation (2.1.15) reduces to

\[ K_{\mu + \frac{1}{2}}[x, y, z] = \frac{Z^{-1/2(\mu+1)}}{2\sqrt{\pi}} \]

\[ \left\{ \Gamma \left[ \frac{1}{2} (\mu + \frac{1}{2}) \right] \Psi_2 \left[ \frac{1}{2} (\mu + \frac{1}{2}); \frac{1}{2}, \frac{1}{2}, -\frac{x^2}{4z}, \frac{y^2}{4z} \right] \right\} \]

\[ \frac{-y}{\sqrt{z}} \Gamma \left[ \frac{1}{2} (\mu + \frac{3}{2}) \right] \Psi_2 \left[ \frac{1}{2} (\mu + \frac{3}{2}); \frac{1}{2}, \frac{3}{2}, -\frac{x^2}{4z}, \frac{y^2}{4z} \right] \right\} \quad (2.1.16) \]

and

\[ L_{\mu + \frac{1}{2}}[x, y, z] = \frac{x z^{-1/2(\mu+1)}}{2\sqrt{\pi}} \]
In particular, \( \mu = 1/2 \) and \( z = 1/4 \), equations (2.1.16) and (2.1.17) reduce to the following known results of Exton [22; p. L76(8),(9)]

\[
K[x,y] = \sum_{\mu=-\infty}^{\infty} \psi_2 \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -x^2, y^2 \end{array} \right] - \frac{2y}{\sqrt{\pi}} \psi_2 \left[ \begin{array}{c} \frac{1}{2}, \frac{3}{2}; -x^2, y^2 \end{array} \right],
\]

(2.1.18)

and

\[
L[x,y] = \frac{2x}{\sqrt{\pi}} \psi_2 \left[ \begin{array}{c} \frac{3}{2}, \frac{1}{2}; -x^2, y^2 \end{array} \right] - 2xy \psi_2 \left[ \begin{array}{c} \frac{3}{2}, \frac{3}{2}; -x^2, y^2 \end{array} \right],
\]

(2.1.19)

respectively.
2.2 Voigt functions of multivariables

In view of the above facts, thus we introduce and study the multivariable Voigt functions of the first kind, and of the form

\[ K[x_1, \ldots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{n-2} \exp(-yt - \frac{1}{4} t^2) \prod_{j=1}^n (\cos(x_j t)) dt \quad (2.2.1) \]

\[ L[x_1, \ldots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{n-2} \exp(-yt - \frac{1}{4} t^2) \prod_{j=1}^n (\sin(x_j t)) dt \quad (2.2.2) \]

(y \in \mathbb{R}^+ and x \in \mathbb{R}).

Obviously

\[ K[x_1, \ldots, x_n, y] \pm iL[x_1, \ldots, x_n, y] = (\pi)^{-n/2} \int_0^\infty t^{n-2} \exp(-yt - \frac{1}{4} t^2) \prod_{j=1}^n (\cos(x_j t) \pm i \sin(x_j t)) dt. \quad (2.2.3) \]

For \( n = 1 \) the above equations (2.2.1) to (2.2.3) reduces to the elementary integrals (2.1.1) to (2.1.4)

From the view point of the relation (1.4.3), we now define the generalized (unified) Voigt functions of multi-variables by means of integral
\[ V_{\mu,v_1,\ldots,v_n}(x_1,\ldots,x_n,y) = \left(\frac{x_1}{2}\right)^{\frac{\mu}{2}} \cdots \left(\frac{x_n}{2}\right)^{\frac{\mu}{2}} \int_0^\infty t^\mu \exp\left(-yt - \frac{1}{4}t^2\right) \prod_{j=1}^n (I_{v_j}(x_j))dt \]

(\( y \in \mathbb{R}^+; x_1,\ldots,x_n \in \mathbb{R} \) and \( \text{Re}(\mu + \sum_{j=1}^n v_j) > -1 \))

so that

\[ K[x_1,\ldots,x_n,y] = V_{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}[x_1,\ldots,x_n,y], \quad L[x_1,\ldots,x_n,y] = V_{\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}}[x_1,\ldots,x_n,y]. \]

(2.2.5)

