5. FIXED POINT THEOREMS FOR HYBRID CONTRACTIONS
CHAPTER - V

FIXED POINT THEOREMS FOR HYBRID CONTRACTIONS

5.1. INTRODUCTION

The study of fixed point theorems for multifunctions using Hausdorff metric was initiated by Markin [70] and Nadler [74]. Since then, various generalizations using different contractive conditions were obtained by Ciric [12], Khan [62], Kubaik [67], Reich [85], Smithson [101], Wegrzyk [106] and others. However, hybrid contractions, viz. contractive conditions involving singlevalued and multivalued mappings, have been studied by Mukherjee [73], Naimpally et al. [76], Rhoades et al. [89], Singh et al. [99] and many others.

Assad-Kirk [4] have used the technique of Nadler [74] to obtain the sufficient condition for non-self multivalued mappings to have a fixed point. Their results were further extended for Hardy-Rogers type generalized contraction by Itoh [49]. Subsequently these results were further extended for a pair of multivalued mappings by Khan [62] and others.

In Section 5.2, we have given necessary definitions and results used in the sequel.

In Section 5.3, we prove a result for hybrid contractions using the contractive condition of Som-Mukherjee [103] which extends their results and also those of Khan [61], Pachpatte [78] and many others.
In Section 5.4, we have obtained fixed point and coincident point results for asymptotically regular mappings which extend the earlier known results of Rhoades et al. [89] and Singh et al. [99].

Section 5.5 is devoted to the study of fixed point theorems for non-self hybrid contractions which extend the earlier known results of Assad-Kirk [4], Hadzic-Gajic [38], Itoh [49], Khan [62] and many others.

5.2. BASIC DEFINITIONS

Let \((X,d)\) be a metric space. Then following Nadler [74], Khan [62] and Singh et al. [89] we shall use the notations:

- \(\text{CL}(X) = \{A : A \text{ is a nonempty closed subset of } X\}\),
- \(\text{BN}(X) = \{A : A \text{ is a nonempty bounded subset of } X\}\),
- \(\text{CB}(X) = \{A : A \text{ is a nonempty closed and bounded subset of } X\}\),
- \(\text{C}(X) = \{A : A \text{ is a nonempty compact subset of } X\}\),
- \(d(A,B) = \inf \{d(a,b) : a \in A, b \in B\}\),
- \(d(x,A) = \inf \{d(x,a) : a \in A\}\),
- \(\delta(A,B) = \sup \{d(a,b) : a \in A, b \in B\}\).

For \(A,B \in \text{CL}(X)\) and \(\epsilon > 0\)

- \(N(\epsilon,A) = \{x \in X : d(x,a) < \epsilon \text{ for some } a \in A\}\),
- \(E_{A,B} = \{\epsilon > 0 : A \subseteq N(\epsilon,B) \text{ and } B \subseteq N(\epsilon,A)\}\).
\[ H(A, B) = \begin{cases} \inf E_{A, B}, & \text{if } E_{A, B} \neq \emptyset, \\ +\infty, & \text{if } E_{A, B} = \emptyset. \end{cases} \]

\( H(A, B) \) is called the generalized Hausdorff distance function induced by \( d \) and \( H \) defined on \( CB(X) \) is said to be the Hausdorff metric induced by \( d \). For \( A, B \in CB(X) \) one can alternatively define as

\[ H(A, B) = \max \{ \sup d(a, B) : a \in A \}, \{ \sup d(A, b) : b \in B \} \].

It is well known (cf. Kuratowski [69]) that \( CB(X) \) is a metric space with the distance function \( H \) and \( (CB(X), H) \) is a complete metric space in the event that \( (X, d) \) is complete.

Following Fisher [26] we record:

**Definition 5.2.1.** The set valued mapping \( F : X \to CB(X) \) is said to be continuous at \( x \in X \) if whenever \( \{ x_n \} \) is a sequence of points in \( X \) converging to \( x \), then the sequence \( \{ Fx_n \} \) converges to \( Fx \). The mapping \( F \) is continuous on \( X \) if it is continuous at every point \( x \in X \). Alternatively a mapping \( F : X \to CB(X) \) is said to be continuous at \( x_0 \in X \) if for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ H(Fx, Fx_0) < \varepsilon \text{ whenever } d(x, x_0) < \delta. \]

If \( F \) is continuous at every point of \( X \), then we say that \( F \) is continuous on \( X \).

Following Hadzic-Gajic [38] and Kaneko-Sessa [55], we have
Definition 5.2.2. Two mappings $F : X \rightarrow CB(X)$ and $T : X \rightarrow X$ are said to be weakly commuting if and only if for each $x,y \in X$ such that $x \in Fy$ and $Ty \in X$, we have

$$d(Tx, FTy) \leq d(Ty, Fy).$$

For $F$, a single valued mapping, this definition reduces to that of Sessa [95] (See Definition 1.5.1).

Definition 5.2.3. Two mappings $F : X \rightarrow CB(X)$ and $T : X \rightarrow X$ are compatible if and only if $TFx \in CB(X)$ for all $x \in X$ and $H(FTx_n, TFX_n) \rightarrow 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $Fx_n \rightarrow M \in CB(X)$ and $Tx_n \rightarrow t \in M$.

The following Lemma due to Nadler [74] is used frequently.

Lemma 5.2.4. Let $A, B$ be in $CB(X)$. Then for all $\varepsilon > 0$ and $a \in A$ there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \varepsilon$. If $A, B$ are compact then one can find $b \in B$ such that $d(a,b) \leq H(A,B)$.

5.3. SOME RESULTS FOR HYBRID CONTRACTIONS

We shall prove some results for hybrid contractions using the contractive condition of Som-Mukherjee [103].

Theorem 5.3.1. Let $(X,d)$ be a complete metric space, $F, G : X \rightarrow CB(X)$ and $T : X \rightarrow X$ satisfying the inequality,

$$H(Fx, Gy) \leq h(d(Tx, Fx))^\alpha (d(Ty, Gy))^\beta (d(Tx, Ty))^\gamma$$

(1)
for all \( x, y \in X, h \in (0,1), 0 < \alpha, 0 \leq \beta \) with \( \alpha + \beta < 1 \).

If the following conditions hold:

(i) \( FX \cup GX \subseteq TX \),

(ii) \( \{F,T\} \) and \( \{G,T\} \) are weakly commuting pairs,

(iii) \( T \) is continuous,

then there exists a unique common fixed point \( z \) in \( X \) such that \( z = Tz \in Fz \cap Gz \).

