Chapter 1
INTRODUCTION, DEFINITIONS
AND NOTATIONS

1.0. Introduction

A wide range of problems exist in classical and quantum physics, engineering and applied mathematics in which special functions arise. Special functions are solutions of a wide class of mathematically and physically relevant functional equations.

Each special function can be defined in a variety of ways and different researchers may choose different definitions (Rodrigous formulas, generating functions, contour integral). At the present time, applied mathematics, physics, and various branch of science and technology involves generating function of special functions and theory of integral transforms.

The aim of the present chapter is to introduce several class of special functions which occur rather more frequently in the study of generating functions and transformations.
1.1. The Gamma Function and Related Functions

The Gamma function has several equivalent definitions, most of which are due to Euler,

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt, \quad \text{Re}(z) > 0 \quad (1.1.1) \]

upon integrating by part equation (1.1.1) yields the recurrence relation

\[ \Gamma(z+1) = z \Gamma(z) \quad (1.1.2) \]

the relation (1.1.2) yields the useful result

\[ \Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots \]

which shows that gamma function is the generalization of factorial function

The Beta function

We define the beta function \( B(p,q) \) by

\[ B(p,q) = \int_{0}^{1} t^{p-1} (1-x)^{q-1} \, dx, \quad \text{Re}(p) > 0, \text{Re}(q) > 0 \quad (1.1.3) \]

Gamma function and Beta function are related by the following relation

\[ B(p,q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}, \quad p, q \neq 0, -1, \ldots \quad (1.1.4) \]
The Pochhammer symbol

The Pochhammer symbol \((\lambda)_n\) is defined by

\[
(\lambda)_n = \begin{cases} 
1, & \text{if } n=0 \\
\lambda(\lambda+1)\ldots(\lambda+n-1), & \text{if } n=1,2,3, 
\end{cases} \tag{1.1.5}
\]

In terms of Gamma function, we have

\[
(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma\lambda}, \quad \lambda \neq 0, -1, -2, \ldots \tag{1.1.6}
\]

\[
(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n \tag{1.1.7}
\]

\[
(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}, \quad n = 1,2,3,\ldots, \quad \lambda \neq 0, \pm 1, \ldots 2, \ldots \tag{1.1.8}
\]

\[
(\lambda)_{n-m} = \frac{(-1)^m (\lambda)_n}{(1-\lambda-n)_m}, \quad 0 \leq m \leq n \tag{1.1.9}
\]

For \(\lambda = 1\), equation (1.1.9) reduces to

\[
(n-m)! = \frac{(-1)^m n!}{(-n)_m}, \quad 0 \leq m \leq n \tag{1.1.10}
\]

Another useful relation of pochhamer symbol \((\lambda)_n\) is included in Gauss’s multiplication theorem:

\[
(\lambda)_{mn} = (m)^m \prod_{j=1}^{m} \frac{\lambda+j-1}{m} \right) _n, n = 0,1,2, \ldots, \tag{1.1.11}
\]

where \(m\) is a positive integer.
For \( m = 2 \) the equation (1.1.11) reduces to Legendre's duplication formula

\[
(\lambda)_{2n} = 2^{2n} \left( \frac{\lambda}{2} \right)_n \left( \frac{\lambda + 1}{2} \right)_n, n = 0, 1, 2, \ldots. \tag{1.1.12}
\]

In particular we have

\[
(2n)! = 2^{2n} \left( \frac{1}{2} \right)_n n! \tag{1.1.13}
\]

and

\[
(2n + 1)! = 2^{2n} \left( \frac{3}{2} \right)_n n! \quad \ldots. \tag{1.1.14}
\]

The error function

The error function \( \text{erf}(z) \) is defined for any complex \( z \)

by

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt \tag{1.1.15}
\]

and its complement by

\[
\text{erfc}(z) = 1 - \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2)dt \tag{1.1.16}
\]

Clearly

\[
\text{erf}(0) = 0, \quad \text{erfc}(0) = 1
\]

\[
\text{erf}(\infty) = 1, \quad \text{erfc}(\infty) = 0
\]
The asymptotic expansion of error function (see [63, p-36]) is obtained after little manipulation.