Making use of the series representation (1.4.2) and expanding the exponential function \( \exp(-yt) \), and then integrating the resulting (absolutely convergent) multiple series term by term, we obtain

\[ V_{\mu,v_1,\ldots,v_n}(x_1,\ldots,x_n,y) = 2^{\mu-1/2}x_1^{\nu_1+1/2} \cdots x_n^{\nu_n+1/2} \]

\[ \sum_{m_1,\ldots,m_n,r=0}^{\infty} \frac{(-x_1^2)^m_1 \cdots (-x_n^2)^m_n (-2y)^r}{(m_1)! \cdots (m_n)! r! \Gamma(v_1 + m_1 + 1) \cdots \Gamma(v_n + m_n + 1)} \]

\[ \cdot \Gamma\left[\frac{1}{2}(\mu + \sum v_j + 2(m_1 + \cdots + m_n) + r + 1)\right]. \]

(2.2.6)

Separating the \( r \)-series into its even and odd terms, we get

\[ \left\{ \Gamma\left[\frac{1}{2}(\mu + \sum v_j + 1)\psi_2^{(n+1)}\right] \frac{1}{2}(\mu + \sum v_j + 1); v_1 + 1,\ldots,v_n + 1, \frac{1}{2}; -x_1^2,\ldots,-x_n^2, y^2 \right\} \]
where $\psi_2^{(n)}$ denotes Humbert's confluent hypergeometric function of $n$ variables defined by (1.3.14)

For $\mu=-\nu_1=-\cdots=-\nu_n=1/2$, equation (2.2.7) reduces to the representation:

$$K[x_1,\ldots,x_n,y] = (\pi)^{-\frac{n}{2}} \left\{ \Gamma\left(\frac{3}{4} - \frac{n}{4}\right)\psi_2^{(n+1)}\left[\frac{3}{4} - \frac{n}{4}, \frac{1}{2}, \frac{1}{2}, -x_1^2, \ldots, -x_n^2, y^2\right] ight.$$

$$- 2y\Gamma\left(\frac{3}{4} - \frac{n}{4}\right)\psi_2^{(n+1)}\left[\frac{3}{4} - \frac{n}{4} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, -x_1^2, \ldots, -x_n^2, y^2\right]\left.\right\}, \quad (2.2.8)$$

while for the special values $\mu=\nu_1=\cdots=\nu_1 = \frac{1}{2}$, equation (2.2.7) yields the representation:

$$I[x_1,\ldots,x_n,y] = \left(\frac{2}{\sqrt{\pi}}\right)^n \prod_{j=1}^n (x_j) \left\{ \Gamma\left(\frac{3}{4} + \frac{n}{4}\right)\psi_2^{n+1}\left[\frac{3}{4} + \frac{n}{4}, \frac{3}{2}, \frac{3}{2}, -x_1^2, -x_n^2, y^2\right] ight.$$

$$- 2y\Gamma\left(\frac{3}{4} + \frac{n}{4}\right)\psi_2^{n+1}\left[\frac{3}{4} + \frac{n}{4} - \frac{3}{4}, \frac{3}{2}, \frac{3}{2}, -x_1^2, -x_n^2, y^2\right]\left.\right\}. \quad (2.2.9)$$

When $n = 1$, equations (2.2.8) and (2.2.9) correspond to results (8) and (9), respectively of Exton [22; p. L 76].
Also equation (2.7) reduces to the known representation of Srivastava and Miller [80; p. 131 (II)].

2.3 **Further representations of** $V_{\mu,\nu} (x,y)$

For convenience, a few conventions and notations of multiindices are recalled here [7; p. 3].

