CHAPTER VI

THE EXPANSION OF DISCRETE FUNCTIONS IN INFINITE SERIES

The discrete power $z^n$ and $(z-z_0)^n$ can be utilized in series of the form

$$
\sum_{j=0}^{\infty} a_j z^j \quad \text{and} \quad \sum_{j=0}^{\infty} a_j (z-z_0)^j.
$$

In this chapter series of the above type are used to obtain discrete analogues of the exponential function and of the general power $(z-z_0)^a$.

Finally it is shown that any $(p,q)$-analytic function can be represented by a discrete analogue of Maclaurin's series.

§ 1. DISCRETE EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS—In the monodiffirc theory, Isaacs [72] obtained, as an analogue of the exponential function, the function $\mathbb{2} (1+i)^x$ which satisfies the equation, $\Delta f(z) = f(z)$, where $\Delta$ is the monodiffirc difference operator.

Duffin [33] derived a similar function $\mathbb{3} (\frac{1+i}{1-i})^y$ which satisfies $\Delta f(z) = f(z)$, where $\Delta$ is the discrete operator of
Apart from a few simple properties, a very limited study has been made of the discrete exponential functions defined by the above authors.

In this section a \((p,q)\)-analytic function \(e(z)\) is defined which is an analogue of the classical function \(e^z\). An inverse function \(E(z)\), satisfying \(e(z) \ast E(z) = 1\), is derived and \((p,q)\)-analytic analogues of the trigonometric functions are obtained.

From (1.3.12), a \(p\)-analogue of the exponential function \(e^x\) is given by \(e_p((1-p)x)\).

The discrete analogue of \(e_p((1-p)x)\) can be found by an application of the continuation operator \(C_x^y\) as follows:

Define the function \(e(z)\) by the equality,

\[
e(z) \equiv C_y \left[ e_p((1-p)x) \right] \quad \ldots (6.1.1)
\]

Hence by (1.3.12),

\[
e(z) = C_y \left[ \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)_{k,p}} x^k \right]; \quad |x| < \frac{1}{(1-p)}
\]

\[
= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \int_{p,x}^{\infty} \left[ \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)_{k,p}} x^k \right]
\]
and so by corollary 2.3.2 and (5.1.3)

\[ e(z) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^j \sum_{k=j}^{\infty} \frac{(1-p)^{k-j}}{(1-p)^{k,p}} \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^j \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)^k} x^{k-j} \]

the two series being absolutely convergent if

\[ |x| < \frac{1}{(1-p)} \quad \text{and} \quad |y| < \frac{1}{(1-q)} \]

or (using the norm of theorem 5.2.2), \( \|(1-p)x + (1-q)iy\| < 1 \). Hence,

\[ e(z) = e_q((1-q)iy) e_p((1-p)x) \quad \ldots (6.1.2) \]

Now the series representation of \( e_q(x) \) has analytic continuation given by the infinite product,

\[ e_q(x) = \frac{1}{(1-x)_{\infty,q}} \]

which converges for all \( x \) such that \( x \neq q \); \( n \) a non-negative integer.

It follows that \( e(z) \), as given by (6.1.2), has analytic continuation.
\[ e(z) = \frac{1}{(1-(1-p)x)^{\infty, p} (1-(1-q)iy)^{\infty, q}} \] ...... (6.1.3)

Since \( y \) is real, the factor \( \frac{1}{(1-(1-q)iy)^{\infty, q}} \) can not contribute a pole and so \( y \) is unrestricted. However, the values of \( x \) are restricted by the condition \( x \neq (1-p)^{-1} \) \( ^{-n} \) a non-negative integer, since there are poles at these points. This problem can be overcome by suitably choosing the lattice to avoid the singular values of \( x \).

The above function is a reasonable analogue of the exponential function as is justified by the following theorem:

**THEOREM 6.1.1.** The function \( e(z) \) is \( (p,q) \)-analytic, it satisfies the equation,

\[ \oint e(z) = e(z) \]

and in fact for \( \| (1-p)x+(1-q)iy \| < 1 \),

\[ e(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)^{j,p}} z^j. \]

**PROOF:** \[ \oint_{p,x} [e(z)] = \frac{e_q((1-q)iy)}{(1-p)x} \left[ \frac{1}{(1-(1-p)x)^{\infty, p}} - \frac{1}{(1-(1-p)px)^{\infty, p}} \right] \]
eq((1-q)iy)
= \frac{e_q((1-q)iy)}{(1-(1-p)x)_\infty\cdot p}
= e_q((1-q)iy)\cdot e_p((1-p)x)
= e(z).

