CHAPTER VI

A STUDY OF q-LAGRANGES POLYNOMIALS
OF THREE VARIABLES

ABSTRACT – The present chapter introduces a q-analogue of Lagranges polynomials of three variables due to Khan and Shukla and gives certain results involving these polynomials.

1. INTRODUCTION: Lagranges polynomials arise in certain problems in statistics. In literature they are denoted by symbol $g_n^{(a,\beta)}(x, y)$ and are defined by means of the following generating relation (see [78-79]):

$$ (1 - xt)^{-\alpha} (1 - yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(a,\beta)}(x, y) t^n $$

(1.1)

Brenke Polynomials [5] are defined as

$$ A(t) \ B(xt) = \sum_{n=0}^{\infty} P_n(x) t^n $$

(1.2)

So that

$$ P_n(x) = \sum_{k=0}^{n} a_k \ b_{n-k} \ x^k $$

(1.3)

where $a_n$ and $b_n$ are arbitrary constants and where $A(i) = \sum a_i x^i, B(x) = \sum b_i x^i$. Furthermore, with the proper choice of the parameters, they form some interesting sets of orthogonal polynomials. These were first encountered by Al-Salam and Chihara [1].
\[ P_n(x; q; a, b, c) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] (q^{a/\lambda})_k (q^{b/\mu})_{n-k} \alpha^{-k} \beta^{-n+k} \]  

(1.4)

where \( 1 - (a + 1)x + at^2 = \left(1 - \frac{t}{\alpha}\right) \left(1 - \frac{t}{\beta}\right) \) and

\[ 1 - (c + 1)x + ct^2 = \left(1 - \frac{t}{\mu}\right) \left(1 - \frac{t}{\nu}\right). \]

Next they appeared as the q-Random Walk Polynomials of Askey and Ismail [2].

\[ F_n(x; q; a, c) = \left[ \frac{1}{(q)_n} \right] \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] (q^{a/\alpha})_k (q^{c/\mu})_{n-k} \nu^{-k} \mu^{-k} \]  

(1.5)

where \( 1 - xt + ct^2 = \left(1 - \frac{t}{\alpha}\right) \left(1 - \frac{t}{\beta}\right) \) and \( 1 - at + bt^2 = \left(1 - \frac{t}{\lambda}\right) \left(1 - \frac{t}{\mu}\right) \).

In 1991, M. A. Khan and A. K. Sharma [28] considered an interesting special case of Brenke Polynomials in the form of q-analogue of Lagranges polynomials (1.1). They defined q-Lagranges polynomials by means of the following generating relation using the notations of Slater [86]:

\[ \phi_0^{q^{a/\alpha}}; x, \frac{d}{dx}; \phi_0^{q^{b/\beta}}; y, \frac{d}{dy} = \sum_{n=0}^{\infty} g_{n, q}^{(a, \beta)}(x, y) t^n \]  

(1.6)

where

\[ \phi_0^{q^{a/\alpha}}; z, \frac{d}{dz}; \phi_0^{q^{b/\beta}}; y, \frac{d}{dy} = \frac{(1 - q^{a/\alpha})_z}{(1 - z)_\alpha} = \frac{1}{(1 - z)_{\alpha, q}} \]  

(1.7)
Recently in 1998, M. A. Khan and A. K. Shuida [56] studied Lagranges Polynomials of three variables. They defined the three variable analogue of Lagrange Polynomials \( g_n^{(\alpha, \beta, \gamma)}(x, y, z) \) by means of the following generating relation:

\[
(1 - xt)^\alpha (1 - yt)^\beta (1 - zt)^\gamma = \sum_{n=0}^{\infty} g_n^{(\alpha, \beta, \gamma)}(x, y, z) t^n .
\] (1.8)

In order to study the above polynomials they introduced the following lemma:

**LEMMA:**

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} A(k, m, n) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{k=0}^{m} A(k, m - k, n - m) .
\] (1.9)

and

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{k=0}^{m} A(k, m, n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A(k, m + k, n + m) .
\] (1.10)

In this chapter we consider the q-analogue of Lagranges polynomials of three variables.