Let $v = (v_1, ..., v_n) \in \mathbb{R}^n$ and $k = (k_1, ..., k_n) \in \mathbb{N}_0^n$ ($n$ factors) where $k_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $j = \{1, 2, ..., n\}$. We have the following abbreviations:

$$k! = k_1! \cdot k_2! \cdot ... \cdot k_n!, \; k \in \mathbb{N}_0^n$$

$$(v)_k = \prod_{j=1}^{n} (v_j)_{k_j} \cdot (v_n)_{k_n}, \; v \in \mathbb{R}^n (or \mathbb{C}^n), \; k \in \mathbb{N}_0^n$$

$$\Gamma(v) = \prod_{j=1}^{n} \Gamma(v_j)_{k_j} \cdot \Gamma(v_n)_{k_n}, \; v \in \mathbb{R}^n (or \mathbb{C}^n)$$

$$\lambda x = \lambda x_1, ..., \lambda x_n, \; \lambda \in \mathbb{R}, x \in \mathbb{R}^n.$$

$\lambda + 1$ means $\lambda_1 + 1, \lambda_2 + 1, ..., \lambda_n + 1$ ($n$-parameters) and

$$\sum_{k \in \mathbb{N}_0^n} f(k) = \sum_{k_1=0} f(k_1, ..., k_2=0, ..., k_n=0) (multiple \; series).$$

We start from the definition of multiindex Laguerre and Hermite polynomials of $n$ variables [7; p. 174].
Laguerre polynomials of \( n \) variables of order \( \nu \in \mathbb{R}^n \) and degree \( k \in \mathbb{N}_0 \simeq \mathbb{N} \cup \{0\} \), is the function \( L_k^\nu(x) \), defined by means of generating function

\[
e^t(x/t)^{\nu/2} \prod_{j=1}^n J_{\nu_j}(2\sqrt{x_jt}) = \sum_{k=0}^{\infty} \frac{(kt)^{n-1}x^\nu}{\Gamma(k + \nu + 1)} L_k^\nu(x) t^k
\]

or, equivalently,

\[
e'(xt)^{-\nu/2} \prod_{j=1}^n J_{\nu_j}(2\sqrt{x_jt}) = \sum_{k=0}^{\infty} \frac{(kt)^{n-1}}{\Gamma(k + \nu + 1)} L_k^\nu(x)t^k,
\]

where for convenience, we have the following the abbreviations

\[
\Gamma'(\nu) = \prod_{j=1}^n \Gamma(\nu_j) = \Gamma(\nu_1)...\Gamma(\nu_n)
\]

\[
\lambda \cdot x = \lambda x_1,...,\lambda x_n, \quad \lambda \in \mathbb{R}
\]

\( \nu \in \mathbb{R}^n, \nu > -1, x \in \mathbb{R}^n \). Furthermore \( k = kI = (k,...,k) \in \mathbb{R}^n \), \( k \in \mathbb{N}_0 \) and \( k! = (k!)^n \).

On replacing \( 2\sqrt{t} \) by \( t \) and \( x_j \) by \( x_j^2 \), \( j = \{1,2,\ldots, n\} \), respectively, in equation (2.3.1), multiplying both the sides by
\[ t^{\mu + \sum v_j} \exp \left( -yt - \frac{t^2}{2} \right) \text{ and now integrating with respect to } t \]

between the limits 0 and \( \infty \), we obtain

\[ V_{\mu, \mathbf{v}}(x, y) = V_{\mu, \mathbf{v}_1, \ldots, \mathbf{v}_n}[x_1, \ldots, x_n, y] = \left( \frac{x}{2} \right)^{\nu + 1/2} \sum_{k=0}^{\infty} \frac{(kl)^{n-1} L_k^v(x^2)}{\Gamma(k + \nu + 1) 2^{2k}} \]

\[ \int_0^\infty t^{\mu + \sum v_j + 2k} \exp \left( -yt - \frac{1}{2} t^2 \right) dt, \quad (2.3.2.) \]

where the integral formula (2.2.4) is applied.

Expand the exponential function \( \exp (-yt) \) in terms of series and then integrate to get

\[ V_{\mu, \mathbf{v}}(x, y) = 2^{(\mu - \sum v_j - 2)/2} \left( \frac{x}{2} \right)^{\nu + 1/2} \sum_{k=0}^{\infty} \frac{(kl)^{n-1} L_k^v(x^2)}{\Gamma(k + \nu + 1) 2^{2k}} \]

\[ \sum_{m=0}^{\infty} \frac{(-\sqrt{2} y)^m}{m!} \Gamma \left[ \frac{1}{2} (\mu + \sum v_j + m + 2k + 1) \right], \quad (2.3.3) \]