Similarly,

\mathcal{J}_{q,y}[e(z)] = e(z).

Hence e(z) is (p,q)-analytic and satisfies,

\mathcal{J}[e(z)] = e(z).

If \| (1-p)x + (1-q)iy \| < 1 then \| x \| < \frac{1}{1-p} and \| y \| < \frac{1}{1-q}

and so each of the series representations of e_p((1-p)x) and e_q((1-q)iy) is absolutely convergent. Hence

\begin{align*}
e(z) &= \left( \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)^{j,p}} \cdot x^j \right) \cdot \left( \sum_{k=0}^{\infty} \frac{(1-q)^k}{(1-q)^{k,q}} \cdot (iy)^k \right) \\
&= \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)^{j,p}} \cdot \sum_{k=0}^{j} \frac{(1-p)^{j,p}(1-q)^k}{(1-p)^{j-k,p}(1-q)^{k,q}(1-p)^k} \cdot x^{j-k} \cdot (iy)^k
\end{align*}
and so by (5.1.7),

\[ e(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_j} z^j, \]  

..... (6.1.4)

the series being absolutely convergent if \(|(1-p)x+(1-q)i y| < 1\).

This proves the theorem.

Note that since \( \lim_{p \to 1} (1-p)^n = 1 \), \( \lim_{q \to 1} (1-q)^k - 1 \),

\[ \lim_{n \to \infty} (n!) = n \]

and \( \lim_{p \to 1} \lim_{q \to 1} z^n = z \), then

\[ \lim_{p \to 1} \lim_{q \to 1} e(z) = e^z. \]

The series in (6.1.4) converges in the region

\[ ||(1-p)x+(1-q)i y|| < 1, \]

whereas the series expansion of \( e^z \) converges everywhere. However for \( p \) and \( q \) close to unity the condition \( ||(1-p)x+(1-q)i y|| < 1 \) is not very restrictive.

From (1.3.12) and (1.3.13) the function \( E_q(x) \) is defined by

\[ E_q(x) = \frac{1}{e_q(x)}. \]

If the discrete function \( E \) is defined by
\[ E(z) = C_y \left[ E_p((1-p)x) \right] \quad \text{...... (6.1.5)} \]

it follows by the definition of the operator * that,

\[(e \ast E)(z) = C_y \left[ e(x,0)E(x,0) \right] \]

and so by (6.1.2),

\[(e \ast E)(z) = C_y \left[ e_p((1-p)x)E_p((1-p)x) \right] = C_y (1), \]

and by (4.3.1),

\[(e \ast E) = 1. \]

Hence with respect to the operator * the function \( \Xi \) is an inverse of the discrete exponential function \( e \).

A discrete series expansion for \( E \) can be derived as follows:

\[ E(z) = C_y \left[ \Xi_p((1-p)x) \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left( iy \right)^j q^j \left[ E_p((1-p)x) \right] \]

and so by (1.3.13),
\[ E(z) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{\infty} \frac{(-1)^j}{(1-p)^{k,j}} \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \sum_{k=0}^{\infty} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{j} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{j} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{j} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{j} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{j} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left[ \sum_{k=0}^{j} \frac{(-1)^j}{(1-p)^{k+j}} \cdot \frac{p^{(k+j)(k+j-1)/2}}{(1-p)^{k+j}} \cdot \frac{(-1)^j (k+j)(k+j-1)/2}{(1-p)^{k+j}} \cdot \frac{1}{(1-p)^{j-k}} \right] \]

the series being absolutely convergent for all \( z \), which justifies the rearrangement of series used above. Also \( E(z) \) is \((p,q)\)-analytic.