**2. q-LAGRANGE POLYNOMIALS OF THREE VARIABLES:** We define the q-analogue of Lagranges polynomials of three variables \( g_n^{(\alpha, \beta, \gamma)}(x, y, z) \) by means of the following generating relation using the notations of Gasper and Rahman [13]:
In order to study such a polynomial we need the following lemma:

**LEMMA 1:** We have

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{n-j} A(k, j; n) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{n-j} A(k, j; n-j-k) \tag{2.2}
\]

and

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{n-j} A(k, j, n) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{n-j} A(k, j, n+j+k). \tag{2.3}
\]

The above lemma is a special case of the following lemma due to Srivastava and Monacha [92]:

**LEMMA 2.** For positive integers \(m_1, \ldots, m_r \geq 1),

\[
\sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r=0}^n \theta(k_1, \ldots, k_r; n) = \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r=0}^n \theta(k_1, \ldots, k_r; n - m_1 k_1 - \ldots - m_r k_r) \tag{2.4}
\]

and

\[
\sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r=0}^n \phi(k_1, \ldots, k_r; n) = \sum_{n=0}^{\infty} \sum_{k_1, \ldots, k_r=0}^n \phi(k_1, \ldots, k_r; n + m_1 k_1 + \ldots + m_r k_r). \tag{2.5}
\]
Expanding the L. H. S. of (2.1) using Lemma 1 and finally equating the coefficient of \(t^n\) on both sides, we get

\[
g_{n,q}^{(α,β,γ)}(x, y, z) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{\left(q^{α}; q\right)_{n-j-k} \left(q^{β}; q\right)_{j} \left(q^{γ}; q\right)_{k} x^{n-j-k} y^{j} z^{k}}{(q, q)_{n-j-k} (q, q)_{j} (q, q)_{k}}. \tag{2.6}
\]

Polynomials \(g_{n,q}^{(α,β,γ)}(x, y, z)\) can be regarded as a generalization of q-Lagrange Polynomials of Khan and Sharma [28] from two to three variables as it can easily be seen that

\[
g_{n,q}^{(α,β,γ)}(x, y, 0) = g_{n,q}^{(α,β)}(x, y) \tag{2.7}
\]

Replacing \(z\) by \(yq^β\) in (2.1), we also get

\[
g_{n,q}^{(α,β,γ)}(x, y, q^{β} y) = g_{n,q}^{(α,β+γ)}(x, y). \tag{2.8}
\]

Similarly,

\[
g_{n,q}^{(α,β,γ)}(x, q^{α} x, z) = g_{n,q}^{(α+β,γ)}(x, y) \tag{2.9}
\]

\[
g_{n,q}^{(α,β,γ)}(x, y, q^{α} x) = g_{n,q}^{(αγ,β)}(x, y) \tag{2.10}
\]

\[
g_{n,q}^{(α,β,0)}(x, y, z) = g_{n,q}^{(α,β)}(x, y). \tag{2.11}
\]

Now consider

\[
\sum_{n=0}^{∞} q^{(a+α+b+β+μ+c+γ+η)}(x, y, z) x^n t^n
\]

\[
= \phi_0(q^{a+α}; -; q, xt) \phi_0(q^{b+β+μ}; -; q, yt) \phi_0(q^{c+γ+η}; -; q, zt)
\]

\[
= \phi_0(q^{a}; -; q, xt) \phi_0(q^{α}; -; q, q^{α} xt) \phi_0(q^{b}; -; q, q^{b+β} yt)
\]

\[
\phi_0(q^{b}; -; q, yt) \phi_0(q^{β}; -; q, q^{b+β} yt) \phi_0(q^{μ}; -; q, q^{b+β} yt)
\]
Using (2.2).