\( y \in \mathbb{R}^+, \mathbf{v}, \mathbf{x} \in \mathbb{R}^n \) and \( \mu + \sum v_j > -1 \)

Separate the m-series into its even and odd terms, to get

\[ V_{\mu, \mathbf{v}}(x, y) = 2^{(\mu - \sum v_j - 2)/2} \left( \frac{x}{2} \right)^{\nu + 1/2} \sum_{k=0}^{\infty} \frac{(kl)^{n-1} L_k^v(x^2)}{\Gamma(k + \nu + 1) 2^{2k}} \]

\[ \left\{ \Gamma \left[ \frac{1}{2} (\mu + \sum v_j + 1) + k \right] \right\} \left[ \Gamma \left[ \frac{1}{2} (\mu + \sum v_j + 1) + \frac{1}{2}; y^2 / 2 \right] \right] \]
\[-\sqrt{2}y \Gamma \left[ \frac{1}{2}(\mu + \sum v_j + 2) + k \right] _1F_1 \left[ \frac{1}{2}(\mu + \sum v_j + 2) + k; \frac{3}{2}; y^2 / 2 \right], \quad (2.3.4)\]

\[(\mu, y \in \mathbb{R}^+, v, x \in \mathbb{R}^n \text{ and } (\mu + \sum v_j) > -1)\]

where \(_1F_1\) denotes the confluent hypergeometric function \([63; p. 123(9)]\).

For particular values \(\mu = -\nu = 1/2\) and \(\mu = \nu = 1/2\), equation (2.3.4) reduces further to new representations of Voigt functions of multivariables

\[K(x,y) = K[x_1,\ldots,x_n,y] = 2^{(n-3)/4} \sum_{k=0}^{\infty} \frac{(k!)^{n-1}}{\Gamma (k + 1/2)} \left\{ \Gamma \left[ k + \frac{3-n}{4} \right] \right\} \]

\[\left\{ \Gamma \left[ k + \frac{5-n}{4} \right] \right\} \left( k! \right)^{-n} \left( \begin{array}{c} 1 \end{array} \right)^{2k} \left( \begin{array}{c} 2k \end{array} \right)^{n} \left( \begin{array}{c} 3-n \end{array} \right)^{2k} \left( \begin{array}{c} 3 \end{array} \right)^{2k} \left( \begin{array}{c} y^2 / 2 \end{array} \right)\right\} \quad (2.3.5)\]

and

\[L(x,y) = L(x_1,\ldots,x_n,y) = 2^{(-n+3)/4} \sum_{K=0}^{\infty} \frac{x^1 L_K^{(1/2)} (x^2)}{\Gamma (k + 3/2)} 2^k \left\{ \Gamma \left[ k + \frac{3+n}{4} \right] \right\} \]

\[\left\{ \Gamma \left[ k + \frac{5+n}{4} \right] \right\} \left( k! \right)^{-n} \left( \begin{array}{c} 1 \end{array} \right)^{2k} \left( \begin{array}{c} 2k \end{array} \right)^{n} \left( \begin{array}{c} 3-n \end{array} \right)^{2k} \left( \begin{array}{c} 3 \end{array} \right)^{2k} \left( \begin{array}{c} y^2 / 2 \end{array} \right)\right\} \quad (2.3.7)\]
where $H_k(x)$ denotes the Hermit polynomials of $n$-variables and the relations [7; p.175].

\[
L_k^{(-1/2)}(x_1^2, x_2^2, \ldots, x_n^2) = \frac{(-1)^k [2k]!}{{2}^{n-1}} H_{2k}(x)
\]

\[
x^l L_k^{(1/2)}(x_1^2, x_2^2, \ldots, x_n^2) = \frac{(-1)^k [2k+1]!}{{2}^{n-1}} H_{2k+1}(x)
\]

\[
\left(\pm \frac{1}{2}, x \in \mathbb{R}^n, x^l = \prod_{j=1}^{n} (x_j)\right)
\]

are used to get equations (5.3.6) and (5.3.8) respectively.

For $n = 1$ ($x = x \in \mathbb{R}^1$), equations (2.3.4) to (2.3.8) reduce to the known results (2.2.4) to (2.2.8) respectively of Kamarujjama and Singh [42].