Using the above definitions of \( e \) and \( E \), discrete
analytic analogues of trigonometric functions can be obtained. The resulting functions have properties in common with the q-analogues of trigonometric functions treated by Hahn [49]. If the analogues of sin and cos are defined as,

\[ s(z) = \frac{1}{2i} [e(iz) - e(-iz)] \]

or

\[ s(z) = \sum_{j=0}^{\infty} \frac{(-1)^j (1-p)^{2j+1}}{(1-p)_{2j+1}p} \frac{(2j+1)}{z^{2j+1}} \text{ if } ||(1-p)x+(1-q)iy|| < 1 \]

and

\[ c(z) = \sum_{j=0}^{\infty} \frac{(-1)^j (1-p)^{2j}}{(1-p)_{2j}p} \frac{(2j)}{z^{2j}} \text{ if } ||(1-p)x+(1-q)|| < 1 \]

then it can easily be verified that \( s(z) \) and \( c(z) \) are \((p,q)\)-analytic and satisfy the p-difference (or q-difference) equation

\[ \nabla^2 f(z) = - f(z) \]

Alternatively, discrete analogues of sin and cos can be defined as
\[ S(z) = \frac{1}{2i} \left[ \mathbb{E}(-iz) - \mathbb{E}(iz) \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(-1)^j p^{j(2j+1)} (1-p)^{2j+1}}{(1-p)^{2j+1},p} \frac{2j+1}{z} \]

and

\[ C(z) = \frac{1}{2} \left[ \mathbb{E}(iz) + \mathbb{E}(-iz) \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(-1)^j p^{j(2j-1)} (1-p)^{2j}}{(1-p)^{2j},p} \frac{2j}{z} \]

It follows that \( S(z), C(z) \) are \((p,q)\)-analytic and are solutions of the \( p \)-difference equation,

\[ \nabla^2 [f(z)] = - p f(pz) \]

\[ \cdots (6.1.1) \]

The following two addition formulae can easily be verified

\[ (C * c)(z) + (S * s)(z) = 1 \]

\[ (C * s)(z) - (S * c)(z) = 0. \]

From the above definitions it is clear that

\[ \lim_{p \to 1} \lim_{q \to 1} e(z) = e^{z} \]
\[
\lim_{\substack{p \to 1 \\ q \to 1}} E(z) = e^{-z}
\]
\[
\lim_{\substack{p \to 1 \\ q \to 1}} s(z) = \sin z
\]
\[
\lim_{\substack{p \to 1 \\ q \to 1}} S(z) = \sin z
\]
\[
\lim_{\substack{p \to 1 \\ q \to 1}} c(z) = \cos z
\]
\[
\lim_{\substack{p \to 1 \\ q \to 1}} C(z) = \cos z.
\]

The series representation of \( e(z) \) given by,
\[
e(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} z^j,
\]
has been shown to be absolutely convergent for \( \| (1-p)x + (1-q)i y \| < 1 \). This leads to convergence conditions for more general power series of the form,
\[
f(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} a_j z^j \quad \ldots \ldots (6.1.11)
\]

**Theorem 6.1.2.** If \( \lim \sup |a_j| = a \), then the series (6.1.11) converges absolutely for all \( z \) such that,
\[ \| (1-p)x + (1-q)i y \| < \frac{1}{a} \]

**PROOF:** From (6.1.4), if \( \lambda \) is a scalar constant then,

\[ e(\lambda z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} \lambda^j z^j \]

\[ = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} \lambda^j z^j \]

is absolutely convergent if \( |\lambda| \| (1-p)x + (1-q)i y \| < 1 \) .... (6.1.12)

Since \( \limsup |a_j|^{1/j} = a \) it follows that given \( \varepsilon > 0 \), there exists an integer \( J \) such that

\[ |a_j| < (a+\varepsilon) \text{ for all } j \geq J \]

...... (6.1.13).

Now,

\[ |f(z)| \leq \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} |a_j||z|^j \text{, and so by (6.1.13),} \]

\[ |f(z)| \leq \sum_{j=0}^{J-1} \frac{(1-p)^j}{(1-p)_{j,p}} |a_j||z|^j + \sum_{j=J}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} (a+\varepsilon)^j |z|^j \]

By (6.1.12) the latter series is absolutely convergent for
(a + \varepsilon) \| (1-p)x + (1-q)iy \| < 1, \text{ or }

\| (1-p)x + (1-q)iy \| < \frac{1}{(a + \varepsilon)} < \frac{1}{a}.

This proves the theorem establishing a useful convergence criterion for series of the form (6.1.11).

§ 2. THE DISCRETE POWER \((z-z_0)^{(a)}\). — As mentioned in the previous chapter, analogues of the function \(z\) where \(n\) is a non-negative integer, have been studied in the theories of Isaacs and Duffin. It is believed that no analogue for the more general power \(z^a\) has been found in the discrete analytic function theory. In fact in Problem 1 of [30] the question of finding an analogue for the function \(z^\frac{1}{2}\) is posed.