**Proof.** Assume \( k = 1/\sqrt{h} \). Let \( x_0 \in X \) and \( y_1 \) an arbitrary point in \( Fx_0 \). Choose \( x_1 \in X \) such that \( y_1 = Tx_1 \). This is possible as \( FX \subseteq TX \). By Lemma 5.2.4, we can find \( y_2 \in Gx_1 \) such that \( d(y_1, y_2) \leq k \cdot H(Fx_0, Gx_1) \). Choose \( x_2 \in X \) such that \( y_2 = Tx_2 \). This is also possible as \( GX \subseteq IX \). Inductively after having selected \( y_{2n} = Tx_{2n} \in Gx_{2n-1} \), choose \( y_{2n+1} = Tx_{2n+1} \in Fx_{2n} \). Then having selected \( y_{2n+1} \), choose \( y_{2n+2} = Tx_{2n+2} \in Gx_{2n+1} \) such that

\[
d(y_{2n+1}, y_{2n+2}) \leq k \cdot H(Fx_{2n}, Gx_{2n+1}).
\]

Thus for \( n \geq 1 \), we have

\[
d(y_{2n}, y_{2n+1}) \leq k \cdot H(Fx_{2n}, Gx_{2n-1})
\]

\[
\leq \sqrt{h} \left( d(Tx_{2n}, Tx_{2n-1}) \right)^\beta \left( d(Tx_{2n}, Fx_{2n}) \right)^{1-\alpha-\beta} \left( d(Tx_{2n-1}, Gx_{2n-1}) \right)^\alpha \left( d(y_{2n}, y_{2n-1}) \right)^\beta \left( d(y_{2n}, y_{2n+1}) \right)^{1-\alpha-\beta} \left( d(y_{2n-1}, y_{2n}) \right)^\alpha
\]
so that

\[(3) \quad d(y_{2n}, y_{2n+1}) \leq \gamma d(y_{2n-1}, y_{2n})^{1/2(\alpha + \beta)}\]

where \(\gamma = (h) < 1\), since \(\alpha + \beta < 1\).

Similarly, we can show that

\[d(y_{2n-1}, y_{2n}) \leq \gamma d(y_{2n-2}, y_{2n-1}).\]

Combining the above two inequalities, we obtain

\[d(y_{n+1}, y_{n+2}) \leq \gamma d(y_n, y_{n+1})\]

\[\leq \gamma^n d(y_1, y_2).\]

This shows that \(\{y_n\} = \{T x_n\}\) is a Cauchy sequence and hence converges to a point \(z\) in \(X\).

Now, since \(T\) is continuous, the sequence \(\{T T x_n\}\) converges to \(Tz\). Using the weak commutativity of \(G\) and \(T\), we have \(T x_{2n} \in G x_{2n-1}\), \(T x_{2n-1} \in X\), it follows that

\[d(T T x_{2n}, G T x_{2n-1}) \leq d(T x_{2n-1}, G x_{2n-1})\]

\[\leq d(T x_{2n-1}, T x_{2n}).\]

On letting \(n \to \infty\), we have

\[d(Tz, GT x_{2n-1}) \to 0 \quad \text{as} \quad n \to \infty.\]

Similarly, using the continuity of \(T\) and weak commutativity of \(F\) and \(T\), we can prove that

\[d(Tz, FT x_{2n}) \to 0 \quad \text{as} \quad n \to \infty.\]
Now consider
\[ d(Tz, Fz) \leq d(Tz, GTx_{2n-1}) + H(GTx_{2n-1}, Fz) \]
\[ \leq d(Tz, GTx_{2n-1}) + h(d(TTx_{2n-1}, Tz))^\beta \]
\[ (d(TTx_{2n-1}, GTx_{2n-1}))^\alpha (d(Tz, Fz))^{1-\alpha-\beta} \]
and on letting \( n \to \infty \), we get \( d(z, Tz) = 0 \) giving thereby \( Tz \in Fz \) as \( Fz \) is closed. Similarly we can show that \( Tz \in Gz \).

Now consider
\[ d(Tx_{2n}, Tz) \leq H(Gx_{2n-1}, Fz) \]
\[ \leq h(d(Tx_{2n-1}, Tz))^\beta (d(Tx_{2n-1}, Gx_{2n-1}))^\alpha \]
\[ (d(Tz, Fz))^{1-\alpha-\beta} \]
and on letting \( n \to \infty \), we get \( d(z, Tz) = 0 \). This implies that \( z = Tz \). Thus we have shown that \( z = Tz \in Fz \cap Gz \).

In order to show that \( z \) is unique, let \( w \) be another point such that \( w = Tw \in Fw \cap Gw \) then
\[ d(z, w) \leq H(Fz, Gw) \]
\[ \leq h(d(Tz, Tw))^\beta (d(Tw, Gw))^\alpha (d(Tz, Fz))^{1-\alpha-\beta} \]
\[ = 0 \]
giving thereby \( w = z \). This completes the proof.

**Theorem 5.3.2.** Let \((X, d)\) be a complete metric space, \( F, G : X \to CB(X) \) and \( T : X \to X \) satisfying inequality
(1) of Theorem 5.3.1 and the following conditions hold:

(i) \( FX \cup GX \subseteq TX, \)

(ii) \( \{ F, T \} \) and \( \{ G, T \} \) are compatible pairs,

(iii) \( F, G \) and \( T \) are continuous,

then there exists a point \( z \) in \( X \) such that \( Tz \in Fz \cap Gz. \)

**Proof.** Proceeding as in Theorem 5.3.1, we can show that \( \{ y_n \} = \{ Tx_n \} \) is a Cauchy sequence and hence converges to some point \( z \) in \( X. \) Further the inequalities (2) and (3) yields that

\[
H(Fx_{2n}, Gx_{2n-1}) \leq \sqrt{\tau} \ d(y_{2n-1}, y_{2n})
\]

which implies that the sequence

\[
\{ Fx_0, Gx_1, Fx_2, \ldots, Gx_{2n-1}, Fx_{2n}, Gx_{2n+1}, \ldots \}
\]

is a Cauchy sequence in the complete metric space \( (CB(X), H) \) and hence converges to some \( M \in CB(X). \) Consequently the subsequences \( \{ Fx_{2n} \} \) and \( \{ Gx_{2n+1} \} \) also converge to \( M. \) Thus

\[
d(z, M) \leq d(z, Tx_{2n}) + H(Tx_{2n}, M)
\]

and on letting \( n \to \infty \) we get \( z \in M \) as \( M \) is closed. Further the compatibility of \( F \) and \( T \) implies that

\[
H(FTx_{2n}, TFx_{2n}) \to 0 \quad \text{as} \quad n \to \infty.
\]

This along with the continuity of \( F \) and \( T \) implies that
\[ d(Tz, Fz) \leq d(Tz, TTx_{2n+1}) + d(TTx_{2n+1}, Fz) \]
\[ \leq d(Tz, TTx_{2n+1}) + H(TFx_{2n}, Fz) \]
\[ \leq d(Tz, TTx_{2n+1}) + H(TFx_{2n}, FTx_{2n}) + H(FTx_{2n}, Fz) \]

and on letting \( n \to \infty \), we get \( Tz \in Fz \) as \( Fz \) is closed.