1.2. Gaussian Hypergeometric Function and Generalization

The second order linear differential equation

\[ z(1-z) \frac{d^2w}{dz^2} - [c - (a + b + 1)z] \frac{dw}{dz} - abw = 0 \] (1.2.1)

has a solution

\[ w = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^n n! \]

where \(a, b, c\) are parameters independent of \(z\) for \(c\) neither zero nor a negative integer and is denoted by \(\text{\(2\)F\(_{1}\) \((a, b; c; z)\)}\) i.e.

\[ \text{\(2\)F\(_{1}\) \((a, b; c; z)\)} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \] (1.2.2)

which is known as hypergeometric function. The special case \(a=c, b=1\) or \(b=c, a=1\) yields the elementary geometric series

\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \], hence the term hypergeometric.

If either of the parameter \(a\) or \(b\) is negative integer, then in this case, equation (1.2.2.) reduces to hypergeometric polynomials.
Generalized Hypergeometric Function

The hypergeometric function defined in equation (1.2.2) can be generalized in an obvious way.

\[
pFq\left[\begin{array}{c}
\alpha_1, \alpha_2, \ldots, \alpha_p; \\
\beta_1, \beta_2, \ldots, \beta_q;
\end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}
\]

\[
= pFq (\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),
\]

(1.2.3)

where \( p,q \) are positive integer or zero. The numerator parameter \( \alpha_1, \ldots, \alpha_p \) and the denominator parameter \( \beta_1, \ldots, \beta_q \) take on complex values, provided that

\[\beta_j \neq 0, -1, -2, \ldots, \quad j = 1, 2, \ldots, q\]

Convergence of \( pFq \)

The series \( pFq \)

(i) converges for all \( |z| < \infty \) if \( p \leq q \)

(ii) converges for \( |z|<1 \) if \( p=q+1 \) and

(iii) diverges for all \( z, z \neq 0 \) if \( p > q+1 \)

Further more if we set

\[\omega = \operatorname{Re}\left(\sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j\right) > 0,\]

\[\text{6}\]
it is known that the $_pF_q$ series with $p = q + 1$, is

(i) Absolutely convergent for $|z| = 1$ if $\text{Re} \ (\omega) > 0$

(ii) Conditionally convergent for $|z| = 1$, $z \neq 1$ if $-1 < \text{Re} \ (\omega) \leq 0$

(iii) Divergent for $|z| = 1$ if $\text{Re} \ (\omega) \leq 1$.

An important special case when $p = q = 1$, the equation (1.2.3) reduces to the confluent hypergeometric series $\text{$_1F_1$}$ named as Kummer’s series [46], see also Slater [69] and is given by

$$\text{$_1F_1$} (a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}$$

(1.2.4)

When $p = 2$, $q = 1$, equation (1.2.3) reduces to ordinary hypergeometric function $\text{$_2F_1$}$ of second order given by (1.2.2).

1.3. Hypergeometric function of Two and several Variables

**Appell Function**

In 1880 Appell [4] introduced four hypergeometric series which are generalization of Gauss hypergeometric function $\text{$_2F_1$}$ and are given below:

$$\text{$_1F_1$}[a, b, b'; c; x, y] = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n x^m y^n}{(c)_{m+n} m! n!}$$

(1.3.1)

$$\max \{|x|, |y|\} < 1$$
\[ F_2[a, b, b'; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_m (c')_n} x^m y^n \frac{(m+n)!}{m!n!} \quad (1.3.2) \]

\[ |x| + |y| < 1 \]

\[ F_3[a, a', b, b'; c; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)(a')_n (b)_m (b')_n}{(c)_{m+n}} x^m y^n \frac{(m+n)!}{m!n!} \quad (1.3.3) \]

\[ \max \{|x|, |y|\} < 1 \]

\[ F_4[a, b; c, c'; x, y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (c')_n} x^m y^n \frac{(m+n)!}{m!n!} \quad (1.3.4) \]

\[ \sqrt{|x|} + \sqrt{|y|} < 1 \]

The standard work on the theory of Appell series is the monograph by Appell and Kampe de Feriet [3]. See also Bailey [6, Ch-9] Slater [70, Ch-8] and Exton [23, p.23-28] for a review of the subsequent work.