In the \(q\)-function theory the function \((x-x_0)^a,q\) is defined by the infinite product,

\[(x-x_0)^a,q = x \frac{a (1-x_0/x)^\infty,q}{(1-q^a x_0/x)^\infty,q}.

This is now used to construct a \((p,q)\)-analytic analogue of the function \((z-z_0)^a\) where \(a\) is an arbitrary scalar constant.
The real function \((x-x_o)_{a,p}\) satisfies the following,

\[
Q_{p,x} (x-x_o)_{a,p} = \frac{(x-x_o)_{a,p} - (px-x_o)_{a,p}}{(1-p)x}
\]

and on simplification

\[
Q_{p,x} (x-x_o)_{a,p} = \frac{(1-p)^a}{(1-p)} (x-x_o)_{a-1,p}
\]  

..... (5.2.1)

The discrete power \((z-z_o)^{(a)}\), \(z \in Q\), is defined as,

\[
(z-z_o)^{(a)} \equiv \sum_{j=0}^{\infty} (1-p)^{a-j} (1-q)^j \frac{1}{(1-p)^{a-j,p} (1-q)^j,q (1-p)^j (y-y_o)_{j,q} (x-x_o)_{a-j,p}}
\]  

..... (6.2.2)

where \(a\) is an arbitrary scalar constant.

It can readily be shown that the series converges absolutely for all \(z \in Q\) such that \(y < |x_o|\). Moreover the function is \((p,q)\)-analytic in the region of convergence and satisfies the conditions,
(i) \[ \mathcal{A}(z-z_0)^{(a)} = \frac{(1-p)^a}{(1-p)} (z-z_0)^{(a-1)} \]

(ii) \[ (z-z_0)^{(0)} = 1 \]

(iii) \[ 0^{(a)} = 0, \quad a > 0. \]

Hence the function is a discrete analytic analogue of the classical power \( (z-z_0)^a \). Furthermore in the special case \( a = n \) where \( n \) is a non-negative integer, the series \((6.2.2)\) can be seen to reduce to the series representation of \((z-z_0)^{(n)}\) given by theorem 5.3.1.

It can easily be shown that

\[
\lim_{p \to 1} \lim_{q \to 1} \frac{(1-p)^a, p}{(1-q)^a, q} (1-p)^j \]

= \[
\frac{a(a-1) \ldots \ldots \ldots (a-j+1)}{j!}
\]

and that \( \lim_{p \to 1} (x-x_0)^{(a-j)}, p = (x-x_0)^{a-j} \). Hence,

\[
\lim_{p \to 1} \lim_{q \to 1} (z-z_0)^{(a)} = (z-z_0)^a.
\]

When \( z_0 \) is real, the function \((z-z_0)^{(a)}\) can be expressed
in terms of the continuation operator $C_y$ as follows.

By (6.2.2), when $y_0 = 0$,

$$
(z-x_0)^{(a)} = \sum_{j=0}^{\infty} \frac{(1-p)^a, p (1-q)^j}{(1-p)^{a-j, p} (1-q)^j, q (1-p)^j} (iy)^j (x-x_0)^{a-j, p}
$$

\[ \ldots \ldots (6.2.3) \]

Now

$$
\frac{(1-p)^{a, p}}{(1-p)^{a-j, p}} = \frac{(1-p)^{a-j+1, p}}{(1-p)^{a+1, p}}
$$

$$
= (1-p)^{a-j+1, p}
$$

and so,

$$
(z-x_0)^{(a)} = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j, q}} (iy)^j (1-p)^{a-j+1, p} (x-x_0)^{a-j, p},
$$

whence by (6.2.1)

$$
(z-x_0)^{(a)} = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j, q}} (iy)^j \rho_p, x^j [(x-x_0)^{a-j, p}]
$$

$$
= C_y [(x-x_0)^{a-j, p}].
$$

\[ \ldots \ldots (6.2.4) \]
The series (6.2.3) converges absolutely for all \( z \in \mathbb{Q} \) such that \( y < |x| \). If \( x = 0 \) it follows that the series diverges for all \( z \in \mathbb{Q} \). Hence the above expression for \((z-z_0)\)^{(a)} is only valid if \( z_0 \neq 0 \).

