CHAPTER III

COMPLEX INTEGRALS OF DISCRETE FUNCTIONS

The concept of a discrete line integral is defined in this chapter which, in a sense, can be regarded as inverse of the $\oint_{p,x}$, $\oint_{q,y}$ operators. Properties studied in this chapter of these discrete line integrals exhibit a close analogy with the theory of continuous variables. Analogues for Green's Formula and Cauchy's Integral formula are also established.

§1. THE DISCRETE LINE INTEGRAL - If $z_j$ and $z_{j+1}$ are two 'adjacent' points of some discrete domain $D$, then $z_{j+1}$ must be one of the following:

$$(p x_j, y_j), (x_j, q y_j), (p^{-1} x_j, y_j) \text{ or } (x_j, q^{-1} y_j).$$

The 'discrete line integral' from $z_j$ to $z_{j+1}$ of a discrete function $f$, is defined as

$$\int_{z_j}^{z_{j+1}} f(t) d(t; p, q) = \begin{cases} (z_{j+1} - z_j) f(z_j); z_{j+1} = (p x_j, y_j) \text{ or } (x_j, q y_j) \\ (z_{j+1} - z_j) f(z_{j+1}); z_{j+1} = (p^{-1} x_j, y_j) \text{ or } (x_j, q^{-1} y_j) \\ \ldots \ldots \quad (3.1.1) \end{cases}$$
In general if $C = \langle z_0, z_1, \ldots, z_n \rangle$ is a discrete curve in $D$, then the discrete line integral from $z_0$ to $z_n$ along $C$ is defined as

$$\int_{z_0}^{z_n} f(t) \, d(t; p, q) = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} f(t) \, d(t; p, q) \quad \ldots \ldots \ (3.1.2)$$

For example, the discrete line integral around the curve $C = \langle (x, y), (px, y), (px, qy), (x, qy), (px, qy), (px, y) \rangle$ is given by

$$(p^{-1}x, y) \int_{(x, y)} f(t) \, d(t; p, q)$$

$$= \left[ \int_{(x, y)} f(x, y) + \int_{(px, y)} f(px, y) + \int_{(px, qy)} f(px, qy) + \int_{(x, qy)} f(x, qy) + \int_{(p^{-1}x, qy)} f(p^{-1}x, qy) \right] f(t) \, d(t; p, q)$$

$$= (p-1)x f(x, y) + (q-1)i y f(px, y) + (1-p)x f(x, qy)$$

$$+ (p^{-1}x) f(p^{-1}x, qy) + (1-q)i y f(p^{-1}x, y)$$

For convenience the discrete line integral around a
discrete curve $C$ will also be denoted as $\int_C f(t) d(t; p, q)$.

§2. PROPERTIES OF THE DISCRETE LINE INTEGRAL— The following elementary properties of the discrete line integral are immediate consequence of its definition:

(i) Let $C_1 = \langle z_0, z_1, \ldots, z_m \rangle$ and $C_2 = \langle z_m, z_{m+1}, \ldots, z_n \rangle$ be two discrete curves in $D$. If $f$ is a discrete function defined on $D$ then

$$\int_{C_1} f(t) d(t; p, q) + \int_{C_2} f(t) d(t; p, q) = \int_{C_1 + C_2} f(t) d(t; p, q) \ldots \ (3.2.1)$$

where $C_1 + C_2 = \langle z_0, z_1, \ldots, z_m, z_{m+1}, \ldots, z_n \rangle$.

(ii) If $C = \langle z_0, z_1, \ldots, z_n-1, z_n \rangle$ then $-C$ denotes the sequence in the reverse order, $\langle z_n, z_{n-1}, \ldots, z_1, z_0 \rangle$.

It follows that

$$\int_C f(t) d(t; p, q) = - \int_{-C} f(t) d(t; p, q) \ldots \ (3.2.2)$$

(iii) If $\alpha$ denotes a scalar constant then,
\[ \int_{C} \alpha f(t) d(t;p,q) = \alpha \int_{C} f(t) d(t;p,q) \quad \ldots \quad (3.2.3) \]

(iv) If \( f, g \) are two discrete functions then

\[ \int_{C} [f(t)+g(t)] d(t;p,q) = \int_{C} f(t) d(t;p,q) + \int_{C} g(t) d(t;p,q) \quad \ldots \quad (3.2.4) \]

Thus the sum of a finite number of discrete functions may be integrated term by term. Under certain conditions, term by term integration can be extended to infinite series.