Similarly using the compatibility and continuity of \( G \) and \( T \), we can show that \( Tz \in Gz \). Thus we have proved that \( Tz \in Fz \cap Gz \). This completes the proof.

In order to obtain the fixed point result for Theorem 5.3.2, we need additional hypothesis given below. We shall use the following Lemma borrowed from [55].

**Lemma 5.3.3.** Let \( F : X \to \text{CB}(X) \) and \( T : X \to X \) be compatible. If \( Tw \in Fw \) for some \( w \in X \), then \( TFw = FTw \).

**Theorem 5.3.4.** Let \( F, G \) and \( T \) have the same meaning as in Theorem 5.3.2. Assume also that for each \( x \in X \), either

(I) \( Tx \nmid T^2x \implies Tx \nmid Fx \cup Gx \) or

(II) \( Tx \in Fx \cup Gx \implies T^n x \to w \) for some \( w \in X \).

Then \( F, G \) and \( T \) have a common fixed point in \( X \).

**Proof.** By Theorem 5.3.2, there exists a point \( z \) in \( X \) such that \( Tz \in Fz \cap Gz \). Suppose \( Tz \in Fz \), we have by Lemma 5.3.3 that \( TFz = FTz \).

Assuming (I), we have \( Tz = T^2z \in TFz = FTz \). Thus \( w = Tz \) is the fixed point of \( T \) and \( F \).
Assuming (II), it is clear that $T^w = w$ by the continuity of $T$. We assert that $T^nz \in FT^{n-1}z$ for each $n$. To see this we have $T^2z = TTz \in TFz = FTz$. Using Lemma 5.3.3 (with $w = Tz$) we have $T^3z = TT^2z \in TFIz = FT^2z$.

Thus inductively we obtain $T^nz \in FT^{n-1}z$ and the continuity of $F$ implies that

$$d(w,Fw) \leq d(w,T^nz) + d(T^nz,Fw) \leq d(w,T^nz) + H(FT^{n-1}z,Fw) \to 0$$

as $n \to \infty$. Hence $w = Tw \in Fw \cap Gw$. If $Tz \in Gz$ then proceeding as above we can show that $w = Tw \in Gw \cap Fw$.

Thus $w$ is the common fixed point of $F, G$ and $T$. This completes the proof.

5.4. A FIXED POINT THEOREM FOR ASYMPTOTICALLY REGULAR MAPPINGS

In this section we have obtained a fixed point and coincidence point result for asymptotically regular mappings which extends the earlier known results of Rhoades et al. [89] and Singh et al. [99]. For proving our main theorem we introduce the following.

Let $f, g$ be the single-valued mappings of $X$ into itself and $A, B$ the multivalued mappings of $X$ into the $2^X$ (the non-empty subsets of $X$).

**Definition 5.4.1.** If, for some $x_0 \in X$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $y_n = fx_n \in Bx_{n-1}$ if
n is even and \( y_n = gx_n \in Ax_{n-1} \) if \( n \) is odd, then

\[ O_{f,g}(x_0) = \{ y_n : n \in \mathbb{N} \} \]

is said to be the orbit for \((A, B ; f, g)\) at \( x_0 \). Further \( O_{f,g}(x_0) \) is said to be regular orbit for \((A, B ; f, g)\) if

\[
d(y_n, y_{n+1}) \leq \begin{cases} 
H(Bx_{n-1}, Ax_n), & \text{if } n \text{ is even} \\
H(Ax_{n-1}, Bx_n), & \text{if } n \text{ is odd}.
\end{cases}
\]

**Definition 5.4.2.** If, for some \( x_0 \in X \), there exists a sequence \( \{x_n\} \) in \( X \) such that every Cauchy sequence of the form \( O_{f,g}(x_0) \) converges in \( X \), then \( X \) is called \((A, B ; f, g)\)-orbitally complete with respect to \( x_0 \) or simply \((A, B ; f, g ; x_0)\)-orbitally complete.

If \( f, g \) are identity mappings then \( O_{f,g}(x_0) \) is denoted by \( O(x_0) \) and \((A, B ; f, g ; x_0)\)-orbitally completeness by \((A, B ; x_0)\)-orbitality completeness.

**Definition 5.4.3.** A pair \((A, B)\) is said to be asymptotically regular at \( x_0 \in X \) if for any sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that \( y_n \in Ax_{n-1} \cup Bx_{n-1}, \lim_{n \to \infty} d(y_n, y_{n+1}) = 0 \).

We remark that Definitions 5.4.1 - 5.4.3 reduce to the definitions 1.1 - 1.3 of Singh et al. [98] for \( f = g \), and definitions 4, 6 and 7 of Rhoades et al. [99] for \( A = B, f = g \).

We further remark that orbital completeness need not imply the completeness of \( X \). However, it is evident that every complete space is orbitally complete.