A product formula in terms of the convolution \( * \) is now deduced for functions of the form \((z-x_0)\)^{(a)}, \( x_0 \) real, 'a' arbitrary scalar constant. From Hahn [50], \((x-x_0)^\alpha (x-q x_0)^\beta = (x-x_0)^{\alpha+\beta}\), for general constants, \( \alpha, \beta \). Hence by (6.2.4) and the definition of \( * \),

\[
(z-x_0)^{(a)} * (z-q x_0)^{(b)} = C_y [(x-x_0)^\alpha (x-q x_0)^\beta]
\]

\[
= C_y [(x-x_0)^{\alpha+\beta}]
\]

\[
= (z-x_0)^{(\alpha+\beta)} \quad \cdots \quad (6.2.5)
\]

This is a generalization of the result obtained in (5.3.3) for the addition of integer powers.

A similar formula applies if \( z_0 \) is purely imaginary, however (6.2.5) does not follow for arbitrary complex \( z_0 \).

Some examples of the general discrete power \((z-z_0)^{(a)}\)
are now considered.

(a) \((z-z_0)^{\frac{1}{2}}\)

By (6.2.2),

\[
(z-z_0)^{\frac{1}{2}} = \sum_{j=0}^{\infty} \frac{(l-p)^{\frac{1}{2}-j,p} (l-q)^{i-j}}{(l-p)^{\frac{1}{2}-j,p} (l-q)^{j-q} (l-p)^{j}} (x-x_0)^{\frac{1}{2}-j,p},
\]

where \((x-x_0)^{\frac{1}{2}-j,p} = \frac{1}{x-j} \frac{(l-x_0)^{\infty,p}}{(l-p)^{\frac{1}{2}-j,p} (l-p)^{\infty,p}}\)

and \(\frac{1}{x} = \frac{1}{l-p^{\frac{1}{2}-j,p}} = (l-p^{\frac{1}{2}-j,p})^{-1}\).

Hence,

\[
(z-z_0)^{\frac{1}{2}} = x^2(1-\frac{x_0}{x})^{\infty,p} \sum_{j=0}^{\infty} \frac{3-j}{3-1} (l-p^{\frac{1}{2}-j,p} (l-q)^{j-j} (y-y_0)^{j,q}}{(l-q)^{j,q}(l-p)^{j} (l-p^{\frac{1}{2}-j,p} \frac{x_0}{x})^{\infty,p}}.
\]

which converges for all \(z \in \mathbb{Q}'\) such that \(y < |x_0|\).

(b) \((z-p x_0)^{-1}\)

With respect to the operator \(\ast\) this function is the
inverse of \((z-x_o)\) since

\[(z-x_o) * (z-p x_o)^{(-1)} = C_y \left[ \left( z-x_o \right) \left( x-p x_o \right)_{-1} \right] \]

Now \((x-p x_o)_{-1} = \frac{1}{(1 - \frac{x}{x_o})} \)

\[= \frac{1}{(x-x_o)} \]

Hence,

\[(z-x_o) * (z-p x_o)^{(-1)} = C_y \left[ 1 \right] \]

\[= 1. \]

A discrete power series can be derived for \((z-p x_o)^{(-1)}\) which is analogous to the power series for \((z-x_o)^{-1}\) given by

\[ (z-x_o)^{-1} = -\sum_{j=0}^{\infty} \frac{z^j}{x_{j+1}} \]

Now,
\[(z-p \, x_o)^{-1} = C_y \left[\frac{1}{(x-x_o)}\right]; \, y < |x_o|\]

\[= C_y \left[\frac{1}{(x-x_o)}\right]\]

\[= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left(\frac{iy}{p,x}\right)^j \left[\frac{1}{(x-x_o)}\right]\]

\[= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} \left(\frac{iy}{p,x}\right)^j \left[\frac{(1-p)^{j,p}}{(1-p)_{j,p}}\right] \left(\frac{x}{x_o-x}\right)_{j+1,p}\]

Or \[(z-p \, x_o)^{-1} = -\sum_{j=0}^{\infty} \frac{(1-p)^{j,p}(1-q)^j}{(1-p)_{j,p}(1-q)_{j,q}} \left(\frac{iy}{x_o-x}\right)_{j+1,p}\]

From Hahn [50],

\[\frac{1}{(1+x)^{\alpha}} = \sum_{j=0}^{\infty} (-1)^j \frac{a^j}{(1-q)_{j,q}} ; \, x , |x| < 1 \quad \quad \quad \quad (6.2.6)\]