(v) A formula for change of variable is given by

\[ \int_{z_0}^{z_n} f(t) d(t;p,q) = \lambda^{m} \int_{\lambda^{m} z_0}^{\lambda^{m} z_n} f(\lambda t) d(t;p,q) \]

where \( \lambda = p \) or \( q \).

(Note that if \( t \in Q' \) then \( (p t, q t) \in Q' \) also).

Another discrete line integral can be defined by,

\[ \int_{z_0}^{z_n} f(t) |d(t;p,q)| = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} f(t) |d(t;p,q)|, \]

where
\[ \int_{z_j}^{z_{j+1}} f(t) |d(t;p,q)| = \begin{cases} |z_{j+1} - z_j| f(z_j); z_{j+1} = (px_j, y_j) & \text{or} (x_j, qy_j) \\ |z_{j+1} - z_j| f(z_{j+1}); z_{j+1} = (px_j, y_j) & \text{or} (x_j, q^{-1} y_j) \\ \end{cases} \]

This integral is evidently invariant under change of direction of the discrete curve \( C \), i.e.,

\[ \int_{z_0}^{z_n} f(t) |d(t;p,q)| = \int_{z_0}^{z_n} f(t) |d(t;p,q)| \]

or \( \int_C = \int_{-C} \)

where \( C = \langle z_0, z_1, \ldots, z_n \rangle \).

Several theorems involving the above definitions of discrete integrals and their properties are now considered.

**THEOREM 3.2.1.** If \( f \) is a discrete function and \( C \) is a discrete curve in \( D \), then

\[ |\int_C f(t) d(t;p,q)| \leq \int_C |f(t)| d(t;p,q)| \]
PROOF - If \( C = \langle z_0, z_1, \ldots, z_n \rangle \) then,

\[
| \int_C f(t) \, d(t; p, q) | = \left| \sum_{j=0}^{n-1} f(t) \, d(t; p, q) \right|
\]

\[
\leq \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} |f(t)| \, d(t; p, q)
\]

Hence by (3.1.1), (3.2.5),

\[
| \int_C f(t) \, d(t; p, q) | \leq \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} |f(t)| \, d(t; p, q)
\]

\[
= \int_C |f(t)| \, d(t; p, q)
\]

COROLLARY 3.2.1. If \( \max_{t \in C} |f(t)| = M \), and \( L \) denotes the 'curve length' \( \sum_{j=0}^{n-1} |z_{j+1} - z_j| \), then

\[
| \int_C f(t) \, d(t; p, q) | \leq ML
\]

PROOF - By theorem 3.2.1,

\[
| \int_C f(t) \, d(t; p, q) | \leq \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} |f(t)| \, d(t; p, q)
\]
Discrete functions involving a parameter can be defined. Some results are now considered.

Let \( P \) be a given set of complex numbers and \( D \) a discrete domain, such that \( f(z; \lambda) \) is a discrete function for each \( \lambda \in P \) and \( z \in D \).

If \( \lim_{\lambda \in P} f(z; \lambda) = \phi(z) \) then the limit will be termed 'uniform' if, given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( |\lambda - \lambda_0| < \delta \) then \( |f(z; \lambda) - \phi(z)| < \varepsilon \) for all \( z \in C \), where \( \delta \) is independent of \( z \).

With this definition the following theorems are obtained.