**Theorem 5.4.4.** Let \((X, d)\) be a metric space,
A, B : X \to \text{CL}(X) \text{ and } f, g : X \to X \text{ such that } AX \subseteq gX, \\
BX \subseteq fX \text{ and }

(1) \quad H(Ax, By) \leq \emptyset (d(fx, gy), d(fx, Ax), d(gy, By), d(fx, By), \\
d(gy, Ax))

holds for all \(x, y \in X\) and for \(t > 0, q \in (0, 1), \)

\(\emptyset (t, t, t, at, bt) < qt, a > 0, b \geq 0, a+b \leq 2 \) and

\(\max (\emptyset (t, t, t, 0, 2t), \emptyset (t, 0, 0, t)) \leq qt, \)

(2) \quad there exists a point \(x_0 \in X\) such that the pair \((A, B)\)

is asymptotically regular at \(x_0, \)

(3) \quad fX \text{ and } gX \text{ are } (A, B ; f, g, x_0)\text{-orbitally complete, then there }

exist \(u, z_1 \text{ and } z_2 \in X\) with \(u = f(z_1) \in Az_1, u = g(z_2) \in Bz_2. \)

Further, if \(u\) is a fixed point of \(f\) and \(f, A\) commute

weakly at \(z_1\) then \(u = fu \in Au\) and if \(u\) is a fixed point

of \(g\) and \(g, B\) commute weakly at \(z_2\) then \(u = gu \in Bu. \)

Proof. Assume \(h = 1/\sqrt{q}. \) Let \(x_0 \in X\) satisfying condition (2)

and \(y_1\) an arbitrary point in \(AX_0. \) Choose \(x_1\) such that \(y_1 = gx_1. \)

Such a choice is possible, since \(AX \subseteq gX. \) By Lemma 5.2.4, we

can find \(y_2 \in BX_1\) such that \(d(y_1, y_2) \leq h H(AX_0, BX_1). \) Set

\(y_2 = fx_2. \) This can also be done as \(BX \subseteq fX. \) In general,

\(y_{2n} = fx_{2n} \in BX_{2n-1}\) has been selected, choose

\(y_{2n+1} = gx_{2n+1} \in AX_{2n}\) such that

\(d(y_{2n}, y_{2n+1}) \leq h H(Ax_{2n}, Bx_{2n-1}). \)

Then having selected \(y_{2n+1}, \) choose \(y_{2n+2} = fx_{2n+2} \in Bx_{2n+1}\)
such that \( d(y_{2n+1}, y_{2n+2}) \leq h H(Bx_{2n+1}, Ax_{2n}) \) for \( n = 1, 2, \ldots \)

We denote \( t_n = d(y_n, y_{n+1}) \). Thus for \( n \geq 1 \), we have

\[
\begin{align*}
d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \
&\leq h H(Ax_{2n}, Bx_{2n-1}) \
&\leq h \phi (d(fx_{2n}, gx_{2n-1}), d(fx_{2n}, Ax_{2n}), \
&\quad d(gx_{2n-1}, Bx_{2n-1}), d(fx_{2n}, Bx_{2n-1}), \
&\quad d(gx_{2n-1}, Ax_{2n})) \
&\leq h \phi (d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \
&\quad d(y_{2n-1}, y_{2n}), 0, d(y_{2n-1}, y_{2n+1})),
\end{align*}
\]

which gives

\[
t_{2n} \leq h \phi (t_{2n-1}, t_{2n}, t_{2n}, 0, (t_{2n-1} + t_{2n})).
\]

If \( t_{2n} > t_{2n-1} \), we have

\[
t_{2n} \leq h \phi (t_{2n}, t_{2n}, t_{2n}, 0, 2t_{2n}) \leq \sqrt{q} t_{2n},
\]

which is inadmissible. Therefore, \( t_{2n} < t_{2n-1} \).

Similarly, we can show that \( t_{2n+1} < t_{2n} \). Thus \( \{t_n\} \)

is a Cauchy sequence.

Now,

\[
t_2 \leq h \phi (t_1, t_1, t_1, 0, 2t_1) \leq \sqrt{q} t_1.
\]

Inductively, it follows that

\[
t_{n+1} \leq q^{n/2} t_1.
\]
Letting $n \to \infty$, we have $\lim_{n \to \infty} t_n = 0$. This shows that $\{y_n\}$ is a Cauchy sequence and has a limit $u$ in $X$. Consequently, the subsequences $\{y_{2n}\} = \{f_{2n}\}$ and $\{y_{2n+1}\} = \{g_{2n+1}\}$ also converge to $u$ in $X$. Evidently $u \in fX \cap gX$. Thus there exist points $z_1$ and $z_2$ in $X$ such that $u = f_{z_1} = g_{z_2}$. By condition (1)

$$d(f_{z_1}A_{z_1}) \leq d(f_{z_1}, f_{2n+2}) + d(f_{2n+2}, A_{z_1})$$

$$\leq d(f_{z_1}, f_{2n+2}) + \mathcal{H}(B_{2n+1}, A_{z_1})$$

$$\leq d(f_{z_1}, f_{2n+2}) + \mathcal{O}(d(f_{z_1}, g_{2n+1}), d(f_{z_1}, A_{z_1}))$$

$$d(g_{2n+1}, B_{2n+1}), d(f_{z_1}, B_{2n+1}), d(g_{2n+1}, A_{z_1})$$

$$\leq d(f_{z_1}, f_{2n+2}) + \mathcal{O}(d(f_{z_1}, g_{2n+1}), d(f_{z_1}, A_{z_1}))$$

$$d(g_{2n+1}, f_{2n+2}), d(f_{z_1}, f_{2n+2}), d(g_{2n+1}, A_{z_1})$$

On letting $n \to \infty$, the inequality yields

$$d(f_{z_1}, A_{z_1}) \leq \mathcal{O}(0, (f_{z_1}, A_{z_1}), 0, 0, d(f_{z_1}, A_{z_1}))$$

$$\leq q \cdot d(f_{z_1}, A_{z_1})$$

a contradiction. Hence $f_{z_1} \in A_{z_1}$ as $A_{z_1}$ is closed. Similarly, we can show that $g_{z_2} \in B_{z_2}$. Thus $z_1$ is the coincidence point of $f$ and $A$ and $z_2$ is the coincidence point of $g$ and $B$.

If we assume that $u = f_{z_1}$ is a fixed point of $f$ then $u = fu = ff_{z_1} \in fA_{z_1}$. If $f$ and $A$ commute weakly at $z_1 \in X$ then $fA_{z_1} = Af_{z_1}$. Since $f_{z_1} \in A_{z_1}$, Therefore,
u = fu = fAz₁ = Afz₁ = Au, that is, u = fu ∈ Au.
Similarly u is the fixed point of g, as g and B commute
weakly at z₂ ∈ X then u = gu ∈ Bu. This completes the proof.

Our Theorem 5.4.4 improves Theorem 2.1 of Singh et al. [99] for f = g. We remark that the condition A X ⊆ g X,
B X ⊆ f X can be replaced by orbital regularity (cf. Def. 5.4.1).
For multivalued mappings A, B : X → C(X) it is well known
that for y₁ = gx₁ ∈ Ax₀ and y₂ = fx₂ ∈ Bx₁ we have
d(y₁, y₂) ≤ H(Ax₀, Bx₁). This suggest that for A, B : X → C(X)
the condition of orbital regularity can be dropped.