It follows then that for \(x < |x_o|\)

\[\frac{1}{(z-p \, x_o)^{-1} = -\sum_{j=0}^{\infty} \frac{(1-p)^{j,p}(1-q)^j}{(1-p)_{j,p}(1-q)_{j,q}} \left(\frac{iy}{x_o-x}\right)_{j+1,p} \left(\frac{1-p}{x} \right)_{j+1-k}}\]
Hence if $y < |x_o|, x < |x_o|$, i.e. $||z|| < |x_o|$, it follows that

$$(z-p \ x_o) \ (-1) = \sum_{j=0}^{\infty} \frac{z}{j+1} \ x_o \ (-1) \ j \ (6.2.7)$$

Using similar methods to (b) above, it can be shown that for $||z|| < |p \ x_o|$, 

$$(z-x_o) \ (-n) = \sum_{j=0}^{\infty} \frac{(-1)^{n-j} \ \frac{1}{2} \ [j(j+1)+n(n+1)]+nj}{(1-p)_{n-j} \ p^j \ (1-q)_j \ q \ (1-p)_{n-j}} \ (-x_o)^{j+1} \ x_o \ z\ j \ (6.2.8)$$

where $n$ is a positive integer.
§3. DISCRETE MACLAURIN SERIES—To include the point \((C, C)\), extend the definition of \(\bar{R}\) (from (4.2.4)) as follows;

\[ \bar{R}_0 = \bar{R} \cup (0,0) \quad \cdots \quad (6.3.1) \]

A discrete function \(f\) is said to be \((p,q)\)-analytic on \(\bar{R}_0\) if it is \((p,q)\)-analytic on \(\bar{R}\) and in addition,

\[ \lim_{(x,y) \to (0,0)} \sum_j^n [f(x,y)] \]

exists. The limit is denoted by \(\sum_j^n f(0,0)\). \(\cdots \quad (6.3.2)\)

Under certain conditions the discrete Maclaurin series can be shown to represent a \((p,q)\)-analytic function, provided the series converges. For example the following theorem holds:

**THEOREM 6.3.1.** Let \(f\) be \((p,q)\)-analytic in \(\bar{R}_0\). If \(f(z) = C_y [f(x,0)] = C_x [f(0,y)]\), the series representations of \(C_y, C_x\) being uniformly and absolutely convergent in \(\bar{R}\), then

\[
\frac{1}{(1-q)^j} \sum_{j=0}^\infty \frac{(1-q)^j}{j!} [f(0,0)] [f(0,0)] z^j,
\]
the series being absolutely convergent for all $z \in \mathbb{R}_0$.

**Proof:**

$$f(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} \sum_{q=0}^{j} \left[ f(0,y) \right].$$

Hence,

$$f(x,C) = \lim_{y \to 0} \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} \sum_{q=0}^{j} \left[ f(0,y) \right].$$

and by uniform convergence,

$$f(x,0) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} \sum_{q=0}^{j} \lim_{y \to 0} \left[ f(0,y) \right].$$

By (6.3.2),

$$f(x,0) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} \sum_{q=0}^{j} \left[ f(0,0) \right].$$

Now $f(z)$ is also given by
\[ f(z) = C_y \left[ f(x,0) \right] \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^{j,q} \left[ f(x,0) \right] \]

and so by the above,

\[ f(z) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^{j,q} \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)^{k,p}} x \]

\[ \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^{j,q} \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)^{k,p}} x \]

By absolute convergence the summation can be rearranged to give,

\[ f(z) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^{j,q} \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)^{k,p}} x \]

\[ = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)^{j,q}} (iy)^{j,q} \left[ f(0,0) \right] z^{j} \]

This proves the theorem.
Theorem 6.1.2 provides as a direct consequence a condition for convergence of the discrete Maclaurin series as follows:

**THEOREM 6.3.2.** If \( \lim \sup \left\{ \left| \sum_{j=0}^{\frac{1}{\frac{1}{f(0,0)}}} j^j \right| \right\} = a \), then the series

\[
\sum_{j=0}^{\infty} \frac{(1-q)^j}{j!} \frac{q}{(1-q)^{j,q}} [f(0,0)] \frac{j}{z} \]

converges absolutely for all \( z \) such that

\[
\| (1-p)x + (1-q)i\gamma \| < \frac{1}{a} .
\]