THEOREM 3.2.2. If \( \lim_{\lambda \in P} f(z; \lambda) = \phi(z) \) is uniform for \( z \in C \), then

\[
\lim_{\lambda \in P} \int_{C} f(t; \lambda) d(t; p, q) = \int_{C} \phi(t) d(t; p, q).
\]
PROOF—By uniformity, given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|\lambda - \lambda_0| < \delta$ then

$$|f(z; \lambda) - \phi(z)| < \frac{\varepsilon}{L} \text{ for all } z \in \mathbb{C}.$$  

By corollary 3.2.1,

$$\left| \int_{\mathbb{C}} [f(t; \lambda) - \phi(t)] d(t; p, q) \right| \leq \frac{\varepsilon}{L} L = \varepsilon,$$

and so

$$\lim_{\lambda \to \lambda_0} \int_{\mathbb{C}} [f(t; \lambda) - \phi(t)] d(t; p, q) = 0.$$  

Also by (3.2.4),

$$\lim_{\lambda \to \lambda_0} \int_{\mathbb{C}} f(t; \lambda) d(t; p, q) = \int_{\mathbb{C}} \phi(t) d(t; p, q).$$

This proves the theorem.

An extension to property (iv) which follows from the above theorem is

THEOREM 3.2.3. If the series $\sum_{j=0}^{\infty} g_j(z)$ of discrete functions
g_j converges uniformly for all points of a discrete curve C, then the series may be integrated term by term along the curve C.

PROOF- If \( f(z; \frac{1}{n}) \) is defined by

\[
f(z; \frac{1}{n}) = \sum_{j=0}^{n} g_j(z)
\]

then it is readily seen that uniform convergence of the series

\[
\sum_{j=0}^{\infty} g_j(z)
\]

is equivalent to the uniform limit of the sequence of functions \( \{ f(z; \frac{1}{n}) \} \).

Therefore,

\[
\lim_{n \to \infty} f(z; \frac{1}{n}) = \sum_{j=0}^{\infty} g_j(z),
\]

and by theorem 3.2.2,

\[
\int_{C} \sum_{j=0}^{\infty} g_j(z)d(z;p,q) = \lim_{n \to \infty} \int_{C} \sum_{j=0}^{n} g_j(z)d(z;p,q)
\]

Hence by property (iv),
\[
\int \sum_{j=0}^{\infty} g_j(z) d(z; p, q) = \lim_{n \to \infty} \sum_{j=0}^{n} \int g_j(z) d(z; p, q) = \sum_{j=0}^{\infty} \int g_j(z) d(z; p, q),
\]

which proves the theorem.

The condition of uniformity in the above theorem is only required when the discrete curve \(C\) consists of an infinite number of points. When the curve is finite, only pointwise convergence is needed.

Multiple integrals of discrete functions can be defined as the following theorem indicates:

**THEOREM 3.2.4.** Let \(C_1 = \langle z_0, z_1, \ldots, z_n \rangle\) and \(C_2 = \langle w_0, w_1, \ldots, w_m \rangle\) be two discrete curves in \(\mathcal{Q}^1\). If \(f(z; w)\) is a discrete function defined for \(z \in C_1, w \in C_2\) then,

\[
\int_{C_2} \left[ \int_{C_1} f(z; w) d(z; p, q) \right] d(w; p, q)
\]

\[= \int_{C_1} \left[ \int_{C_2} f(z; w) d(w; p, q) \right] d(z; p, q).\]
**PROOF**—If $z_{j}, z_{j+1}$ and $w_{k}, w_{k+1}$ are adjacent points of $C_{1}$, $C_{2}$ respectively, then $z_{j+1}$ is one of

$$(px_{j}, y_{j}), (x_{j}, qy_{j}), (p x_{j}, y_{j}) \text{ or } (x_{j}, q y_{j})$$

and $w_{k+1}$ is one of

$$(pu_{k}, v_{k}), (u_{k}, qv_{k}), (p u_{k}, v_{k}) \text{ or } (u_{k}, q v_{k})$$

where $z_{j} = (x_{j}, y_{j})$ and $w_{k} = (u_{k}, v_{k})$.

Hence

$$\int_{w_{k}}^{w_{k+1}} \left[ \int_{z_{j}}^{z_{j+1}} f(z; w) d(z; p, q) \right] d(w; p, q)$$

admits the possibility of sixteen pairings for $w_{k+1}, w_{k}$ and $z_{j+1}, z_{j}$.

