4. COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES UNDER WEAK CONDITIONS OF COMMUTATIVITY
CHAPTER - IV

COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES UNDER WEAK CONDITIONS OF COMMUTATIVITY

4.1. INTRODUCTION

Gähler [32] initiated the study of 2-metric spaces and continued its extensive study in a series of papers ([33], [34], [35]). Among other significant contributors, White [107], Freese [28], Andalafte and Freese [29] deserve special mention.

Perhaps it was Iséki [46], who for the first time proved a fixed point theorem in 2-metric spaces which has set out a tradition of proving fixed point theorem in 2-metric spaces employing various known contractive conditions. For the work of this kind one can be referred to Iséki [47], Singh et al. [97] etc.

In this chapter we prove some fixed point theorems employing a contractive condition due to Delbosco [17] which can not be regarded as a restriction of some contractive condition like Husain-Sehgal [42], whose 2-metric space version can be found in Imdad et al. [44]. In doing so we are amply motivated by Fisher-Sessa [27], Naidu-Prasad [75] and Jungck [53].

4.2. AN EXTENSION OF DELBOSCO FIXED POINT THEOREM IN 2-METRIC SPACES

Following Delbosco [17], we consider the set $\{y$ of
all real valued continuous functions \( g : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) satisfying the conditions given in (c)-(1.3.3), we prove the following:

**Theorem 4.2.1.** Let \((X,d)\) be a complete 2-metric space and \(S,T,I\) and \(J\) are mappings of \(X\) into itself satisfying the inequality

\[
(1) \quad d(Sx,Ty,a) \leq g(d(Ix,Jy,a), d(Ix,Sx,a), d(Jy,Ty,a))
\]

for all \(x,y,a\) in \(X\), where \(g\) is in \(\mathcal{K}\). Further if the following holds:

1. \(TX \subseteq IX\) and \(SX \subseteq JX\),
2. \(\{S,I\}\) and \(\{T,J\}\) are weakly commuting pairs,
3. any one of \(S,I,T\) or \(J\) is continuous.

Then \(S,T,I\) and \(J\) have a unique common fixed point \(z\). Further, \(z\) is the unique common fixed point of \(S\) and \(I\) and of \(T\) and \(J\).

**Proof.** Let \(x_0\) be an arbitrary point in \(X\) and \(x_1\) be a point such that \(Sx_0 = Jx_1\). This can be done as \(SX \subseteq JX\). Let \(x_2\) be a point such that \(Tx_1 = Ix_2\). This can also be done as \(TX \subseteq IX\). In general, we can choose \(x_{2n}, x_{2n+1}\) and \(x_{2n+2}\) such that \(Sx_{2n} = Jx_{2n+1}\) and \(Tx_{2n+1} = Ix_{2n+2}\) for \(n = 0,1,2, \ldots\)

Using inequality (1), we have

\[
d(Sx_{2n}, Tx_{2n+1},a) \leq g(d(Ix_{2n}, Jx_{2n+1},a), d(Ix_{2n}, Sx_{2n},a),
\]

\[
d(Jx_{2n+1}, Tx_{2n+1},a))
\]

\[
= g(d(Tx_{2n-1}, Sx_{2n},a), d(Tx_{2n-1}, Sx_{2n},a),
\]

\[
d(Sx_{2n}, Tx_{2n+1},a))
\]
which implies by (1.3.3) - (ii)

\[ d(Sx_{2n}, Tx_{2n+1}, a) \leq \delta d(Sx_{2n}, Tx_{2n-1}, a). \]

Similarly, we can show that

\[ d(Sx_{2n}, Tx_{2n-1}, a) \leq \delta d(Sx_{2n-2}, Tx_{2n-1}, a) \]

and so, we have

\[
\begin{align*}
    d(Sx_{2n}, Tx_{2n+1}, a) & \leq \delta d(Sx_{2n}, Tx_{2n-1}, a) \\
    \leq \delta^2 d(Sx_0, Tx_1, a) \\
    \leq \ldots \leq \delta^n d(Sx_0, Tx_1, a)
\end{align*}
\]

for \( n = 1, 2, \ldots \). Since \( \delta < 1 \), we have that the sequence

\[ \{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\} \]

is a Cauchy sequence and so has a limit \( z \) in \( X \), since \( X \) is complete. Hence the subsequences

\[ \{Sx_{2n}\} = \{Jx_{2n+1}\} \text{ and } \{Tx_{2n-1}\} = \{Ix_{2n}\} \]

also converge to the point \( z \).

