CHAPTER V

ON TRIPLE TRANSFORMATION OF CERTAIN
HYPERGEOMETRIC FUNCTIONS

ABSTRACT – In 1965, R. P. Singh [85] gave a double transformation of certain hypergeometric functions which augments parameters in the $pF_q$ function. The present chapter deals with a study of triple transformations of certain hypergeometric functions which may be regarded as a generalization of double transformations of certain hypergeometric functions due to R. P. Singh.


$$\int_0^\infty \int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}dx \, dy = B(\alpha, \beta) \int_0^\infty \phi(z)z^{\alpha+\beta-1}dz,$$  \hspace{1cm} (1.1)

where $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$.

Using the method of term by term integration, he showed that, if $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, and if $k$ and $s$ are non-negative integers, then inside the region of convergence of resulting series, the following result holds:

$$\int_0^\infty \int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}pF_q \left[ a_1, a_2, \ldots, a_p; t x^s y^k \right] dx \, dy$$

$$= B(\alpha, \beta) \int_0^\infty \phi(z)z^{\alpha+\beta-1}pF_{q+s+k} \left[ a_1, a_2, \ldots, a_p, \Delta(s, \alpha), \Delta(k, \beta); b_1, b_2, \ldots, b_q, \Delta(s+k, \alpha+\beta) \right] dz$$

$$\hspace{1cm} (1.2)$$
where \( \delta = \frac{k^k s^s}{(k+s)^{k+s}} \), \( \Re \alpha > 0, \Re \beta > 0 \), \( k \) and \( s \) are positive integers.

In particular, he let \( \phi(z) = e^{-z} z^\mu \) and evaluated the above integral and obtained the following:

\[
\int_0^\infty \int_0^\infty e^{-(x+y)} (x+y)^\mu x^{a-1} y^{\beta-1} \binom{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \left[ x^s, y^k \right] \mathrm{d}x \mathrm{d}y
\]

\[
= B(\alpha, \beta) \Gamma(\alpha + \beta + \mu)
\]

\[
\times \binom{a_1, a_2, \ldots, a_p, \Delta(s, \alpha), \Delta(k, \beta), \Delta(s + k, \alpha + \beta + \mu)}{b_1, b_2, \ldots, b_q, \Delta(s + k, \alpha + \beta + \mu)} \quad (1.3)
\]

where \( \Re(\alpha + \beta + \mu) > 0 \).

For brevity he used the operator notation

\[
\Omega_{(\alpha, \beta, \mu)} = \left[ B(\alpha, \beta) \Gamma(\alpha + \beta + \mu) \right]^{-1} \int_0^\infty \int_0^\infty e^{-(x+y)} (x+y)^\mu x^{a-1} y^{\beta-1} \mathrm{d}x \mathrm{d}y \quad (1.4)
\]

in which \( \Re \alpha > 0, \Re \beta > 0 \) and \( \Re(\alpha + \beta + \mu) > 0 \).

The present chapter deals with a study of generalization of (1.4) to triple transformation of certain hypergeometric functions.

Frequently occurring definitions, notations and results used in this paper are as given below:

The gamma function is defined as
\[ \Gamma(z) = \begin{cases} \int_{0}^{\infty} t^{z-1} e^{-t} \, dt, \quad R\ell(z) > 0 \\ \Gamma(z+1), \quad R\ell(z) < 0, \quad z \neq 0, -1, -2, -3, \end{cases} \] (1.5)

The Beta function \( B(\alpha, \beta) \) is a function of two complex variables \( \alpha \) and \( \beta \), defined by

\[ B(\alpha, \beta) = \begin{cases} \int_{0}^{1} t^{\alpha-1} (1-t)^{\beta-1} \, dt, \quad R\ell(\alpha) > 0, R\ell(\beta) > 0 \\ \Gamma(z+1), \quad R\ell(\alpha) < 0, R\ell(\beta) < 0, \quad \alpha, \beta \neq -1, -2, -3, \end{cases} \] (1.6)

The generalized hypergeometric function is defined as

\[ _pF_q \left[ \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \] (1.7)

where the series in (1.7)

(i) Converges for \( |z| < \infty \) if \( p \leq q \)

(ii) Converges for \( |z| < 1 \) if \( p = q+1 \) and

(iii) Diverges for all \( z, z \neq 0 \), if \( p > q+1 \).

Further a symbol of the type \( \Delta \ (k, \alpha) \) stands for the set of \( k \) parameters

\[ \frac{\alpha}{k}, \frac{\alpha+1}{k}, \ldots, \frac{\alpha+k-1}{k}. \]

Thus

\[ _{p+k}F_{q+k} \left[ \begin{array}{c} (a_p)_p \Delta(k, \alpha); \\ (b_q)_q \Delta(r, \beta); \end{array} z \right] \]
\[ \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n \left( \frac{\alpha}{k} \right)_n \left( \frac{\alpha+k}{k} \right)_n \ldots \left( \frac{\alpha+k-1}{k} \right)_n}{(b_1)_n(b_2)_n \ldots (b_q)_n \left( \frac{\beta}{r} \right)_n \left( \frac{\beta+1}{r} \right)_n \ldots \left( \frac{\beta+r-1}{r} \right)_n} \frac{z^n}{n!} \]

\[ \text{(1.8)} \]

2. SOME GENERAL FORMULAE: Using the transformations \( x + y + z = u \), \( y + z = uv \) and \( z = uvw \), it can easily be verified that

\[ \int \int \int \phi(x + y + z) x^{\alpha-1} y^{\beta-1} z^{\gamma-1} \, dx \, dy \, dz = \frac{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma)}{\Gamma(\alpha + \beta + \gamma)} \int_0^\infty \phi(u) u^{\alpha + \beta + \gamma - 1} \, du \]

\[ (2.1) \]

where \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \).