The case $z_{j+1} = (px_{j}, y_{j}), w_{k+1} = (pu_{k}, v_{k})$ is now considered. By (3.1.1),

$$\int_{w_{k}}^{w_{k+1}} \left[ \int_{z_{j}}^{z_{j+1}} f(z; w) d(z; p, q) \right] d(w; p, q)$$
\[ \int_{w_k}^{w_{k+1}} \left[ \int_{z_j}^{z_{j+1}} f(z;w) \, d(w;p,q) \right] \, d(z;p,q) \]

\[ = (w_{k+1} - w_k) \int_{z_j}^{z_{j+1}} f(z;w_k) \, d(w;p,q) \]

\[ = (p-1)^2 u_k x_j f(z_j;w_k) \cdot \]

In a similar manner,

\[ \int_{z_j}^{z_{j+1}} \int_{w_k}^{w_{k+1}} f(z;w) \, d(w;p,q) \, d(z;p,q) \]

\[ = (p-1)^2 u_k x_j f(z_j;w_k) \]

and so

\[ \int_{w_k}^{w_{k+1}} \int_{z_j}^{z_{j+1}} f(z;w) \, d(w;p,q) \, d(z;p,q) = \int_{w_k}^{w_{k+1}} \int_{z_j}^{z_{j+1}} f(z;w) \, d(w;p,q) \]

\[ \ldots (3.2.6) \]

This result holds for the other fifteen pairings of \( w_{k+1}, w_k \) and \( z_{j+1}, z_j \) as can easily be checked as above.

Now

\[ \int_{C_2}^{C_1} \left[ \int_{C_1}^{C_2} f(z;w) \, d(z;p,q) \right] \, d(w;p,q) = \]
\[
\sum_{k=0}^{n-1} \sum_{j=0}^{z_j+1} w_k \int_{C_1} f(z;w) d(z;p,q) d(w;p,q)
\]

from (3.2.4). Using (3.2.6) and interchanging the two sums, the result of the theorem follows.

§ 3. INTEGRALS OF (p,q)-ANALYTIC FUNCTIONS—The discrete line integral defined in (3.1.1) is related to the monodiffric integral defined by Isaacs [72], and as such has similar properties. The following theorems are analogues of Isaac's results and are given for completeness.

The analogue of Cauchy's integral theorem is

**THEOREM 3.3.1.** If \( f \) is (p,q)-analytic in \( D \) then the discrete line integral around every discrete closed curve in \( D \) is zero.
**PROOF**— The discrete domain $D$ consists of a union of basic sets (see (2.1.2)). Let $S(z)$ be an arbitrary basic set in $D$. The integral around this set is given by

$$\int_{\partial(S)} f(t) d(t; p, q)$$

$$= \left[ \int_z^{(px, y)} + \int_{(px, qy)} + \int_{(x, qy)} \right] f(t) d(t; p, q),$$

and by (3.1.1)

$$\int_{\partial(S)} f(t) d(t; p, q) = (p-1)x f(z) + (q-1)iy f(px, y) + (1-p)x f(x, qy)$$

$$+ (1-q)iy f(z)$$

$$= - \left\{ (1-p)x-(1-q)iy \right\} f(z) - (1-p)x f(x, qy) + (1-q)iy f(px, y)$$

$$= - R_{p, q}[f(z)]$$

and since $f$ is $(p, q)$-analytic at $z$ so by (2.4.2) $R_{p, q}[f(z)]$; and hence

$$\int_{\partial(S)} f(t) d(t; p, q) = 0$$
Now if \( S_1 = S(z_1) \) and \( S_2 = S(z_2) \) are two adjoining basic sets in \( D \) (see fig.3), then since

\[
\int_{z_2}^{z_5} f(t)d(t;p,q) = -\int_{z_5}^{z_2} f(t)d(t;p,q)
\]

it follows that,

\[
\int_{\delta(S_1)} f(t)d(t;p,q) + \int_{\delta(S_2)} f(t)d(t;p,q) = \int_{\delta(S_1)} f(t)d(t;p,q)
\]