We further emphasize that if the assumption u is a
fixed point of f (resp. u is a fixed point of g) is dropped
then A and f (resp. B and g) need not have a fixed point as
is evident from the following example given by Singh et al.
[99] for f = g.

Example 5.4.5. Let X = [0, 1] and Ax = Bx = {0, 1}, fx = 1-x
for all x ∈ X. Since Ax = {0, 1} ⊆ fX = X, H(Ax, Ay) = 0 for
all x, y ∈ X. fAx = {0, 1} = Afx and f₀ = 1 ∈ A₁, f₁ = 0 ∈ A₀,
all hypothesis of Theorem 5.4.4 are satisfied for f = g except
that none of the coincidence values, viz., f₀ or f₁ is a fixed
point of f. Evidently, f and A are continuous, and the only
fixed point of f is 1/2 which is not the fixed point of A.

5.5. COMMON FIXED POINT THEOREMS FOR NON-SELF HYBRID CONTRACTIONS

This section is devoted to the study of some common
fixed point theorems for non-self hybrid contractions in
metrically convex metric spaces which extends the earlier known result of Assad-Kirk [4], Hadzic-Gajic [38], Itoh [49], Khan [62] and many others. Some related results have also been obtained.

Following Assad-Kirk [4] we recall:

**Definition 3.5.1.** A metric space $(X,d)$ is said to be metrically convex if for every $x,y$ in $X$ (with $x \neq y$) there exists $z$ in $X$ ($x \neq y \neq z$) such that

$$d(x,z) + d(z,y) \leq d(x,y).$$

Further, if $K$ is nonempty closed subset of $X$ and $x \in K$ and $y \notin K$ then there exists a point $z \in \partial K$ (the boundary of $K$) such that $d(x,z) + d(z,y) = d(x,y)$

The following lemmas are borrowed from Rus [90] and Khan [62].

**Lemma 3.3.2.** Let $A$ be in $CB(X)$ and $0 < \theta < 1$, then for any $x \in A$ there exists $a \in A$ such that $d(x,a) \geq \theta \delta(x,A)$ and $d(x,a) \geq \theta H(x,A)$.

**Lemma 3.3.3.** For any $A,B$ in $CB(X)$ and $x \in X$,

$$|d(x,A) - d(x,B)| \leq H(A,B).$$

**Lemma 3.3.4.** For any $x,y$ in $X$ and $A \subseteq X$,

$$|d(x,A) - d(y,A)| \leq d(x,y).$$

In an attempt to extend the concepts of weak commutativity
of Sessa [95] and compatibility of Jungck [53] for non-self multivalued mappings, Hadzic [37] and Hadzic-Gajic [38] introduced the following.

**Definition 5.5.5.** Let $K$ be a non-empty subset of a metric space $(X, d)$, $F : K \to \text{CB}(X)$ and $T : K \to X$. Then the pair \{$F, T$\} is said to be weakly commuting if for every $x, y$ in $K$ such that $x \in Fy$ and $Ty \in K$, $d(Tx, FTy) \leq d(Ty, Fy)$.

**Definition 5.5.6.** Let $K$ be a nonempty subset of a metric space $(X, d)$, $F : K \to \text{CB}(X)$ and $T : K \to X$. Then the pair \{$F, T$\} is said to be compatible if for every sequence \{$_n^x$\} from $K$ and from the relation $\lim_{n \to \infty} d(F^n_x, Tx_n) = 0$ and $Tx_n \in K$ it follows that $\lim_{n \to \infty} d(Ty_n, FTy_n) = 0$, for every sequence \{$_n^y$\} from $K$ such that $y_n \in Fx_n$.

For $K = X$ and $F$ single valued the definitions 5.5.5 and 5.5.6 reduce to those of Sessa [95] and Jungck [53] respectively (see Def. 1.5.1 and 1.5.3).

Following Khan [62] we introduce the following:

**Definition 5.5.7.** Let $K$ be a non-empty closed subset of a metric space $(X, d)$, $F : K \to \text{CB}(X)$ and $T : K \to X$. Then $F$ is said to be generalized $T$-contraction of $K$ into $\text{CB}(X)$ if there exists non-negative reals $\alpha, \beta, \gamma$ with $\alpha + 2\beta + 2\gamma < 1$ such
that for all \(x, y \in K\),

\[
H(Fx, Fy) \leq \alpha d(Tx, Ty) + \beta (d(Tx, Fx) + d(Ty, Fy)) + \gamma (d(Tx, Fy) + d(Ty, Fx)).
\]

**Definition 3.3.8.** If \(K, F\) and \(T\) have the same meaning as in Definition 3.5.7. Then \(F\) is generalized \(T\)-contractive mapping of \(K\) into \(\text{CB}(X)\) if there exists non-negative reals \(\alpha, \beta, \gamma\) with \(0 < 2\alpha + 2\beta + 4\gamma \leq 1\) such that for any \(x, y\) in \(X\) with \(x \neq y\), the inequality (\(\star\)) holds.

**Remark.** If \(F\) is single valued, then we simply say that \(F\) is generalized \(T\)-contraction (\(T\)-contractive) mapping of \(K\) into \(X\).

Now, we prove the following.

**Theorem 3.3.9.** Let \((X, d)\) be a complete metrically convex metric space and \(K\) a non-empty closed subset of \(X\). If \(F\) is a generalized \(T\)-contraction mapping of \(K\) into \(\text{CB}(X)\) satisfying

\[
(\alpha + \beta + \gamma) \frac{(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1
\]

and

(I) \(\partial K \subseteq TK, FK \subseteq TK\);

\(Tx \in \partial K \Rightarrow Fx \subseteq K\),

(II) \([F, T]\) is a weakly commuting pair,

(III) \(T\) is continuous on \(K\),

then there exists a point \(z\) in \(K\) such that \(z = Tz \in Fz\).