**Humbert Function**

In 1920 Humbert [34] has studied seven confluent form of the four Appell functions and denoted by

\[ \Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, E_1, E_2 \]

four of them are given below: [see, 79]:

\[ \sqrt{|x|} + \sqrt{|y|} < 1 \]
\[ \Phi_1[\alpha, \beta; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, \quad (1.3.5) \]

\[ |x|<1, \quad |y|<\infty \]

\[ \Phi_2[\beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, \quad (1.3.6) \]

\[ |x|<\infty, \quad |y|<\infty \]

\[ \Psi_1[\alpha, \beta; \gamma', \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m}(\gamma')_n} \frac{x^m y^n}{m! n!}, \quad (1.3.7) \]

\[ |x|<1, \quad |y|<\infty \]
\[ \Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m y^n}{m! n!}, \]  

(1.3.8)

\(|x|<\infty, \ |y|<\infty.\)

**Lauricella Function**

In 1893 Lauricella [48] generalized the four Appell function \(F_1, F_2, F_3, F_4\) to functions of \(n\) variables defined as [see 79]

\[ F_{(n)}^A[a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1 \ldots x_n] = \sum_{m_1 \ldots m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \cdots m_n!} (1.3.9) \]

\(|x_1| + \cdots + |x_n| < 1\)

\[ F_{(n)}^B[a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1 \ldots x_n] = \sum_{m_1 \ldots m_n = 0}^{\infty} \frac{(a_1)_{m_1} \ldots (a_n)_{m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1 + \ldots + m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \cdots m_n!} (1.3.10) \]

\(\max \{ |x_1|, \ldots, |x_n| \} < 1\)

\[ F_{(n)}^C[a, b; c_1, \ldots, c_n; x_1 \ldots x_n] = \sum_{m_1 \ldots m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \cdots m_n!} (1.3.11) \]

\(\sqrt{|x_1|} + \cdots + \sqrt{|x_n|} < 1\)
\[
F_D^{(n)}[a,b_1,\ldots, b_n; c; x_1\ldots x_n] \\
= \sum_{m_1\ldots m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \ldots (b_n)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n} m_1! \cdots m_n!} \\
\text{max } \{|x_1|, \ldots, |x_n|\} < 1
\] (1.3.12)

Clearly we have

\[F_A^{(2)}=F_2, \quad F_B^{(2)}=F_3, \quad F_C^{(2)}=F_4, \quad F_D^{(2)}=F_1,\]

where \(F_1, F_2, F_3, F_4\), are Appell function defined by (1.3.1) to (1.3.4) and

\[F_A^{(1)}=F_B^{(1)}=F_C^{(1)}=F_D^{(1)}=2F_1\]

A summary of Lauricella’s work is given by Appell and Kampe de Feriet [3]. See also Carlson [9] and Srivastava [8].

A unification of Lauricell 14–hypergeometric functions \(F_1, \ldots, F_{14}\) of three variables [48], and the additional functions \(H_A, H_B, H_C\) [73] was introduced by Srivastava [72, p.428] who defined a general triple hypergeometric series \(F^{(3)}[x,y,z]\)

**Confluent form of Lauricella function**

\(\Phi_2^{(n)}\) and \(\Psi_2^{(n)}\) are two important confluent form of Lauricella functions are given by

11
\[ \Phi^{(n)}_2 [b_1, \ldots, b_n; c; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1 + \cdots + m_n} m_1! \cdots m_n!} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}, \quad (1.3.13) \]

and
\[ \Psi^{(n)}_2 [a; c_1, \ldots, c_n; x_1, \ldots, x_n] = \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n} m_1! \cdots m_n!} \frac{x_1^{m_1} \cdots x_n^{m_n}}{m_1! \cdots m_n!}. \quad (1.3.14) \]

In terms of \( \Psi^{(n)}_2 \), the multivariable extension of Whittaker's \( M_{\kappa, \mu} \) function was defined by Humbert [34] in the following form:

\[ M_{\kappa, \mu_1, \ldots, \mu_n} (x_1, \ldots, x_n) = x_1^{\mu_1 + 1/2} \cdots x_n^{\mu_n + 1/2} \exp \left[ \frac{-1}{2} (x_1 + \cdots + x_n) \right], \]

\[ \Psi^{(n)}_2 [\mu_1 + \ldots + \mu_n - \kappa + n/2; 2\mu_1 + 1, \ldots, 2\mu_n + 1; x_1, \ldots, x_n]. \quad (1.3.15) \]

1.4 Bessel Function and Hyper Bessel Function

Bessel's equation of order \( n \) is
\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 + n^2)y = 0 \quad (1.4.1) \]

where \( n \) is non-negative integer. The series solution of the equation (1.4.1) is
\[ J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{r! \Gamma n + r + 1}. \quad (1.4.2) \]
In particular,

\[ J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \quad \text{and} \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \]  

(1.4.3)

the series (1.4.2) converges for all \( x \).