But by the above \( \int_{\delta(S_1)} f(t)d(t;p,q) = \int_{\delta(S_2)} f(t)d(t;p,q) = 0 \),

giving

\[
\int_{\delta(S_1)} f(t)d(t;p,q) = 0
\]

similarly if \( D = \bigcup_{j=1}^{N} S(z_j) \), it follows that

\[
\int_{\delta(D)} f(t)d(t;p,q) = \sum_{j=1}^{N} \int_{\delta(S_j)} f(t)d(t;p,q)
\]

\[
= 0
\]

This proves the theorem.
\[ C = \{ z_1, z_2, z_3, z_4, z_5, z_6, z_7 \} \]

\[ \int_{z_2}^{z_5} - \int_{z_3}^{z_2} \]

\[ \sum_{\partial(S_1)} \cdot \sum_{\partial(S_2)} = \sum_{\partial(C)} \]

FIG. 21: FIG 3
The converse of the above is an analogue of Morera's Theorem given by

**THEOREM 3.3.2.** If the discrete integral around every discrete closed curve in $D$ is zero, then $f$ is $(p,q)$-analytic in $D$.

**PROOF**— From the previous theorem, if $S(z)$ is a basic set of $D$ then

$$R_{p,q}[f(z)] = -\int_{\delta(S)} f(t) d(t;p,q)$$

$$= 0 \quad \text{by assumption.}$$

Hence by (2.3.2) $f$ is $(p,q)$-analytic at $z$, and similarly at all other points of $D$, proving the theorem.

The 'discrete indefinite integral' is defined as

$$F(z) = \int_{a}^{z} f(t) d(t;p,q) \quad \ldots. \quad (3.3.1)$$

where $a,z$ belong to some discrete domain $D$ and $a$ is fixed.

**THEOREM 3.3.3.** (The fundamental theorem).
If $f$ is $(p,q)$-analytic in $D$, then

$$F(z) = \int_a^z f(t) d(t; p, q)$$

is independent of the discrete curve in $D$ from $a$ to $z$.

**PROOF—** Let $C_1 = \langle a = z_0, z_1, z_2, \ldots, z_n = z \rangle$

$$C_2 = \langle a = z_0, z_1, \ldots, z_m = z \rangle$$

be two discrete curves in $D$.

By (3.2.1), (3.2.2),

$$\int_{C_1} f(t) d(t; p, q) - \int_{C_2} f(t) d(t; p, q) = \int_{C_1 - C_2} f(t) d(t; p, q)$$

where $C_1 - C_2 = \langle a = z_0, z_1, \ldots, z_n = z = Z_m, Z_{m-1}, \ldots, Z_0 = a \rangle$.

Hence $C_1 - C_2$ is a discrete closed curve and by theorem 3.3.1,

$$\int_{C_1} f(t) d(t; p, q) - \int_{C_2} f(t) d(t; p, q) = 0$$

which proves the theorem.
From the above theorems several corollaries follow.

**COROLLARY 3.3.1.** If \( f \) is \((p, q)\)-analytic in \( D \) and \( C = \langle z_0, z_1, \ldots, z_n \rangle \) is a discrete curve in \( D \) then

\[
\int_{z_0}^{z_n} f(t) d(t; p, q) = F(z_n) - F(z_0)
\]

**PROOF-**

\[
F(z_n) - F(z_0) = \int_{a}^{z_n} f(t) d(t; p, q) - \int_{a}^{z_0} f(t) d(t; p, q)
\]

\[
= \left[ \int_{a}^{z_n} + \int_{z_0}^{a} \right] f(t) d(t; p, q)
\]

and so by (3.2.1) and theorem 3.3.3.,

\[
F(z_n) - F(z_0) = \int_{z_0}^{z_n} f(t) d(t; p, q).
\]

**COROLLARY 3.3.2.** If \( f \) is \((p, q)\)-analytic in \( D \) then

(i) \( F \) is \((p, q)\)-analytic in \( D \) and

(ii) \( \delta F = f \).