Now equating the coefficient of $t^n$, we get

\[
g_{n,q}^{(a+b+c)}(x,y,z) g_{j,q}^{(a,b,c)}(q^a x, q^b y, q^c z) g_{k,q}^{(l,m,n)}(q^{a+b} x, q^{b+c} y, q^{c+a} z) l^{n+j+k}
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{l} g_{n-j-k,q}^{(a,b,c)}(x,y,z) g_{j,q}^{(a,b,c)}(q^a x, q^b y, q^c z) g_{k,q}^{(l,m,n)}(q^{a+b} x, q^{b+c} y, q^{c+a} z) l^n
\]

As a particular case of (2.12) it can easily be verified that

\[
g_{n,q}^{(a,b,c)}(x,y,z) = \sum_{r=0}^{n} g_{m-r,q}^{(a,b,c)}(x,y,z) g_{r,q}^{(a,b,c)}(q^a x, q^b y, q^c z). \quad (2.13)
\]

3. MAIN RESULTS: The first formula to be proved is

\[
\sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+n}{n} g_{m+n,q}^{(a,b,c)}(x,y,z) l^m v^m
\]

\[
= \frac{(x(v+t);q)_\infty (y(v+t);q)_\infty (z(v+t);q)_\infty}{(x(v+t);q)_\infty (y(v+t);q)_\infty (z(v+t);q)_\infty} \quad (3.1)
\]
Proof: \[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+n}{n} g_{n+m,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^n v^m = \sum_{n=0}^{\infty} g_{n,q}^{(\alpha, \beta, \gamma)}(x, y, z) (v+t)^n
\]

\[
\phi_0(q^\alpha; q, x(v+t), y) \phi_0(q^\beta; q, y(v+t), z) \phi_0(q^\gamma; q, z(v+t))
\]

\[
= \frac{(x(v+t)q^\alpha; q)_\infty (y(v+t)q^\beta; q)_\infty (z(v+t)q^\gamma; q)_\infty}{(x(v+t)q; q)_\infty (y(v+t)q; q)_\infty (z(v+t)q; q)_\infty}.
\]

The second formula to be proved is

\[
\sum_{n=0}^{\infty} F_0 \left[ -n; \omega \right] g_{n,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^n
\]

\[
= \frac{(x(1-\omega)q^\alpha; q)_\infty (y(1-\omega)q^\beta; q)_\infty (z(1-\omega)q^\gamma; q)_\infty}{(x(1-\omega)q; q)_\infty (y(1-\omega)q; q)_\infty (z(1-\omega)q; q)_\infty}.
\]

The proof of (3.2) is similar to that of (3.1).

The third formula to be proved is

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+m+k}{n} g_{n+m+k,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^n v^m \omega^k
\]

\[
= \frac{(x(t+v+\omega)q^\alpha; q)_\infty (y(t+v+\omega)q^\beta; q)_\infty (z(t+v+\omega)q^\gamma; q)_\infty}{(x(t+v+\omega)q; q)_\infty (y(t+v+\omega)q; q)_\infty (z(t+v+\omega)q; q)_\infty}.
\]

The proof of (3.3) is similar to that of (3.1)

The fourth formula to be proved is
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+n}{n} g_{m+n,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^n v^m q^\frac{1}{2} q^{m(m-1)}
\]

\[= \phi^{(3)} \left[ (v+t); -; -; q^\alpha; q^\beta; q^\gamma; -; -; -; -; q, x, y, z \right] \quad (3.4)
\]

Proof:
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+n}{n} g_{m+n,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^n v^m q^\frac{1}{2} q^{m(m-1)}
\]

\[= \sum_{n=0}^{\infty} \frac{(q)_n}{(q)_m} g_{n,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^{n-m} v^m q^\frac{1}{2} q^{m(m-1)}
\]

\[= \sum_{n=0}^{\infty} \frac{(q^n)_m}{(q)_m} (-1)^m q^{mn} g_{n,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^{n-m} v^m
\]

\[= \sum_{n=0}^{\infty} g_{n,q}^{(\alpha, \beta, \gamma)}(x, y, z) t^n q^{-n} \phi_0 \left[ q^{-n}; -; q, -; \frac{vq^n}{t} \right]
\]

\[= \phi^{(3)} \left[ (v+t); -; -; q^\alpha; q^\beta; q^\gamma; -; -; -; -; q, x, y, z \right].
\]

**CONCLUDING REMARK:**

It is proposed to study q-Lagrange polynomials of n-variables in future research work.