We call \( J_n(x) \) as Bessel function of first kind. The generating function for the Bessel function is given by

\[ \exp \left[ \frac{x}{2} \left( t - \frac{1}{t} \right) \right] = \sum_{n=-\infty}^{\infty} t^n J_n(x) \]  

(1.4.4)

Bessel function is connected with hypergeometric function by the relation

\[ J_n(x) = \frac{(x/2)^n}{\Gamma(1+n)} \, {}_0\!F_1 \left[ \left. -;1+n;i; \frac{-x^2}{4} \right] \right. \]  

(1.4.5)

Bessel functions are of most frequent use in the theory of integral transform. For fuller discussion of the properties of Bessel function [see, 93]

**Modified Bessel’s Function**

Bessel’s modified equation is

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2) y = 0 \]  

(1.4.6)

the series solution of the equation (1.4.6) is

\[ I_n(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{2r+n}}{r! \Gamma(n+r+1)}. \]  

(1.4.7)
where \( n \) is non negative integer.

We call \( I_n(x) \) as modified Bessel function. The function \( I_n(x) \) is related to \( J_n(x) \) in much the same way that the hyperbolic function is related to trigonometric function, and we have

\[
I_n(x) = i^{-n} J_n(ix)
\]

The generalized Bessel functions (GBF) have been the topic of a recent study by the authors [13]. This research activity was stimulated by the number of problems in which this type of functions is an essential analytical tool and by their intrinsic mathematical importance. The GBF have many properties similar to those of conventional Bessel function (BF).

As far as the application of GBF are concerned they frequently arise in physical problems of quantum electrodynamics and optics, the emission of electromagnetic radiation, scattering of laser radiation from free or weekly bounded electrons ([12],[13]).

**Hyper Bessel Function**

The Hyper Bessel function \( J_{m,n}(z) \) of order 2 is defined by (see Humbert [35])
\[ J_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} {}_0F_2 \left[ -; m+1, n+1; -\left(\frac{z}{3}\right)^3 \right] \]  

(1.4.8)

and its generating function is defined by

\[ \exp \left[ \frac{z}{3} \left( x + y - \frac{1}{xy} \right) \right] = \sum_{m,n=-\infty}^{\infty} x^m y^n \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} {}_0F_2 \left[ -; m+1, n+1; \left(\frac{z}{3}\right)^3 \right] \]  

(1.4.9)

**Modified Hyper Bessel Function**

The modified Hyper Bessel Function \( I_{m,n}(z) \) of order 2 is defined by (Delerue [14])

\[ I_{m,n}(z) = \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} {}_0F_2 \left[ -; m+1, n+1; \left(\frac{z}{3}\right)^3 \right] \]  

(1.4.10)

and its generating function is defined by

\[ \exp \left[ \frac{z}{3} \left( x + y + \frac{1}{xy} \right) \right] = \sum_{m,n=-\infty}^{\infty} x^m y^n \frac{(z/3)^{m+n}}{\Gamma(m+1)\Gamma(n+1)} {}_0F_2 \left[ -; m+1, n+1; \left(\frac{z}{3}\right)^3 \right] \]  

(1.4.11)

the generating function of Hyper Bessel function \( J_{m_1,\ldots,m_n}(z) \) of order \( n \) and its modified case \( I_{m_1,\ldots,m_n}(z) \) are given by
\[
\exp \left[ \frac{z}{n+1} \left( x_1 + \ldots + x_n - \frac{1}{x_1 \ldots x_n} \right) \right] = \sum_{m_1, \ldots, m_n = -\infty}^{\infty} x_1^{m_1} \ldots x_n^{m_n} J_{m_1, \ldots, m_n} (z),
\]