Using the method of term by term integration it can be shown that, if \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \), and if \( k, r \) and \( s \) are non-negative integers, then inside the region of convergence of the resulting series, the following result holds

\[ \int \int \int \phi(x + y + z) x^{\alpha-1} y^{\beta-1} z^{\gamma-1} \, dx \, dy \, dz \]

\[ \times \int \, \phi(u) u^{\alpha + \beta + \gamma - 1} \, du \]

\[ \times \sum_{p_1, \ldots, p_r, k} F_{q+r+k} \left[ \left( \frac{a_p}{b_q}, \Delta(s, \alpha), \Delta(r, \beta), \Delta(k, \gamma); s + r + k \right) \right] \]

\[ \text{(2.2)} \]

where \( (\alpha_p) \) denotes a sequence of parameters \( a_1, a_2, \ldots, a_p \).

In particular, by taking \( \phi(u) = e^{-u} u^a \), the above integral on the right can be evaluated to obtain
\[
\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x+y+z)} (x+y+z)^\mu x^{a-1} y^{\beta-1} z^{\gamma-1} F_q \left[ \begin{array}{c} a_p; \\ b_q; \\ t x^\lambda y^\rho z^k \end{array} \right] dx dy dz
\]

\[
= \frac{\Gamma(\alpha \Gamma(\beta \Gamma(\gamma)) \Gamma(\mu + \alpha + \beta + \gamma)}{\Gamma(\alpha + \beta + \gamma)}
\]

\[
\times \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha \Gamma(\beta \Gamma(\gamma)) \Gamma(\mu + \alpha + \beta + \gamma)}
\]

\[
\times \int_0^\infty \int_0^\infty \int_0^\infty e^{-(x+y+z)} (x+y+z)^\mu x^{a-1} y^{\beta-1} z^{\gamma-1} \{dx dy dz\}
\]

(2.3)

where \( R(\alpha + \beta + \gamma + \mu) > 0 \).

For brevity, the following operator notation will be used:

\[
\Omega_{(\alpha, \beta, \gamma, \mu)} \{ \}
\]

\[
= \frac{\Gamma(\alpha \Gamma(\beta \Gamma(\gamma)) \Gamma(\mu + \alpha + \beta + \gamma)}{\Gamma(\alpha \Gamma(\beta \Gamma(\gamma)) \Gamma(\mu + \alpha + \beta + \gamma)}
\]

(2.4)

in which \( R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0 \) and \( R(\alpha + \beta + \gamma + \mu) > 0 \).

The following results are immediate consequences of the relation (2.4)

\[
\Omega_{(\alpha, \beta, \gamma, \mu)} \{1\} = 1
\]

(2.5)

\[
\Omega_{(\alpha, \beta, \gamma, \mu)} \{x^\lambda y^\rho z^k (x+y+z)^\eta \}
\]

\[
= \frac{\Gamma(\alpha + \beta + \gamma) \Gamma(\xi + \alpha) \Gamma(\lambda + \beta) \Gamma(\nu + \gamma) \Gamma(\xi + \alpha + \lambda + \beta + \nu + \gamma + \mu + \eta)}{\Gamma(\alpha \Gamma(\beta \Gamma(\gamma)) \Gamma(\mu + \alpha + \beta + \gamma) \Gamma(\xi + \alpha + \lambda + \beta + \nu + \gamma)}
\]

(2.6)

\[
\Omega_{(\alpha, \beta, \gamma, \mu)} \{e^{\alpha x + \beta y + \gamma z} \}
\]

(2.7)
In particular, if $\lambda = -n$ and $\mu = 0$, (2.10), (2.11) and (2.12) reduce to

\begin{align}
\Omega_{(\alpha, \beta, \gamma, 0)} \left\{ (1 - xt)^{-\lambda} \right\} &= F_0 \left[ -n, \alpha; -; t \right] \\
\Omega_{(\alpha, \beta, \gamma, 0)} \left\{ (1 - xy)^{-\lambda} \right\} &= F_0 \left[ -n, \alpha, \beta; -; t \right] \\
\Omega_{(\alpha, \beta, \gamma, 0)} \left\{ (1 - xyt)^{-\lambda} \right\} &= F_0 \left[ -n, \alpha, \beta, \gamma; -; t \right]
\end{align}

Further, we have

\begin{align}
\Omega_{(\alpha, \beta, \gamma, 0)} \left\{ \begin{array}{c}
\alpha, \mu, \alpha + \beta + \gamma; \\
a, \alpha + \beta + \gamma
\end{array} \right\} &= F_2 \left[ \begin{array}{c}
\alpha, \mu + \alpha + \beta + \gamma; \\
a, \alpha + \beta + \gamma
\end{array}; t \right] \\
\Omega_{(\alpha, \beta, \gamma, 0)} \left\{ \begin{array}{c}
\alpha, \beta, \Delta(2, \mu + \alpha + \beta + \gamma); \\
a, \Delta(2, \alpha + \beta + \gamma)
\end{array} \right\} &= F_3 \left[ \begin{array}{c}
\alpha, \beta, \Delta(2, \mu + \alpha + \beta + \gamma); \\
a, \Delta(2, \alpha + \beta + \gamma)
\end{array}; t \right]
\end{align}
Proof of (2.5) to (2.21): Results (2.5) and (2.6) follow from (1.5), (2.1) and (2.4). Results (2.6) to (2.21) follow from the method of term by term integration and making use (1.5), (1.6), (1.7), (1.8), (2.1) and (2.4).