Let us now suppose that the mapping \( I \) is continuous, so that the sequences \( \{I^2x_{2n}\} \) and \( \{ISx_{2n}\} \) converge to the point \( Iz \). Since \( S \) and \( I \) are weakly commuting, we have

\[ d(SIx_{2n}, ISx_{2n}, a) \leq d(Ix_{2n}, Sx_{2n}, a) \]

and so the sequence \( \{SIx_{2n}\} \) also converges to the point \( Iz \).

We therefore have
Letting $n \to \infty$, we have
\[ d(Iz, z, a) \leq g(d(Iz, z, a), 0, 0) \]
which implies by (1.3.3) - (ii) that
\[ d(Iz, z, a) \leq 0. \]
This gives $Iz = z$.

Further,
\[
\begin{align*}
& d(Sz, Tx_{2n+1}, a) \leq g(d(Iz, Jx_{2n+1}, a), d(Iz, Sz, a), \\\n& \quad d(Jx_{2n+1}, Tx_{2n+1}, a))
\end{align*}
\]
and letting $n \to \infty$, we get
\[ d(Sz, z, a) \leq g(0, d(z, Sz, a), 0) \]
which implies by (1.3.3) - (ii) that
\[ Sz = z. \]
This means that $z$ is in the range of $S$ and $Sx \subseteq Jx$, there exists a point $z'$ in $X$ such that $Jz' = z$. Thus
\[
\begin{align*}
d(z, Tz', a) &= d(Sz, Tz', a) \\
& \leq g(d(Iz, Jz', a), d(Iz, Sz, a), d(Jz', Tz', a)) \\
& = g(0, 0, d(z, Tz', a))
\end{align*}
\]
which implies by (1.3.3) - (ii) that
\[ Tz' = z. \]

Since \( T \) and \( J \) are weakly commuting, we have
\[
\begin{align*}
d(Tz, Jz, a) &= d(TJz', JTz', a) \\
&\leq d(Jz', Tz', a) \\
&= d(z, z, a) \\
&= 0.
\end{align*}
\]
Thus we get \( Tz = Jz \) and so,
\[
\begin{align*}
d(z, Tz, a) &= d(Sz, Tz, a) \\
&\leq g(0, d(z, Tz, a), 0)
\end{align*}
\]
which implies by (1.3.3) - (ii) that
\[ z = Tz = Jz. \]

We have therefore proved that \( z = Iz = Sz = Tz = Jz \), and so \( z \) is a common fixed point of \( S, T, I \) and \( J \).

If the mapping \( J \) is continuous instead of \( I \) then we can similarly prove that \( z \) is again a common fixed point of \( S, T, I \) and \( J \).

Now suppose that \( S \) is continuous, so that the sequences \( \{S^2x_{2n}\} \) and \( \{SIX_{2n}\} \) converge to the point \( Sz \). Since \( S \) and \( I \) are weakly commuting, it follows as above that the sequence \( \{ISx_{2n}\} \) also converges to the point \( Sz \). Now,
\[
\begin{align*}
d(S^2x_{2n}, Tx_{2n+1}, a) &\leq g(d(ISx_{2n}, Tx_{2n+1}, a), d(ISx_{2n}, S^2x_{2n}, a) \\
&\quad + d(Jx_{2n+1}, Tx_{2n+1}, a)).
\end{align*}
\]
Letting $n \to \infty$, we get