(1.4.12)

where

\[
J_{m_1, \ldots, m_n} (z) = \frac{(z/n + 1)^{\sum_{j=1}^{n} m_j}}{m_1! \ldots m_n!} F_{n} \left[ \begin{array}{c}
-m_1 - 1, \ldots, -m_n - 1; \left( \frac{z}{n+1} \right)^{n+1}
\end{array} \right]
\]

(1.4.13)

and

\[
\exp \left[ \frac{z}{n+1} \left( x_1 + \ldots + x_n + \frac{1}{x_1 \ldots x_n} \right) \right] = \sum_{m_1, \ldots, m_n = -\infty}^{\infty} x_1^{m_1} \ldots x_n^{m_n} l_{m_1, \ldots, m_n} (z)
\]

(1.4.14)

where

\[
l_{m_1, \ldots, m_n} (z) = \frac{(z/n + 1)^{\sum_{j=1}^{n} m_j}}{m_1! \ldots m_n!} F_{n} \left[ \begin{array}{c}
-m_1 - 1, \ldots, -m_n - 1; \left( \frac{z}{n+1} \right)^{n+1}
\end{array} \right]
\]

(1.4.15)

For \(n=1\), these functions coincide with the Bessel functions.

1.5. The Classical Orthogonal Polynomial

The hypergeometric representation of classical orthogonal polynomial such as Jacobi polynomial,
Gegenbauer polynomial, Legendre polynomial, Hermite polynomial and Laguerre polynomial and their orthogonality properties, Rodrigues formula, recurrence relation and the differential equation satisfied by them are given in detail in Szego, [87], Reinville [63], Lebedev [49], Luke [50], Carlson [10, Ch-7], Chihara [11]. We mention few of them:

**Jacobi Polynomial**

The Jacobi Polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by generating relation.

$$
\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n = [1+1/2(x+1)t]^\alpha [1+1/2(x-1)t]^\beta \quad (1.5.1)
$$

$\text{Re } (\alpha) > -1, \text{Re } (\beta) > -1$

The Jacobi Polynomials have a number of finite series representation [63 p.255] one of them is given below

$$
P_n^{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k!(n-k)! (1+\alpha)_k (1+\alpha+\beta)_n} \left( \frac{x-1}{2} \right)^k \quad (1.5.2)
$$

For $\beta = \alpha$ the Jacobi Polynomial $P_n^{(\alpha,\alpha)}(x)$ is known as ultraspherical polynomial which is connected with the Gegenbauer polynomial $C_n^{(\alpha)}(x)$ by the relation [2, p.191]
\[ P_n^{(\alpha, \alpha)}(x) = \frac{(1 + \alpha)_n C_n^{\alpha+1/2}(x)}{(1 + 2\alpha)_n} \quad (1.5.3) \]

For \( \alpha = \beta = 0 \), equation (1.5.2) reduces to Legendre Polynomial \( P_n(x) \)

**Hermite Polynomial**

Hermite Polynomial are defined by means of generating relation

\[ \exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \quad (1.5.4) \]

Valid for all finite \( x \) and \( t \) and we can easily obtained

\[ H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k} \quad (1.5.5) \]

**Associated Laguerre Polynomial**

The associated Laguerre Polynomial \( L_n^\alpha(x) \) are defined by means of generating relation.

\[ \sum_{n=0}^{\infty} L_n^\alpha(x)t^n = (1-t)^{-(\alpha+1)} \exp \left( \frac{x}{t-1} \right) \quad (1.5.6) \]

For \( \alpha = 0 \), the above equation (1.5.6) yield the generating function for simple Laguerre Polynomial \( L_n(x) \).
A series representation of $L_n^{(\alpha)}(x)$ for non negative integers $n$ is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (n + \alpha)! x^k}{k! (n-k)! (\alpha + k)!}$$

for $\alpha = 0$, equation (1.5.8) gives the definition of Laguerre polynomial.