3. SOME CLASSICAL AND NEW POLYNOMIALS: From (2.13), we have

\[ \Omega_{(a+1, \beta, \gamma, 0)} \left( \left(1 + \frac{1}{2} xt \right)^n \right) = y_n(t) \]  

\[ \Omega_{(a-n+1, \beta, \gamma, 0)} \left( \left(1 + \frac{xt}{b} \right)^n \right) = y_n(a, b, t) \]

\[ \Omega_{(\alpha+1, \beta+1, \gamma, 0)} \left( \left(1 + \frac{1}{2} xt \right)^n \right) = y_n^{(\alpha, \beta)}(t) \]

where \( y_n(t) \) and \( y_n(a, b, t) \) are respectively Bessel’s simple and generalized polynomials [84] and \( y_n^{(\alpha, \beta)}(t) \) is a special case of Bessel
polynomial studied by M. A. Khan and K. Ahmad [55] and is as given below:

\[ y_{n}^{(\alpha, \beta)}(x) = \binom{-n, \alpha n + \beta + 1}{-x; \frac{r}{2}}. \]

Writing the series for \( _3F_0 \) function in (2.14) in reverse order, we obtain

\[ \Omega_{(\alpha, \beta, r, 0)}(l - x y) = (\alpha)_{n}(\beta)(-1)^n \binom{-n}{-n; 1} \binom{-n}{-n; 1}. \]  

(3.4)

Also Bateman polynomials [8, p. 193] are defined by

\[ z^{-\mu} j_n^{\mu, s}(z) = \frac{\Gamma(1 + n + \nu + \mu)}{n! \Gamma(\mu + 1) \Gamma(1 + \nu + \mu)} \binom{-n}{1, 1 + \nu; \frac{z^2}{2}}. \]  

(3.5)

More particularly

\[ j_n^{0, s}(z) = \frac{(1 + \alpha)_n}{n!} \binom{-n; 1, 1 + \alpha; \frac{z^2}{2}}. \]  

(3.6)

From (3.4) and (3.5), we have

\[ \Omega_{(\alpha, \beta, r, 0)}(l - x y) = n!(-1)^n \binom{(n - s)\alpha}{\frac{n - s}{2}} \Gamma(1 - \alpha)_{n,-\frac{s}{2}} \frac{1}{\sqrt{t}}. \]  

(3.7)

Further, since we have

\[ \Omega_{0, \alpha, \beta, 0} \left\{ \binom{-n; xt}{1, 1 + \alpha; t} \right\} = \binom{-n; t}{1 + \alpha}. \]

therefore from the above relation, we obtain

\[ \Omega_{0, \alpha, \beta, 0} \left\{ j_n^{0, s}(\sqrt{xt}) \right\} = t_n^{(s)}(t) \]  

(3.8)
writing the series for \( _4F_0 \) function in (2.15) is reverse order, we obtain

\[
\Omega_{(\alpha,\beta,\gamma,0)} \left[ (1 - x y z t)^n \right] = (\alpha)_n (\beta)_n (\gamma)_n (-1)^n _1F_3 \left[ \begin{array}{c} -n \\ 1 - \alpha - n, 1 - \beta - n, 1 - \gamma - n, \frac{1}{t} \end{array} \right]
\]

(3.9)

where the polynomial \( _1F_3 \) on the right of (3.9) is believed to be a new polynomial. Since we have

\[
\left[ \begin{array}{c} -n \\ \lambda, 1 - \alpha - n, 1 - \beta - n, \frac{1}{x t} \end{array} \right] = _1F_2 \left[ \begin{array}{c} -n \\ 1 - \alpha - n, 1 - \beta - n, \frac{1}{t} \end{array} \right]
\]

therefore from the above relation, we obtain

\[
\Omega_{(\lambda,\alpha,\beta,0)} \left[ _1F_3 \left[ \begin{array}{c} -n \\ \lambda, 1 - \alpha - n, 1 - \beta - n, \frac{1}{x t} \end{array} \right] \right] = _1F_2 \left[ \begin{array}{c} -n \\ 1 - \alpha - n, 1 - \beta - n, \frac{1}{t} \end{array} \right]
\]

\[
= \frac{n! \Gamma(1 - \alpha) \frac{(\alpha + n)}{2} \beta^{\alpha - n} \frac{\alpha - n}{2} \frac{1}{\sqrt{t}}}{(\alpha)_n (\beta)_n}
\]

(3.10)