**Proof.** If \(\Theta = (\alpha + \beta + \gamma) \frac{(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} = 0\), then the theorem holds trivially. Thus without loss of generality, we assume...
Let $x \in \partial K$, then there exists a point $x_0 \in K$ such that $x = Tx_0$ as $\partial K \subseteq TK$. From $Tx_0 \in \partial K$ and the implication $Tx \in \partial K \implies Fx \subseteq K$, we conclude that $Fx_0 \in K \cap FK \subseteq TK$. Let $x_1 \in K$ be such that $y_1 = Tx_1 \in Fx_0 \subseteq K$. Since $y_1 \in Fx_0$ there exists a point $y_2 \in Fx_1$ such that

$$d(y_1, y_2) \leq H(Fx_0, Fx_1) + (\frac{1-\beta-\gamma}{1+\beta+\gamma}) \theta.$$

Suppose $y_2 \in K$, then $y_2 \in K \cap FK$ which implies that there exists a $x_2 \in K$ such that $y_2 \in Tx_2$. Suppose $y_2 \notin K$ then there exists a point $q \in K$ such that

$$d(Tx_1, q) + d(q, y_2) = d(Tx_1, y_2).$$

Since $q \in \partial K \subseteq TK$, there exists a point $x_2 \in K$ such that $q = Tx_2$ and so

$$d(Tx_1, Tx_2) + d(Tx_2, y_2) = d(Tx_1, y_2).$$

Let $y_3 \in Fx_2$ such that

$$d(y_2, y_3) \leq H(Fx_1, Fx_2) + (\frac{1-\beta-\gamma}{1+\beta+\gamma}) \theta^2.$$

Thus repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$(1) \quad y_{n+1} = Fx_n,$$
\[(II')\quad y_n \in K \implies y_n = T_{x_n} \text{ or} y_n \notin K \implies T_{x_n} \in \partial K \quad \text{and} \quad d(T_{x_{n-1}}, T_{x_n}) + d(T_{x_n}, y_n) = d(T_{x_{n-1}}, y_n).
\]

\[(III')\quad d(y_n, y_{n+1}) \leq H(F_{x_{n-1}}, F_{x_n}) + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n.
\]

We denote
\[P = \{T_{x_1} \in \{T_{x_n}\} : T_{x_1} = y_1\},\]
\[Q = \{T_{x_1} \in \{T_{x_n}\} : T_{x_1} \neq y_1\}.
\]

Obviously two consecutive terms of \[\{T_{x_n}\}\] can not lie in \[Q\].
We consider the following three cases:

\textbf{Case 1.} If \((T_{x_n}, T_{x_{n+1}}) \in P \times P\) then

\[d(T_{x_n}, T_{x_{n+1}}) = d(y_n, y_{n+1}) \leq H(F_{x_{n-1}}, F_{x_n}) + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n\]

\[\leq \alpha d(T_{x_{n-1}}, T_{x_n}) + \beta (d(T_{x_{n-1}}, F_{x_n}) + d(T_{x_n}, F_{x_{n-1}})) + \gamma (d(T_{x_{n-1}}, F_{x_{n-1}}) + d(T_{x_n}, F_{x_{n-1}})) + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n\]

\[\leq \alpha d(T_{x_{n-1}}, T_{x_n}) + \beta (d(T_{x_{n-1}}, T_{x_n}) + d(T_{x_n}, T_{x_{n+1}})) + \gamma (d(T_{x_{n-1}}, T_{x_{n+1}}) + d(T_{x_n}, T_{x_n})) + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n\]
which, on using the triangle inequality, gives

\[ d(T_{x_n}, T_{x_{n+1}}) \leq \left( \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \right) d(T_{x_{n-1}}, T_{x_n}) + \frac{\theta^n}{1 + \beta + \gamma}. \]

**Case-2.** If \((T_{x_n}, T_{x_{n+1}}) \in P \times Q\), then by \((II')\), we obtain

\[ d(T_{x_n}, T_{x_{n+1}}) \leq d(T_{x_n}, y_{n+1}) = d(y_n, y_{n+1}) \]

and like case-1, we have

\[ d(T_{x_n}, T_{x_{n+1}}) \leq \left( \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \right) d(T_{x_{n-1}}, T_{x_n}) + \frac{\theta^n}{1 + \beta + \gamma}. \]

**Case-3.** If \((T_{x_n}, T_{x_{n+1}}) \in Q \times P\), then

\[ T_{x_{n-1}} = y_{n-1}. \]

Hence

\[ d(T_{x_n}, T_{x_{n+1}}) \leq d(T_{x_n}, y_n) + d(y_n, y_{n+1}) \]

\[ \leq d(T_{x_n}, y_n) + H(F_{x_{n-1}}, F_{x_n}) + \left( \frac{1 - \beta - \gamma}{1 + \beta + \gamma} \right) \theta^n \]

\[ \leq d(T_{x_n}, y_n) + \alpha d(T_{x_{n-1}}, T_{x_n}) + \beta d(T_{x_{n-1}}, F_{x_{n-1}}) + d(T_{x_n}, F_{x_n}) + \gamma (d(T_{x_{n-1}}, F_{x_n}) + d(T_{x_n}, F_{x_{n-1}})) \]

\[ + \left( \frac{1 - \beta - \gamma}{1 + \beta + \gamma} \right) \theta^n \]

\[ \leq d(T_{x_n}, y_n) + \alpha d(T_{x_{n-1}}, T_{x_n}) + \beta d(T_{x_{n-1}}, y_n) + d(T_{x_n}, T_{x_{n+1}}) + \gamma (d(T_{x_{n-1}}, T_{x_{n+1}}) + d(T_{x_n}, y_n)) \]

\[ + \left( \frac{1 - \beta - \gamma}{1 + \beta + \gamma} \right) \theta^n \]
\[ \leq (1 + \gamma) d(Tx_n, y_n) + (\alpha + \gamma) d(Tx_{n-1}, Tx_n) \]
\[ + (\beta + \gamma) d(Tx_n, Tx_{n+1}) + \beta d(Tx_{n-1}, y_n) \]
\[ + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n. \]

Since \( d(Tx_{n-1}, Tx_n) + d(Tx_n, y_n) = d(Tx_{n-1}, y_n) \) and \( 0 < \alpha < 1 \), we can write
\[ d(Tx_n, Tx_{n+1}) \leq (1 + \gamma) d(Tx_{n-1}, y_n) + \beta d(Tx_{n-1}, y_n) \]
\[ + (\beta + \gamma) d(Tx_n, Tx_{n+1}) + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n, \]

which yields
\[ d(Tx_n, Tx_{n+1}) \leq \left( \frac{1+\beta+\gamma}{1-\beta-\gamma} \right) d(Tx_{n-1}, y_n) + \left( \frac{1-\beta-\gamma}{1+\beta+\gamma} \right) \theta^n. \]

Since \( Tx_{n-1} = y_{n-1} \), as in case-2, we obtain
\[ d(Tx_{n-1}, Tx_n) \leq \left( \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} \right) d(Tx_{n-2},Tx_{n-1}) + \frac{\theta^{n-1}}{1+\beta+\gamma}. \]

Combining the foregoing two inequalities, we obtain
\[ d(Tx_n, Tx_{n+1}) \leq \frac{(\alpha+\beta+\gamma)(1+\beta+\gamma)}{(1-\beta-\gamma)^2} d(Tx_{n-2},Tx_{n-1}) \]
\[ + \frac{\theta^{n-1}}{1-\beta-\gamma} + \frac{\theta^n}{1+\beta+\gamma}. \]

Thus, in any case, we have
Now, proceeding on the lines of Itoh [49], we can show that \( \{T_n\} \) is a Cauchy sequence and so converges to some point \( z \) in \( X \).