**PROOF-**

(i) Let \( a, z \in D \). By (2.3.1)

\[
R_{p, q}[f(z)] = (1-p)x - (1-q)iy \left\{ F(z) - (1-p)xF(x, qy) + (1-q)iyF(px, y) \right\}
\]
\[
\frac{(x, qy)}{(a, a)} - \frac{(px, y)}{(a, a)} \int_{a}^{z} f(t) dt (t; p, q) + (1-q) iy \int_{z}^{a} f(t) dt (t; p, q)
\]

\[
= -(1-p)x \int_{z}^{a} f(t) dt (t; p, q) + (1-q) iy \int_{z}^{a} f(t) dt (t; p, q)
\]

\[
= -(1-p)x [(q-1) iy f(z)] + (1-q) iy [(p-1) xf(z)]
\]

\[
= 0.
\]

Hence \( F(z) \) is \((p, q)\)-analytic in \( D \).

(ii) Since \( F(z) \) is \((p, q)\)-analytic,

\[
\begin{align*}
\frac{\partial}{\partial p} F(z) &= \frac{\partial}{\partial p, x} F(z) \\
&= \frac{F(z) - F(px, y)}{(1-p)x} \\
&= \frac{1}{(1-p)x} \left[ \int_{a}^{z} f(t) dt (t; p, q) - \int_{a}^{px, y} f(t) dt (t; p, q) \right] \\
&= \frac{1}{(1-p)x} \int_{(px, y)}^{z} f(t) dt (t; p, q) \quad \text{from (3.1.1)} \\
&= \frac{1}{(1-p)x} (1-p)x f(z) \\
&= f(z).
\end{align*}
\]
**COROLLARY 3.3.3.** If \( f \) is \((p,q)\)-analytic and \( C = \langle z_0, z_1, \ldots, z_n \rangle \) in \( D \) then

\[
\int_{z_0}^{z_n} f(t) d(t;p,q) = f(z_n) - f(z_0).
\]

The proof uses the methods similar to the above and so is omitted.

**COROLLARY 3.3.4.** If \( F_1(z) = \int_{a_1}^{z} f(t) d(t;p,q) \) and \( \]

\[
F_2(z) = \int_{a_2}^{z} f(t) d(t;p,q)
\]

then \( F_1(z) = F_2(z) + W(z) \) where \( W \) is arbitrary function, \( p \)-periodic in \( x \)-component and \( q \)-periodic in \( y \) component i.e.

\[
W(z) = W(px, y) = W(x, qy)
\]

\[\cdots \quad (3.3.2)\]

**PROOF-** Let \( W(z) = F_1(z) - F_2(z) \), then,

\[
\int_{r}^{q} W(z) = \int_{r}^{q} F_1(z) - \int_{r}^{q} F_2(z)
\]

\[
= f(z) - f(z)
\]

\[
= 0.
\]
Hence,
\[ q_{p,x} W(z) = \frac{W(z) - W(px,y)}{(1-p)x} = 0 \]
\[ q_{q,y} W(z) = \frac{W(z) - W(x,qy)}{(1-q)iy} = 0 \]

These give
\[ W(z) = W(px,y) = W(x,qy) \]

which is the required result.

Such a function is
\[ W(z) = \phi_p(x, iy) \]

where \( \phi_p(x) \) is Pincherle's p-periodic function defined by
\[
\phi_p(x) = x^{\alpha-\beta} \sum_{n=0}^{\infty} \frac{x^{\alpha+n}}{(1-q)x} \frac{x}{(1-q^{-1}\beta-n)x^{-1}}
\]
and \( \phi_q(iy) \) is Pincherle's q-periodic function.
§ 4. **CAUCHY'S INTEGRAL FORMULA**— In order to develop discrete analogue of Cauchy's integral formula it is also necessary to introduce the concept of \((r,s)\)-analytic function.

If \(r = p\) and \(s = q\) then \(r > 1\) and \(s > 1\).

Let the operators \(\Theta_r, z, \Theta_s, y\) be defined in a manner similar to the operators \(\gamma_{p, x}', \gamma_{q, y}'\)

\[
\Theta_r, x[f(z)] = \frac{f(z) - f(rx, y)}{(1-r)x}
\]

\[
\Theta_s, y[f(z)] = \frac{f(z) - f(x, sy)}{(1-s)iy}
\]

A discrete function \(f\), defined on \(Q'\) is said to be \'(r,s)\)-analytic' at \(z\) if

\[
\Theta_{r, x}[f(z)] = \Theta_{s, y}[f(z)] \quad \cdots \quad (3.4.1)
\]

and the common operator is denoted by \(\Theta\).