From (2.19), we have

\[
\Omega_{(\xi,\beta,1 - \beta,0)} \left[ f^{(p-1)}(x) \right] = \frac{n!}{(p)_n} H_n (\xi, p, v)
\]

(3.11)

where \( 0 < \xi + \beta < 1 \) and \( H_n (\xi, p, v) \) denotes the polynomial due to Rice defined by

\[
H_n (\xi, p, v) = _1F_2 \left[ \begin{array}{c} -n, n + 1, \xi, 1 , p, v \end{array} \right]
\]

and \( L_n^{(\alpha)}(x) \) denotes Laguerre polynomials defined as

\[
L_n^{(\alpha)}(x) = \frac{(1 + \alpha)_n}{n!} _1F_1 \left[ \begin{array}{c} -n \\ 1 + \alpha, x \end{array} \right]
\].
Similarly, for $0 < \xi < 1$, we have

$$\Omega_{(\alpha, 1-\alpha, \beta, 1)}\left\{L_n^{(\alpha)}(x)\right\} = \frac{(p)_n}{(1 + \alpha)_n} H_n^{(\alpha, \beta)}(\xi, p, \nu)$$

(3.12)

where $H_n^{(\alpha, \beta)}(\xi, p, \nu)$ denotes Khandekar's generalization of Rice's polynomials and is defined by

$$H_n^{(\alpha, \beta)}(\xi, p, \nu) = \frac{(1 + \alpha)_n}{n!} \binom{-n, n + \alpha + \beta + 1, \xi;}{1 + \alpha, \nu}. \quad (3.13)$$

$$\Omega_{(\alpha, 1-\alpha, \beta, \nu, 0)}\left\{L_n^{(\alpha-1)}(x)\right\} = \frac{\alpha n}{n!} P_n(t),$$

(3.14)

where $0 < \alpha + \beta < 1$, $P_n(t)$ denotes Legendre's polynomials and $L_n(x)$ denotes simple Laguerre polynomials.

$$\Omega_{(2\nu + n, \beta, \gamma, 0)}\left\{L_n^{(\nu+1/2)}(x)\right\} = \frac{(\nu + 1/2)_n}{(2\nu)_n} C_n^\nu(t),$$

(3.15)

where $C_n^\nu(t)$ denotes Gegenbauer polynomials.

$$\Omega_{(\nu + 2\alpha + 1, \beta, \gamma, 0)}\left\{L_n^{(\alpha)}(x)\right\} = P_n^{(\alpha, \nu)}(t),$$

(3.16)

where $P_n^{(\alpha, \nu)}(t)$ denotes Ultraspherical polynomials.

$$\Omega_{(\nu + \alpha + \beta + 1, \gamma, 0)}\left\{L_n^{(\alpha)}(x)\right\} = P_n^{(\alpha, \beta)}(t),$$

(3.17)

where $P_n^{(\alpha, \beta)}(t)$ denotes Jacobi polynomials.
Also,

\[ \Omega_{(1+\alpha-\beta-\gamma,\beta,\gamma,\mu+\beta)} \left\{ L_n^{(\alpha-\beta-\gamma)} \left( \frac{1}{2} (1-t) \right) x \right\} = \frac{(1+\alpha-\beta-\gamma)n}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(t) \]  

(3.18)

where \( \Re (1+\alpha-\beta-\gamma) > 0 \).

Again from (2.19), we have

\[ \Omega_{(\alpha,1-\alpha-\beta-\mu+\nu)} \left\{ \frac{1}{\alpha} \binom{-\nu}{\frac{1}{2}} (1-t)^{\frac{1}{2}} \right\} = \frac{1}{2} \binom{-\nu}{\frac{1}{2}} \binom{1+\nu}{1-\mu} \frac{1}{2} \binom{-\nu, 1+\nu; 1-\mu}{1-\mu; 2}, \]

where \( \Re (1-\alpha-\beta-\mu) > 0 \). Since [8, p. 143] associated Legendre’s functions

\[ P_\nu^\mu (x) = \frac{1}{\Gamma(1-\mu)} \left( \frac{1+x}{1-x} \right)^{-\nu/2} \binom{-\nu}{\frac{1}{2}} \binom{-\nu, 1+\nu; 1-\mu}{1-\mu; 2}, \]

and Laguerre functions [8, p. 268]

\[ L_\nu^{(\alpha)}(x) = \frac{1}{\Gamma(1+\nu)} \phi(-\nu;1+\alpha;x), \]

therefore, we have

\[ \Omega_{(\alpha,\beta,1-\alpha-\beta-\mu,\nu+\mu)} \left\{ L_\nu^{(\alpha-\beta-\gamma)} \left( \frac{1}{2} (1-t) \right) x \right\} = \frac{\Gamma(1-\mu)/(1+\nu)}{\Gamma(1+\nu)/(1+\nu)} \frac{1}{\sqrt{2}} P_\nu^\mu (t) \]  

(3.19)

where \( \Re (1-\alpha-\beta-\mu) > 0 \).