Laguerre Polynomial $L_n^{(\alpha)}(x)$ is also the limiting case of Jacobi Polynomial

$$L_n^{(\alpha)}(x) = \lim_{|\beta| \to \infty} \left\{ p_n^{(\alpha,\beta)} \left( 1 - \frac{2x}{\beta} \right) \right\}$$

Hypergeometric representations

Some of the orthogonal polynomials and their connections with hypergeometric function used in our work are given below

**Jacobi Polynomial**

(1) $p_n^{\alpha,\beta}(z) = \binom{\alpha + n}{n} \left[ \begin{array}{c} -n, \alpha + \beta + n + 1; \\ \alpha + 1 ; \frac{1-z}{2} \end{array} \right]$

**Gegenbauer Polynomial**

(2) $C_n^{\gamma}(z) = \binom{n + 2\gamma - 1}{n} \left[ \begin{array}{c} -n, 2\gamma + n; \\ \gamma + 1/2 ; \frac{1-z}{2} \end{array} \right]$
**Legendre Polynomial**

(3) \[ P_n(z) = P_n^{(0,0)}(z) = \binom{-n, n+1; 1; \frac{1-z}{2}}{2} \] \hspace{1cm} (1.5.11)

**Hermite Polynomial**

(4) \[ H_n(z) = (2z)^n \binom{-n, \frac{1}{2}, -\frac{n}{2}, -z^{-2}}{2} \] \hspace{1cm} (1.5.12)

**Laguerre Polynomial**

(5) \[ L_n^{(\alpha)}(z) = \frac{(1 + \alpha)_n}{n!} \binom{-n, 1 + \alpha; z}{1} \] \hspace{1cm} (1.5.13)

**Generalized Polynomials**

The Gould–Hopper [27] generalization of the Hermite polynomial is in the form

\[ g_n^m(x, h) = \sum_{k=0}^{[n/m]} \frac{n!}{k!(n-mk)!} h^k x^{n-mk} \]

\[ = x^n \binom{\Delta(m; -n); -h\left(-\frac{m}{x}\right)^m}{m} \] \hspace{1cm} (1.5.14)

where \( m \) is positive integer and \( \Delta (m; -n) \) abbreviates the array of \( m \) parameter.

For \( m=2, h= -1 \), these polynomials reduce to classical Hermite polynomials (1.5.12) and we have
\[ H_n(x) = g_n^2(2x,-1) \quad (1.5.15) \]

Other familiar generalization (and unification) of the various polynomial are studied by Srivastava and Singhal [86], Srivastava and Joshi [75] Srivastava and Panda [82] Srivastava and Pathan [83] and Shahabuddin [66].

1.6. Generating Functions

A generating function may be used to define a set of functions, to determine a differential recurrence relation or pure recurrence relation to evaluate certain integrals etc. We define a generating function for a set of function \( \{f_n(x)\} \) as follows [79, p. 78–82].

**Definition:** Let \( G(x,t) \) be a function that can be expended in powers of \( t \) such that

\[
G(x,t) = \sum_{n=0}^{\infty} c_n f_n(x)t^n \quad (1.6.1)
\]

where \( c_n \) is a function of \( n \), independent of \( x \) and \( t \). Then \( G(x,t) \) is called a generating function of the set \( \{f_n(x)\} \). If the set of function \( \{f_n(x)\} \) is also defined for negative integers \( n=0, \pm 1, \pm 2, \ldots \), the definition (1.6.1) may be extended in terms of the Laurentz series expansion
\[ G(x,t) = \sum_{n=-\infty}^{\infty} c_n f_n(x) t^n \] (1.6.2)

where \( \{c_n\} \) is independent of \( x \) and \( t \). The above definition of generating function used earlier by Rainville [63, p. 129] and McBride [51, p. 1] may be extended to include generating functions of several variables.

Definition: Let \( G(x_1,\ldots,x_k,t) \) be a function of \((k+1)\) variable, which has a formal expansion in power of \( t \) such that

\[ G(x_1,\ldots,x_k,t) = \sum_{n=0}^{\infty} c_n f_n(x_1,\ldots,x_k) t^n \] (1.6.3)

where the sequence \( c_n \) is independent of the variable \( x_1,\ldots,x_k \) and \( t \). Then we shall say that \( G(x_1,\ldots,x_k,t) \) is multivariable generating function for the set \( \{f_n(x_1,\ldots,x_k)\}_{n=0}^{\infty} \) corresponding to non–zero coefficient \( \{c_n\} \).