Much of the theory that has been, and is subsequently developed for \((p,q)\)-analytic functions, applies directly to
(r,s)-analytic functions.

Now equation (3.4.1) is equivalent to \( B_{p,q}[g(z)] = 0 \) where the operator \( B_{p,q} \) is defined as

\[
B_{p,q}[g(z)] = \left\{ (1-r)x - (1-s)i y \right\} g(z) - (1-r)x g(x, sy) + (1-s)i y f(rx, y)
\]

\[
= \left\{ (1-p)x - (1-q)i y \right\} g(z) - (1-p)x g(x, qy) - (1-q)i y g(p x, y)
\]

...... (3.4.2)

The following definition of a 'conjoint line integral' is the \((p,q)\)-function analogue of the one introduced by Isaacs [72].

If \( C = \langle z_0, z_1, \ldots, z_n \rangle \) is a discrete curve in \( D \), and if \( f \) and \( g \) are two discrete functions, then the conjoint line integral along \( C \) is defined as

\[
\int_{z_0}^{z_n} (f \oplus g)(t) d(t; p, q) = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} (f \oplus g)(t) d(t; p, q)
\]
where

\[
\left\{ \begin{array}{l}
(z_{j+1}-z_j)f(z_j)g(z_{j+1}); \text{ for } z_{j+1} = (px_j, y_j) \text{ or } (x_j, qy_j) \\
(z_{j+1}-z_j)f(z_{j+1})g(z_j); \text{ for } z_{j+1} = (p^{-1}x_j, y_j) \text{ or } (x_j, q^{-1}y_j)
\end{array} \right.
\]

\(\ldots (3.4.3)\)

The following two theorems are \((p,q)\)-analogues of non-diffraqic results given by Kurowski [84] and Berzsenyi [18]. The proofs are essentially the same and so are omitted.

**Theorem 3.4.1.** If \(f\) is \((p,q)\)-analytic and \(g\) is \((r,s)\)-analytic in \(D\) then,

\[
\int_{C} (f \oplus g)(t)d(t;p,q) = 0
\]

where \(C\) is any closed curve in \(D\).

**Theorem 3.4.2.** If \(D\) is a finite discrete domain and if \(f\) and \(g\) are discrete functions defined on \(D\), then

\[
\int_{\delta(D)} (f \oplus g)(t)d(t;p,q) = \sum_{t \in D} [f(t) B_{p,q} \cdot g(p^{\xi}, q^{\sigma}) = \sum_{t \in D} [f(t) B_{p,q} g(p^{\xi}, q^{\sigma}) - g(p^{\xi}, q^{\sigma}) B_{p,q} f(t)]
\]
where $D_N$ is given by (2.3.3) and $t = \xi + i\sigma$ so that $f(t) \equiv f(\xi, \sigma)$.

The latter theorem is the $(p,q)$-analogue of Green's Identity.

If $f$ is $(p,q)$-analytic, then since $R_{p,q} f = 0$ the following holds.

**COROLLARY 3.4.1.** If $f$ is $(p,q)$-analytic on some finite discrete domain $D$ then

$$\int_{\partial(D)} (f \oplus g)(t) d(t;p,q) = \sum_{t \in D_N} f(t) B_{p,q} g(p\xi, q\sigma).$$

A discrete function $G_a$ is called a 'singularity function' if it satisfies

$$B_{p,q} [G_a(t)] = \begin{cases} 1 & \text{if } t = a, a = a_1 + i a_2 \\ 0 & \text{if } t \neq a, a \in \mathbb{Q} \end{cases} \quad (3.4.4)$$

If such a function can be found then corollary 3.4.1 reduces to

$$\int_{\partial(D)} (f \oplus G_a)(t) d(t;p,q) = f(p^{-1} a_1, q^{-1} a_2),$$

an analogue of Cauchy's integral formula.