Again since [8, p. 148]

\[ P_\nu^m (x) = \frac{(-2)^{-m} \Gamma(v+m+1)}{m! \Gamma(v-m+1)} (1-x^2)^{v/2} \binom{1+m+v, m-v; 1-x}{1+m; 2}, \]

therefore, using (2.7), we obtain
\[ \Omega_{(m-v, \beta, 1+v - \beta, v)} \left\{ \frac{x(t \mu)}{t^{(1-t)}} \right\} = \frac{(-2)^m \Gamma(v-m+1)m!}{\Gamma(v+m+1)} \left( 1 - t^2 \right)^{v/2} P_v^n(t) \]  

(3.20)

where \( 0 < R \mu (m - v) < 1 \) and \( R \mu (1 + v - \beta) > 0 \).

Further, since [8, p. 128 (24)]

\[ P_{v}^\mu (z) = \frac{2^\mu z^{\mu} (z^2 - 1)^{\mu/2}}{\Gamma(1 - \mu)} \frac{_{2}F_{1}}{_{1}} \left[ \begin{array}{c} {\nu \mu} \\ {2} \mu \mu - 2 \mu \mu - 1 \frac{1 - 1}{z^2} \end{array} \right] \]

where \( R \mu > 0 \). Taking \( \mu = 0 \) and replacing \( \left( 1 - \frac{1}{z^2} \right) \) by \( \left( - \frac{x}{t^2} \right) \), we have

\[ P_n \left( \frac{t}{\sqrt{x + t^2}} \right) = \left( \frac{t}{\sqrt{x + t^2}} \right)^n \frac{_{2}F_{1}}{_{1}} \left[ \begin{array}{c} {n - n - 1} \\ {2} \frac{1}{2} \frac{1}{2} \frac{1}{t^2} \end{array} \right] \]

Now we can easily obtain the relation

\[ \Omega_{(1, \beta, n, 0)} \left\{ (x + t^2)^n P_n \left( \frac{t}{\sqrt{x + t^2}} \right) \right\} = \frac{H_n(t)}{2^n} \]

(3.21)

where \( H_n(t) \) denotes Hermite polynomials defined as

\[ H_n(x) = (2x)^n \frac{_{2}F_{1}}{_{0}} \left[ \begin{array}{c} {n - n - 1} \\ {2} \frac{1}{2} \frac{1}{x^2} \end{array} \right] . \]

Again, from (2.19), we have

\[ \Omega_{(z, \beta, 1-\beta - z, n)} \{ L_n(x) \} = F_n (2z - 1), \]

(3.22)

where \( R \mu (z) > 0 \), \( R \mu (1-\beta - z) > 0 \) and \( F_n (z) \) is another Bateman’s polynomial defined by
\[ F_n(z) = _3F_2 \left[ \begin{array}{c} -n, n+1, \frac{1}{2}(1+z) \\ l, 1 \end{array} ; 1 \right]. \]

\[ \Omega_{(z,\alpha,1-z,n)} \begin{Bmatrix} n! \\ (p-1)_n \end{Bmatrix} L^{(p-1)}_n(x) = F^{(\alpha,\alpha)}_n(p, 2z - 1), \quad (3.23) \]

where \( 0 < \Re(z) < 1 \) and \( F^{(\alpha,\alpha)}_n(p, z) \) is an ultraspherical type generalization of Bateman’s polynomial \( F_n(z) \), first considered by M. A. Khan and A. K. Shukla [60] and is defined as

\[ F^{(\alpha,\alpha)}_n(p, z) = _3F_2 \left[ \begin{array}{c} -n, n+2\alpha + 1, \frac{1}{2}(1+z) \\ l + \alpha, p \end{array} ; 1 \right], \]

which for \( \alpha = 0 \) and \( p = 1 \) reduces to Bateman’s polynomial \( F_n(z) \)

\[ \Omega_{(z+m,\beta,1-\beta-z-m,n)} \begin{Bmatrix} n! \\ (m)_n \end{Bmatrix} L^{(m)}_n(x) = F^{m}_n(2z - 1), \quad (3.24) \]

where \( \Re(z+m) > 0, \Re(1-\beta-z-m) > 0 \) and \( F^{m}_n(z) \) denotes Pasternak’s polynomial defined by

\[ F^{m}_n(z) = _3F_2 \left[ \begin{array}{c} -n, n+1, \frac{1}{2}(1+z+m) \\ l, m+1 \end{array} ; 1 \right]. \]

\[ \Omega_{(z+m,\alpha,1-z-m,n+\alpha)} \begin{Bmatrix} n! \\ (m)_n \end{Bmatrix} L^{(m)}_n(x) = F^{(\alpha,\alpha)}_{n,2m}(2z - 1), \quad (3.25) \]
where $0 < \Re(z+\alpha) < 1$ and $F_{n,m}^{(\alpha,\beta)}(z)$ denotes an ultraspherical type generalization of Pasternak’s polynomial, first considered by M. A. Khan and A. K. Shukla [60] and is defined as

$$F_{n,m}^{(\alpha,\beta)}(z) = \frac{1}{(n!)^2} \binom{2n}{n} L_n^{(\alpha,\beta)}(z),$$

which for $\alpha = 0$ reduces to Pasternak’s polynomial $\Omega(z,\alpha)$

$$\Omega(z,\alpha) = \frac{n!}{(n-1)!} L_n^{(\alpha)}(x) = F_{n,m}^{(\alpha,\beta)}(p,2z-1),$$

where $0 < \Re(z) < 1$ and $F_{n,m}^{(\alpha,\beta)}(p,z)$ denotes a Jacobi type generalization of Bateman’s polynomial $F_n(z)$, first considered by M. A. Khan and A. K. Shukla [60] and is defined as

$$F_{n,m}^{(\alpha,\beta)}(p,z) = \frac{1}{(n!)^2} \binom{2n}{n} L_n^{(\alpha,\beta)}(z),$$

which for $\beta = \alpha$ reduces to $F_{n,m}^{(\alpha,\alpha)}(p,z)$ and for $\alpha = \beta = 1, p = 1$ becomes Bateman’s polynomial $F_n(z)$.