Thus, there exists a subsequence \( \{x_{n_k}\} \) such that each term \( T_{n_k} \) is in \( P \), i.e. \( y_{n_k} = T_{n_k} = F_{n_k} \).

For convenience, we denote \( T_{n_k} \) as \( T_n \).

Since \( T \) is continuous, the sequence \( \{TTx_n\} \) converges to \( Tz \). Thus using the weak commutativity of \( F \) and \( T \). We have

\[
T_n \in Fx_{n-1} \cap K \quad \text{and} \quad T_{n-1} \in K, \quad \text{so}
\]

\[
\begin{align*}
\theta d(Tx_n, Tx_{n-1}) &+ \frac{\theta^n}{1+\beta + \gamma}, \text{ or } \\
\theta d(Tx_{n-2}, Tx_{n-1}) &+ \frac{\theta^{n-1}}{1-\beta - \gamma} + \frac{\theta^n}{1+\beta + \gamma}.
\end{align*}
\]

On letting \( n \to \infty \), we get

\[
d(Tz, FTx_{n-1}) \to 0.
\]

Now, consider

\[
d(TTx_n, Fz) \leq d(TTx_n, FTx_{n-1}) + H(FTx_{n-1}, Fz)
\]

\[
\leq d(TTx_n, FTx_{n-1}) + \alpha d(TTx_{n-1}, Tz)
\]

\[
+ \beta (d(TTx_{n-1}, FTx_{n-1}) + d(Tz, Fz))
\]

\[
+ \gamma (d(TTx_{n-1}, Fz) + d(Tz, FTx_{n-1})),
\]

and on letting $n \to \infty$, we get
\[ d(Tz, FT_{x_n}) \to 0. \]

Now, consider
\[
\begin{align*}
d(Tx_n, Fz) &\leq d(Tx_n, FT_{x_n}) + H(FT_{x_n}, Fz) \\
&\leq d(Tx_n, FT_{x_n}) + \alpha d(Tx_n, Tz) \\
&\quad + \beta (d(Tx_{n-1}, FT_{x_{n-1}}) + d(Tz, Fz)) \\
&\quad + \gamma (d(Tx_{n-1}, Fz) + d(Tz, FT_{x_{n-1}})),
\end{align*}
\]
and on letting $n \to \infty$, we get
\[ d(Tz, Fz) \leq (\beta + \gamma) d(Tz, Fz) \]
giving thereby $Tz \in Fz$, as $Fz$ is closed.

Now, consider
\[
\begin{align*}
d(Tx_n, Tz) &\leq H(Fx_{n-1}, Fz) \\
&\leq \alpha d(Tx_{n-1}, Tz) + \beta (d(Tx_{n-1}, Fx_{n-1}) + d(Tz, Fz)) \\
&\quad + \gamma (d(Tx_{n-1}, Fz) + d(Tz, Fx_{n-1})) \\
&\leq \alpha d(Tx_{n-1}, Tz) + \beta (d(Tx_{n-1}, Tx_n) + d(Tz, Fz)) \\
&\quad + \gamma (d(Tx_{n-1}, Tz) + d(Tz, Tx_n)),
\end{align*}
\]
On letting $n \to \infty$, we get
\[ d(z, Tz) \leq (\alpha + 2\gamma) d(z, Tz). \]
This gives $z = Tz$.

Thus we have proved that $z = Tz \in Fz$. This completes the proof.
Theorem 5.5.10. Let $F$ and $T$ be the same as defined in Theorem 5.5.9. If condition II and III are replaced by

(II') \{F,T\} is a compatible pair,

(III') $F$ and $T$ are continuous on $K$,

then there exists a point $z$ in $K$ such that $Tz \in Fz$.

Proof. Proceeding as Theorem 5.5.9, we have that the sequence $\{T_n\}$ is Cauchy and therefore converges to some point $z$ in $K$.

So, as argued there, there exists a subsequence $\{T_{n_k}\}$ in $P$ i.e. $y_{n_k} = T_{n_k}$. Again, for convenience, we denote $T_{n_k}$ as $T_{n_k}$.

Now, we use the compatibility of $F,T$ to show that $Tz \in Fz$. Since $T_{n_k} \in F_{n-1} \cap K$ and $T_{n-1} \in K$, we have

$$d(F_{n-1}, T_{n-1}) \leq d(T_{n_k}, T_{n-1}) \to 0 \quad \text{as} \quad n \to \infty,$$

it follows from the compatibility of $\{F, T\}$ that

$$\lim_{n \to \infty} d(TT_{n_k}, FT_{n-1}) = 0,$$

from the inequality

$$d(TT_{n_k}, Fz) \leq d(TT_{n_k}, FT_{n-1}) + H(FT_{n-1}, Fz),$$

and since $F$ is $H$-continuous and $T$ is continuous, on letting $n \to \infty$ it follows that $d(Tz, Fz) = 0$ giving thereby $Tz \in Fz$ as $Fz$ is closed.

This completes the proof.
Related Result. Our next theorem generalizes results of Itoh [49] and Assad [3].

Theorem 5.5.1. Let \((X,d)\) be a complete metrically convex metric space and \(K\) a non-empty closed subset of \(X\). Let \(F\) be a generalized \(T\)-contractive mapping of \(K\) into \(CB(X)\). If 
\[
(a+\beta+\gamma)(1+\beta+\gamma)/(1-\beta-\gamma) < 1
\]
and \(F\) continuous and conditions (I) and (III) of Theorem 5.5.9 holds, then there exists a point \(z\) in \(K\) such that \(z = Tz \in Fz\).

Proof. As in the proof of Theorem 5.5.9 we construct the sequence \(\{x_n\}\) and \(\{y_n\}\). Let \(\{Tx_n\}\) be the subsequence whose each element is in \(P\) i.e. \(y_n = Tx_n\). For the sake of convenience, we denote \(Tx_n\) as \(Tx_k\).