$$d(Sz,z,a) \leq g(d(Sz,z,a),0,0)$$

and so by (1.3.3) - (ii) we have

$$Sz = z.$$ 

This again means that there exists a point $z'$ in $X$ such that $Jz' = z$. Thus

$$d(S^2x_{2n}Tz',a) \leq g(d(ISx_{2n}Jz',a), d(ISx_{2n}S^2x_{2n},a),$$

$$d(Jz',Tz',a))$$

and letting $n \to \infty$, it follows that

$$d(z,Tz',a) \leq g(0,0,d(z,Tz',a))$$

and so by (1.3.3) - (ii) we have

$$z = Tz'.$$

Since $T$ and $J$ are weakly commuting, it again follows that $Tz = Jz$. Further

$$d(Sx_{2n}Tz,a) \leq g(d(Ix_{2n}Jz,a), d(Ix_{2n}Sx_{2n},a), d(Jz,Tz,a).$$

Letting $n \to \infty$, we get

$$d(z,Tz,a) \leq g(d(z,Tz,a),0,0)$$

and so by (1.3.3) - (ii), we have

$$Tz = Jz = z.$$ 

It follows that the point $z$ is in the range of $T$ and since $TX \subseteq IX$, there exists a point $z''$ in $X$ such that $Iz'' = z$. Thus
\[ d(Sz'', z, a) = d(Sz'', Tz, a) \]
\[ \leq g(d(Iz'', Jz, a), d(Iz'', Sz'', a), d(Jz, Tz, a)) \]
\[ = g(0, d(z, Sz'', a), 0) \]

and so by (1.3.3) - (ii), we have

\[ Sz'' = z. \]

Again, since \( S \) and \( I \) are weakly commuting, we have

\[ d(Sz, Iz, a) = d(Slz'', ISz'', a) \]
\[ \leq d(Iz'', Sz'', a) \]
\[ = d(z, z, a) = 0. \]

Thus,

\[ Sz = Iz = z. \]

We have therefore shown that \( z \) is a common fixed point of \( S, T, I \) and \( J \).

If the mapping \( T \) is continuous instead of \( S \) then the proof that \( z \) is a common fixed point of \( S, T, I \) and \( J \) is similar.

Now, let \( w \) be a second common fixed point of \( S \) and \( I \) then

\[ d(w, z, a) = d(Sw, Tz, a) \]
\[ \leq g(d(Iw, Jz, a), d(Iw, Sw, a), d(Jz, Tz, a)) \]
\[ = g(d(w, z, a), 0, 0) \]

and it follows from (1.3.3) - (ii) that \( w = z \). Thus \( z \) is a unique common fixed point of \( S \) and \( I \). Similarly, it can be proved that \( z \) is a unique common fixed point of \( T \) and \( J \). This completes the proof.
Remark. We refer to the examples of [72] where it is shown that weak commutativity of T and J (also of S and I) and the range TX ⊆ IX (also SX ⊆ JX) and the continuity of any one of S, I, T or J is a necessary condition of Theorem 1 of [72] and therefore also in Theorem 4.2.1.

Now using the compatibility condition on mappings, we prove the following:

Theorem 4.2.2. Let (X, d) be a complete 2-metric space and S, T, I and J are mappings of X into itself satisfying the inequality

\[(1) \quad d(Sx, Ty, a) \leq g(d(Ix, Jy, a), d(Ix, Sx, a), d(Jy, Ty, a))\]

for all x, y, a in X, where g is in \(\mathcal{D}\). Further if the following holds:

(i) \(TX \subseteq IX\) and \(SX \subseteq JX\),
(ii) \(\{S, I\}\) and \(\{T, J\}\) are compatible pairs,
(iii) I and J are continuous.

Then S, T, I and J have a unique common fixed point z. Further z is a unique common fixed point of S and I and of T and J.

Proof. Proceeding as in Theorem 4.2.1, we can show that the sequence

\[\{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}\]

is a Cauchy sequence and therefore converges to a point z in X. Consequently the subsequences
\[ \{Sx_{2n}\} = \{Jx_{2n+1}\} \text{ and } \{Tx_{2n-1}\} = \{Ix_{2n}\} \]

also converge to \( z \).

Since \( I \) is continuous, it follows that the sequences \( \{ISx_{2n}\} \) and \( \{I^2x_{2n}\} \) converge to \( Iz \). Since \( S \) and \( I \) are compatible, we have \( \lim_{n \to \infty} d(SIx_{2n}, ISx_{2n}, a) = 0 \). It follows that \( \{SIx_{2n}\} \) also converge to \( Iz \). Thus using the inequality (1), we have

\[
d(SIx_{2n}, Tx_{2n+1}, a) \leq g(d(I^2x_{2n}, Jx_{2n+1}, a), d(I^2x_{2n}, SIx_{2n}, a),
\]

\[
d(Jx_{2n+1}, Tx_{2n+1}, a)).
\]

Letting \( n \to \infty \), we have

\[
d(Iz, z, a) \leq g(d(Iz, z, a), 0, 0)
\]

and so by (1.3.3) - (ii), we have

\( Iz = z \).