$$\Omega(z+m,\alpha,\beta) = \frac{n!}{(m!)} L_n^{(\alpha,\beta)}(x) = F_{n,m}^{(\alpha,\beta)}(2z-1)$$

where $0 < \Re(z+m) < 1$ and $F_{n,m}^{(\alpha,\beta)}(z)$ denotes a Jacobi type generalization of Pasternak’s polynomial, first considered by M. A. Khan and A. K. Shukla [60] and is defined as
which for $\beta = \alpha$ reduces to $F_{n,m}^{(a,a)}(z)$ and for $\alpha = \beta = 0$ reduces to Pasternak’s polynomial $F_n^m(z)$.

Further, we have

$$\Omega_{(a,\beta,1-a,-\beta,\alpha)}\left\{ _1F_2 \left[ \begin{array}{c} -n; \\ \alpha,1; \\ xt \end{array} \right] \right\} = Z_n(t).$$

(3.28)

where $\Re(l(1-\alpha-\beta)) > 0$ and $Z_n(x)$ denotes Bateman’s polynomial defined by

$$Z_n(x)={_2F_2}\left[ \begin{array}{c} -n,n+1; \\ 1,1; \\ x \end{array} \right]$$

$$\Omega_{(a,\nu-\frac{1}{2},1-a,\nu-\frac{1}{2},1)}\left\{ _1F_2 \left[ \begin{array}{c} -n; \\ \alpha,1+b; \\ xt \end{array} \right] \right\} = Z_n^{(\nu)}(b, t)$$

(3.29)

where $\Re(l(\nu - \frac{1}{2})) > 0, 0 < \Re(l(\alpha)) < 1$ and $Z_n^{(\nu)}(b, t)$ denotes generalization of Bateman’s polynomial $Z_n(t)$ given by Bateman himself but this notation was adopted by M. A. Khan and A. K. Shukla [60] and is defined as

$$Z_n^{(\nu)}(b, t)={_2F_2}\left[ \begin{array}{c} -n,n+2\nu; \\ \nu+\frac{1}{2},1+b; \\ t \end{array} \right]$$

which for $\nu = \frac{1}{2}$, $b = 0$ reduces to $Z_n(t)$.
\[ \Omega_{(a, \beta, \gamma, \mu)} \left\{ \begin{array}{c} p\, F_q \left[ \begin{array}{c} -n, a_1, a_2, \ldots, a_p, x t \end{array} \right] \\ b_1, b_2, \ldots, b_q \end{array} \right) = f_n \left[ \begin{array}{c} \alpha, \beta, \gamma, a_1, a_2, \ldots, a_p, t \end{array} \right] \\
1, 1, b_1, b_2, \ldots, b_q \end{array} \right) 
\]

(3.30)

where the polynomial on the R. H. S. of (3.30) is Sister Celine’s polynomial which is defined by

\[ f_n \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, x t \\ b_1, b_2, \ldots, b_q \end{array} \right] = p+2 F_q \left[ \begin{array}{c} -n, n+1, a_1, a_2, \ldots, a_p, x t \\ 1, 1, b_1, b_2, \ldots, b_q \end{array} \right] \]

Finally, we have

\[ \Omega_{(a, \beta, \gamma, \mu)} \left\{ \begin{array}{c} p\, F_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, x t \end{array} \right] \\ b_1, b_2, \ldots, b_q \end{array} \right) = p+2 F_q \left[ \begin{array}{c} \alpha, \beta, \gamma, a_1, a_2, \ldots, a_p, t \end{array} \right] \\
\alpha + \beta + \gamma, b_1, b_2, \ldots, b_q \end{array} \right) 
\]

(3.31)

\[ \Omega_{(a, \beta, \gamma, \mu)} \left\{ \begin{array}{c} p\, F_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, x t \end{array} \right] \\ b_1, b_2, \ldots, b_q \end{array} \right) = p+4 F_q \left[ \begin{array}{c} \alpha, \beta, \delta(2, \mu + \alpha + \beta + \gamma), t \end{array} \right] \\
\Delta(2, \alpha + \beta + \gamma) \end{array} \right) \]

(3.32)

\[ \Omega_{(a, \beta, \gamma, \mu)} \left\{ \begin{array}{c} p\, F_q \left[ \begin{array}{c} a_1, a_2, \ldots, a_p, x t \end{array} \right] \\ b_1, b_2, \ldots, b_q \end{array} \right) = p+6 F_q \left[ \begin{array}{c} \alpha, \beta, \gamma, \delta(3, \mu + \alpha + \beta + \gamma), t \end{array} \right] \\
\Delta(3, \alpha + \beta + \gamma) \end{array} \right) \]

(3.33)