Let \(f : K \to \mathbb{R}_+\) (non-negative reals) be defined as

\[f(x) = d(Tx,Fx).\]

Then using Lemma 5.5.3 and 5.5.4 and the continuity of \(T\) and \(F\), we have for \(x,y\) in \(K\),

\[
|f(x) - f(y)| \leq |d(Tx,Fx) - d(Ty,Fx)| + |d(Ty,Fx) - d(Ty,Fy)| < d(Tx,Ty) + H(Fx,Fy).
\]

Thus \(f\) is continuous on the compact subset \(K\). Let \(z \in K\) be such that \(f(z) = \inf \{f(z) : z \in K\}\). Then proceeding as in Theorem 3.4 of Khan [62], we can show that \(f(z) = 0\). Thus \(0 = f(z) = d(Tz,Fz)\) giving thereby \(Tz \in Fz\) as \(Fz\) is closed.

Again if we take the subsequence \(\{Tx_{n_k}\}\) of \(\{Tx_n\}\) such that \(\{Tx_{n_k}\} \subseteq Q\) i.e. \(Tx_{n_k} \notin K\). Then \(z \notin K\). For the sake of convenience we again denote \(Tx_{n_k}\) as \(Tx_k\).
Then by Definition 5.5.1, we have for each $n = 1, 2, \ldots$
there exists $q_n \in \partial K$ such that $d(Tx_n, q_n) + d(q_n, z) = d(Tx_n, z)$.
As $K$ is compact $F(q_n) \subseteq K$ and there exist (Lemma 5.5.2) $w_n \in F(q)$ such that
\[ d(Tx_n, w_n) \leq H(Tz, Fy_n) + \varepsilon. \]
We also assume that $\{q_n\}$ converges to some point $p$ in $\partial K$.
Then again on the lines of Theorem 3.4 of Khan [62] we can show that
\[ f(q) < \left( \frac{1+\beta+\gamma}{1-\beta-\gamma} \right) f(z) - 2\varepsilon. \]
Now choose $u \in K$ such that $Tu \in F(q_0)$ satisfying the condition $d(Tq_0, Fq_0) = d(Tq_0, Tu)$. As $F(z) > 0$ we note that $Tu \notin Tq_0$. Then following Khan [62] we obtain $f(u) < f(z)$ contradicting the minimality of $z$. Hence $f(z) = 0$ and as $f(z)$ is closed, we find that $Tz \in Fz$.

Now
\[ d(Tx_n, Fz) \leq H(Fx_{n-1}, Fz) \]
\[ \leq \alpha d(Tx_{n-1}, Tz) + \beta (d(Tx_{n-1}, Fx_{n-1}) + d(Tz, Fz)) \]
\[ + \gamma (d(Tx_{n-1}, Fz) + d(Tz, Fx_{n-1})) \]
\[ \leq \alpha d(Tx_{n-1}, Tz) + \beta (d(Tx_{n-1}, Tz_n) + d(Tz, Fz)) \]
\[ + \gamma (d(Tx_{n-1}, Tz) + d(Tz, Tz_n)). \]
On letting $n \to \infty$, we get
\[ d(z,Tz) \leq (\alpha+2\gamma) d(z,Tz) \]

which implies that \( z = Tz \).

Thus we have shown that \( z = Tz \in Fz \). This completes the proof.

**Theorem 5.5.12.** Let \( K \) be a non-empty complete subset of a metric space \((X,d)\). Let \( F : K \rightarrow CB(X) \) and \( T : K \rightarrow X \) be such that

\[
\delta(Fx,Gy) \leq \alpha d(Tx,Ty) + \beta(\delta(Tx,Fx) + \delta(Ty,Fy)) + \gamma(\delta(Tx,Gy) + \delta(Ty,Fx))
\]

where \( \alpha, \beta, \gamma \geq 0 \), \( \alpha + 2\beta + 2\gamma < 1 \) and conditions I - III of Theorem 5.5.9 holds, then \( F \) and \( T \) have a unique common fixed point.

**Proof.** Let us put \( \Theta = (\alpha+2\beta+2\gamma)^{1/2} \), then \( \Theta \) is positive. Define single valued mapping \( F_1 \) such that \( F_1x \in Fx \) for all \( x,y \in K \) and

\[ d(Tx,F_1x) \geq \Theta \delta(Tx,Fx) \] for all \( x \in K \).

Our choice of \( F_1 \) is justified by Lemma 5.5.2. Further, in view of condition (I) - (III) of Theorem 5.5.9 \( \{F_1,T\} \) is weakly commuting and the range of \( T \) contains the range of \( F_1 \). Now, we get

\[ d(F_1x,F_1y) \leq \delta(Fx,Gy) \]

\[ \leq \alpha d(Tx,Ty) + \beta(\Theta^{-1} d(Tx,F_1x) + \Theta^{-1} d(Ty,F_1y)) + \gamma(\Theta^{-1} d(Tx,F_1y) + \Theta^{-1} d(Ty,F_1x)). \]
As \( \theta^{-1}(2\beta+2\gamma) + \alpha \leq \theta^{-1}(\alpha+2\beta+2\gamma) \leq (\alpha+2\beta+2\gamma) < 1 \) and \( K \) is complete, proceeding on the lines of Theorem 1 of [20] with suitable modification and Theorem 1 of Wong [108], it follows that \( T \) and \( F \) have a unique common fixed point, say \( z \) in \( K \). Now consider,

\[
0 = d(z, F_z) \geq \theta \delta(z, F_z)
\]

which show that \( \delta(z, F_z) = 0 \) giving thereby \( z = Tz \in Fz \).

This completes the proof.

The foregoing method of Theorem 5.5.12 can also be used to prove the following.

**Theorem 5.5.13.** Let \( K \) be a non-empty complete subset of a metric space \( (X, d) \). Let \( F : K \rightarrow BN(X) \) and \( T : K \rightarrow X \) be such that

\[
\delta(Fx, Fy) \leq \alpha d(Tx, Ty) + \beta(H(Tx, Fx) + H(Ty, Gy))
\]

\[
+ \gamma(H(Tx, Fy) + H(Ty, Fx))
\]

where \( \alpha+2\beta+2\gamma < 1 \), \( \alpha, \beta, \gamma \geq 0 \) and conditions I-III of Theorem 5.5.9 hold then \( F \) and \( T \) have a unique common fixed point.