Again,

\[
d(Sz, Tx_{2n+1}, a) \leq g(d(Iz, Jx_{2n+1}, a), d(Iz, Sz, a),
\]

\[
d(Jx_{2n+1}, Tx_{2n+1}, a))
\]

and letting \( n \to \infty \), we get

\[
d(Sz, z, a) \leq g(0, d(Sz, z, a), 0)
\]

and so by (1.3.3) - (ii), we have

\( Sz = z \).

In the same way using the continuity of \( J \) and compatibility of \( T \) and \( J \), it can be proved that \( Jz = Tz = z \). Thus \( z \)
is a common fixed point of S, T, I and J. The uniqueness can be shown as in Theorem 4.2.1. This completes the proof.

Motivated from Jungck et al. [54], we further extend Theorem 4.2.2 employing the compatibility of type (A) which further improves various known results.

**Theorem 4.2.3.** Let S, T, I and J be the same as defined in Theorem 4.2.1. If the condition (ii) is replaced by (ii)′ \{S, I\} and \{T, J\} are compatible pairs of Type (A), then S, T, I and J have a unique common fixed point z in X. Further z is a unique common fixed point of S and I and of T and J.

**Proof.** We have seen that the sequences

\[
\{Sx_{2n}\} = \{Jx_{2n+1}\} \text{ and } \{Tx_{2n+1}\} = \{Ix_{2n}\}
\]

converge to some point z in X.

Now, suppose J is continuous. Since T and J are compatible mappings of Type (A) and \{Jx_{2n+1}\}, \{Tx_{2n+1}\} converge to the point z, by proposition 1.5.18, we have

\[
TJx_{2n+1}, JJx_{2n+1} \longrightarrow Jz \text{ as } n \rightarrow \infty.
\]

Putting \(x = x_{2n}\) and \(y = Jx_{2n+1}\) in inequality (1), we have

\[
d(Sx_{2n}, TJx_{2n+1}, a) \leq g(d(Ix_{2n}, JJx_{2n+1}, a), d(Ix_{2n}, Sx_{2n}, a),
\]

\[
d(JJx_{2n+1}, TJx_{2n+1}, a)).
\]

Letting \(n \rightarrow \infty\), by (1.3.3) - (ii), we have

\[
d(z, Jz, a) \leq g(d(z, Jz, a), 0, 0) = 0
\]
giving thereby \( \text{Jz} = z \). Again by putting \( x = x_{2n} \) and \( y = z \) in inequality (1), we have

\[
d(Sx_{2n}, Tz, a) \leq g(d(Ix_{2n}, Jz, a), d(Ix_{2n}, Sx_{2n}, a), \\
d(Tz, Jz, a))
\]

and letting \( n \to \infty \), by (1.3.3) - (ii), we have

\[
d(z, Tz, a) \leq g(0, 0, d(z, Tz, a)) = 0
\]

which implies that \( Tz = z \).

Since \( TX \subseteq IX \), there exists a point \( z' \) in \( X \) such that \( z = Iz' \) and so by using inequality (1), we have

\[
d(Sz', z, a) = d(Sz', Tz, a) \leq g(d(Iz', Sz', a), d(Iz', Sz', a), d(Jz, Tz, a)) = g(d(z, Sz', a), d(z, Sz', a), 0) \leq h d(z, Sz', a)
\]

which implies that \( Sz' = z \).

But, since \( S \) and \( I \) are compatible mappings of Type (A) and \( Sz' = Iz' = z \), we have by proposition 1.5.17, \( d(Siz', Iiz', a) = 0 \) giving thereby \( Sz = Siz' = Iiz' = Iz \).

Again, using inequality (1), by (1.3.3) - (ii), we have

\[
d(Sz, z, a) = d(Sz, Tz, a) \leq g(d(Iz, Jz, a), d(Iz, Sz, a), d(Jz, Tz, a)) = g(d(Sz, z, a), 0, 0) = 0 .
\]
giving thereby $Sz = z$.

Thus, we have proved that $Sz = Tz = Iz = Jz = z$, that is, $z$ is a common fixed point of $S, T, I$ and $J$.

Similarly, we can complete the proof when $S$ or $T$ or $I$ is continuous. The uniqueness of $z$ and the fact that $z$ is a unique common fixed point of $S, I$ and of $T, J$ follows from Theorem 4.2.1.

This completes the proof.