4. GENERATING FUNCTIONS AND EXPANSIONS: The operator \( \Omega_{(a, \beta, \gamma, \mu)} \{ \} \) is sometimes useful in deriving the generating functions and expansions one function from the known generating function and expansion of another function. We mention here a few such cases.
In view of the relation [10, p. 267(22)], the generating function for $J^0_{\alpha}(\sqrt{xt})$ may be written as

\[(1-u)^{-1-\alpha}e^{-\frac{tu}{1-u}} = \sum_{n=0}^{\infty} J^0_{\alpha}(\sqrt{xt}) u^n, \tag{4.1}\]

Operating on both sides by $\Omega_{(1,\alpha,\beta,0)}$ and using (3.8), we obtain the generating function for Laguerre polynomials

\[(1-u)^{-1-\alpha}e^{\frac{tu}{1-u}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(t) u^n. \tag{4.2}\]

Again replacing $\beta$ by $\beta - n$ in the relation (3.17), we have

\[\Omega_{(1+\alpha+\beta,\gamma,\delta,0)} \left\{ \sum_{n=0}^{\infty} L_n^{(\alpha)} \left( \frac{1-t}{2} \right) x^n \right\} = P_n^{(\alpha,\beta-n)}(t),\]

which may also be written as

\[\Omega_{(1+\alpha+\beta,\gamma,\delta,0)} \left\{ \sum_{n=0}^{\infty} L_n^{(\alpha)} \left( \frac{1-t}{2} \right) x^n \right\} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(t) u^n.\]

Using (4.2) and finally (2.7), we obtain the generating function for Jacobi polynomials

\[2^{1+\alpha+\beta} (1-u)^{\beta} (2-u-ut)^{(1+\alpha+\beta)} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(t) u^n \tag{4.3}\]

Starting with (3.17), we may also obtain

\[2^{1+\alpha+\beta} (1+u)^{\beta} (2+u-ut)^{(1+\alpha+\beta)} = \sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta)}(t) u^n. \tag{4.4}\]

Again, (3.2) may be adjusted as follows:
Thus, after performing the operation, we obtain the generating function for Bessel polynomials

\[
e^{u} \left(1 - \frac{ut}{b}\right)^{1-a} = \sum_{n=0}^{\infty} y_n(a - n, b, t) \frac{u^n}{n!}
\]

(4.5)

In view of the relation [84, p. 207 (2)], we have

\[
\left(\frac{1-t}{2}\right)^n = \frac{n!}{k!(n-k)!} \frac{(-1)^k}{(1+\alpha)_k} P_k^{(\alpha,\beta-k)}(t)
\]

(4.6)

Operating on both sides of (4.6) by \( \Omega_{(1+\alpha+\beta,\gamma,0)} \{ \} \) and using (3.17), we obtain the expansion for Jacobi polynomials

\[
\left(\frac{1-t}{2}\right)^n = \sum_{k=0}^{n} \frac{(-1)^k}{n!} \frac{(1+\alpha)_n}{(1+\alpha)_k} \frac{(1+\beta)_n}{(1+\beta)_n} P_k^{(\alpha,\beta-k)}(t)
\]

(4.7)

Replacing \( v \) by \( v - \frac{n}{2} \) in (3.15), we have

\[
\Omega_{(2v,\beta,\gamma,0)} \left\{ L_n^{(\nu+\frac{1}{2},\beta-\frac{1}{2})} \left(\frac{1-t}{2}\right) x \right\} = \frac{\left(\frac{1}{2} - v\right)}{\gamma_2'} \frac{\left(\frac{1}{2} + v\right)}{\gamma_2'} \frac{(-1)^{\nu_2}}{(1-2v)_n} C_n^{\nu_2 - \frac{n}{2}}(t)
\]

which may be written as

\[
\Omega_{(2v,\beta,\gamma,0)} \left\{ \sum_{n=0}^{\infty} L_n^{(\nu+\frac{1}{2},\beta-\frac{1}{2})} \left(\frac{1-t}{2}\right) x^n \right\}
\]
Using (4.2) and (2.7), we get

\[
2^{2v} (1-u)^{v+\frac{n-3}{2}} (2-u - ut)^{-2v} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - v\right)_{\frac{n}{2}} \left(\frac{1}{2} + v\right)_{\frac{n}{2}} (-1)^{\frac{n}{2}}}{\left(1 - 2v\right)_n} C_{n_v}^{\frac{n}{2}} (t) u^n
\]

(4.8)

Replacing \(a\) by \(a - \frac{n}{2}\) in (3.16), we have

\[
\Omega_{(2a+1,\beta,\gamma,0)} \left\{ L_n^{\left(\frac{a-n}{2}\right)} \left(\frac{1-t}{2}, x\right) \right\} = P_n^{\left(\frac{a-n}{2}-\frac{a-n}{2}\right)} (t)
\]

which may be written as

\[
\Omega_{(2a+1,\beta,\gamma,0)} \left\{ \sum_{n=0}^{\infty} L_n^{\left(\frac{a-n}{2}\right)} \left(\frac{1-t}{2}, x\right) u^n \right\} = \sum_{n=0}^{\infty} P_n^{\left(\frac{a-n}{2}-\frac{a-n}{2}\right)} (t) u^n
\]

using (4.2) and (2.7), we obtain

\[
2^{1+2a} (1-u)^{\frac{a+n}{2}} (2-u - ut)^{-2a-1} = \sum_{n=0}^{\infty} P_n^{\left(\frac{a-n}{2}-\frac{a-n}{2}\right)} (t) u^n . \quad (4.9)
\]