**Bilinear and Bilateral Generating Functions**

A multivariable generating function \( G(x_1\ldots x_k,t) \) given by (1.8.3) is said to be multilateral generating function if

\[ f_n(x_1\ldots x_k) = g_1 \alpha_1(n)(x_1)\ldots g_k \alpha_k(n)(x_k) \] (1.6.4)
where \( \alpha_j(n), j=1,2, \ldots, k \) are functions of \( n \) which are not necessarily equal. Moreover, if the functions occurring on the right hand side of (1.6.4) are equal the equation (1.6.3) are called multilinear generating function.

In particular if

\[
G(x, y; t) = \sum_{n=0}^{\infty} c_n f_n(x) g_n(y) t^n \quad (1.6.5)
\]

and the sets \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(y)\}_{n=0}^{\infty} \) are different the function \( G(x,y; t) \) is called bilateral generating function for the sets \( \{f_n(x)\}_{n=0}^{\infty} \) or \( \{g_n(y)\}_{n=0}^{\infty} \).

If \( \{f_n(x)\}_{n=0}^{\infty} \) and \( \{g_n(y)\}_{n=0}^{\infty} \) are same set of functions then in that case we say that \( G(x,y; t) \) is bilinear generating function for the set \( \{f_n(x)\}_{n=0}^{\infty} \) or \( \{g_n(y)\}_{n=0}^{\infty} \).

Example of Bilinear Generating Function

\[
(1 - t)^{-1 - \alpha} \exp \left( \frac{-(x + y)t}{1 - t} \right) \text{}_0\text{F}_1 \left( \begin{array}{c} -; \\ 1 + \alpha; \frac{xyt}{(1 - t)^2} \end{array} \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{n! L_n^\alpha(x)L_n^\alpha(y)t^n}{(1 + \alpha)_n} \quad (1.6.6)
\]
Example of bilateral generating function

\[
(1 - t)^{-1 - c - \alpha}(1 - t + yt^{-c})\exp\left(\frac{-xt}{1 - t}\right)_{1F1} \left[\begin{array}{c} \frac{c}{1 + \alpha} \\ \frac{xyt}{(1 - t)(1 - t + yt)} \end{array}\right]
\]

\[
= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{array}{c} -n, c \\ 1 + \alpha \end{array}; y \right] L_n^\alpha(x) t^n
\]  

(1.6.7)

1.7. Integral Transforms

Integral transforms play an important role in various fields of physics. The method of solution of problems arising in physics lie at the heart of the use of integral transform.

\[
T[f(x); \xi] = \int_{\Omega} F(\xi) k(x; \xi) d\xi,
\]

(1.7.1)

where \( \xi \) takes values from \( \Omega \) and the function \( k(x; \xi) \) is called Kernal. When a function \( f(x) \) is defined in terms of \( F(\xi) \) by means of an integral relation (1.7.1), we say that \( f(x) \) is the integral transform of \( F(\xi) \) for the Kernal \( k(x; \xi) \).

Taking different values of the Kernal we get different transform like Fourier, Laplace, Hankel and Mellin transforms.
Fourier Transform

We call

\[ \mathcal{F} [f(x); \xi] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx \quad (1.7.2) \]

the Fourier transform of \( f(x) \) and regard \( \xi \) as complex variable.

Laplace transform

We call

\[ \mathcal{L} [f(t); p] = \int_0^{\infty} f(t) e^{-pt} dt \quad (1.7.3) \]

the Laplace transform of \( f(t) \) and regard \( p \) as complex variable.

Hankel transform

We call

\[ \mathcal{H}_\nu [f(t); \xi] = \int_0^{\infty} f(t) \, t^\nu J_\nu(\xi t) dt \quad (1.7.4) \]

the Hankel transform of \( f(t) \) and regard \( \xi \) as complex variable.
Mellin transform

We call

\[ \mathcal{M}[f(x); s] = \int_0^\infty f(x) x^{s-1} \, dx \] (1.7.5)

the Mellin transform of \( f(x) \) and regard \( s \) as complex variable.

The most complete set of integral transforms are given in Erdelyi et al ([19], [20]) Ditkin and Prudnikov [15] and Prudnikov et al. ([61], [62]).

Other integral transforms have been developed for various purposes and they have limited use in our work so their properties and application are not mentioned in detail here.