5. SOME FORMULAE INVOLVING APPELL'S FUNCTIONS:

Appell’s functions of two variables [8, p. 224] are defined as follows:

\[
F_1(a; b; b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (b')_{n} x^m y^n}{(c)_{m+n} m! n!}
\]

(5.1)
Now it follows from the identity
\[ e^{(x+iy)v} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(xu)^m (yv)^n}{m! n!} \]
and the formula (2.6) that
\[ \Omega_{(\alpha,\beta,\gamma,\mu)} \left\{ e^{(x+iy)v} \right\} = F_{\gamma}(\alpha + \beta + \gamma + \mu; \alpha, \beta; \alpha + \beta + \gamma; u, v) \] (5.5)

The confluent form of (5.1) is [93, p. 25]
\[ \phi_{\gamma}(a, b, c, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m x^m y^n}{(c)_{m+n} m! n!} \] (5.6)

This form is readily obtained from the identity
\[ e^{xv} I_{(\alpha-1)}(2\sqrt{xv})(vx)^{1-\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(uv)^m (vx)^n}{(a)_{m+n} m! n!} \]

where \( I_{(\alpha)}(z) \) is the modified Bessel function, and the formula (2.6) giving
\[ \Omega_{(\alpha,\beta,\gamma,\mu)} \left\{ e^{xv} I_{(\alpha-1)}(2\sqrt{xv})(vx)^{1-\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \right\} = \phi_{\gamma}(\alpha + \beta + \gamma + \mu, \alpha, \beta; \alpha + \beta + \gamma; u, v) \] (5.7)
We also obtain Appell's functions from the confluent series [93, p. 25-26] as follows:

\[
\Omega_{(\beta,\beta',\delta,0)}(\phi_2(\alpha,\gamma,\gamma',ux,vy)) = F_2[\alpha;\beta,\beta',\gamma,\gamma';u,v] \tag{5.8}
\]

Further we have [8, p. 264] that

\[
e^x \frac{1}{2} - \mu \ M_{k,\mu}(x) = \phi(\alpha;\gamma;x); k = \frac{c}{2} - a, \mu = \frac{c}{2} - \frac{1}{2},
\]

where \(M_{k,\mu}(x)\) is Whittaker's M-function. Now consider the identity

\[
e^x \frac{1}{2} - \mu \ M_{k,\mu}(x) = \phi(\alpha;\gamma;x) \phi(\alpha;1 + a + b - c;vy)
\]

Operating on both sides by \(\Omega_{(b,b,d,0)}\) and observing that [8, p. 269]

\[
F_4[\alpha;\beta;\gamma,1 + a + b - c);x(1-y),y(1-x)] = _2F_1[\frac{a,b}{c};x] _2F_1[1 + a + b - c;y],
\]

we immediately obtain

\[
\Omega_{(b,b,d,0)}(e^x \frac{1}{2} - \mu \ M_{k,\mu}(x)) = F_4[\alpha;\beta;\gamma,1 + a + b - c);u(1-v),v(1-u)] \tag{5.10}
\]

6. AN IDENTITY: The operator \(\Omega_{(a,b,y,\mu)}\) may also be used for establishing certain identities. We illustrate the method by considering
the expansion for Laguerre polynomials. We have the relation [3, p. 142] that

\[ L_n^{(\lambda)}(x + y) = \sum_{r=0}^{n} \frac{(-1)^{r}}{r!} y^{r} L_{n-r}^{(\lambda)}(x) \]

which may also be written as

\[
\frac{(1 + \lambda)_{n}}{n!} \sum_{r=0}^{n} \frac{(-1)^{r}}{r!} (1 + \lambda + r)_{n-r} \frac{(-n+r)_{k} x^{k}}{(n-r)!(1+\lambda+r)_{k}}
\]

\[
= \sum_{r=0}^{n} \sum_{r=0}^{n-r} \frac{(-1)^{r}(1 + \lambda + r)_{n-r}}{r! k! (n-r)!(1+\lambda+r)_{k}} (-n+r)_{k} y^{r}
\]

Operating on both sides by = \( \Omega_{(a, \beta, \gamma, \mu)} \), we obtain

\[
\frac{(1 + \lambda - \alpha - \beta - \gamma - \mu)_{n}}{n!} = \sum_{r=0}^{n} \frac{(-1)^{r}(1 + \lambda + r)_{n-r}}{r! (n-r)!(\alpha + \beta + \gamma + \mu)_{r}} x^{r}
\]

\[
\times _{3}F_{2}\left[ \begin{array}{c}
-n+r, \alpha, \alpha + \beta + \gamma + \mu + r; \\
\alpha + \beta + \gamma + r, 1+\lambda + r
\end{array} : \right]
\]

(6.1)

Taking \( \lambda = \alpha + \mu - n \), the right hand side becomes Saalschutz form and finally reduces to \( (-1)^{n} \frac{\left(\beta + \gamma\right)_{n}}{n!} \).

**CONCLUDING REMARK:** Results of this chapter has been accepted for publication in Acta Ciencia Indica.