CHAPTER V

DISCRETE POWERS AND POLYNOMIALS

So far only general properties of \((p,q)\)-analytic functions have been examined and in this chapter explicit examples of \((p,q)\)-analytic functions are considered. Using the continuation operator \(C_y\) and the operator *, discrete analogues of \(z^n\) and of polynomials \(\sum_{n=0}^{m} a_n z^n\) are obtained. Results comparable to the classical powers and polynomials are determined and certain fundamental differences are noted.

\[ \text{§1. DISCRETE POWERS} \quad \text{The function } z \quad \text{is of basic importance in complex analysis since its use in infinite series leads to the Weierstrassian concept of an analytic function. It is desirable then to obtain a discrete analytic analogue of } z \quad \text{and use it to expand discrete functions in power series.} \]

Duffin [33] defined the discrete power \(z^{(n)}\) by the iterative procedure

\[ z^{(n+1)} = (n+1) \int_0^z z^{(n)} \Delta z ; \quad z^{(0)} = 1, \quad \ldots \quad (5.1.1) \]
where $\int_0^z \delta z$ is the discrete integral occurring in the theory of discrete analytic functions of the second kind. However a closed form of the function $z^{(n)}$ does not seem possible. This makes a study of series of the form $\sum a_n z^{(n)}$ rather difficult, although some results have been obtained by Duffin and Peterson [39].

Isaacs [72] obtained the monodiffric analogue $z^{(n)}$ which is defined by,

$$z^{(n)} = \sum_{j=0}^n \binom{n}{j}(x)^{n-j}(y)_{n-j} \quad \ldots \quad (5.1.2)$$

where $(x)_j = x(x-1)(x-2) \ldots \ldots (x-n+1)$

$(y)_j = y(y-1)(y-2) \ldots \ldots (y-n+1)$.

He obtained certain results for polynomials and power series of the form $\sum a_n z^{(n)}$.

A discrete analogue $z^{(n)}$ of the classical function $z^n$ is now obtained for the $(p,q)$-analytic function theory.

The operator $\oint$ satisfies
and so if \( n \) is a non-negative integer, a \((p,q)\)-analytic function \( z^{(n)} \) will denote the discrete analogue of \( z \) if it satisfies the following conditions

1. (i) \[ \mathcal{J} \left[ z^{(n)} \right] = \frac{(1-p^n)}{(1-p)} z \]
2. (ii) \[ z^{(0)} = 1 \] ..... (5.1.4)
3. (iii) \[ z^{(n)} = 0, \ n > 0 \]

Such a function is obtained by applying the operator \( C_{y^n} \) from (4.2.14) to the real function \( x \).

In fact \( z^{(n)} \) is defined by

\[
z^{(n)} = C_{y^n} (x); \ n \ \text{a non-negative integer}
\]

\[
= \sum_{j=0}^{\infty} \frac{(1-q)^{j}}{(1-q)_{j,q}} (iy)^{j} \mathcal{J}^{j}_{p,x} (x) \]

Now from (5.1.3),
\[ \mathcal{L}_{p,x}(x) = \frac{(1-p^n)}{(1-p)} x^{n-1} \]

Similarly,
\[ \mathcal{L}_{p,x}^2(x) = \frac{(1-p)(1-p)}{(1-p)^2} x^{n-2} \]

and in general,
\[ \mathcal{L}_{p,x}^j(x) = \begin{cases} 
\frac{(1-p^{j+1})(1-p^{j+2}) \ldots (1-p^n)(1-p)x}{(1-p)^j (1-q)^j} ; & j \leq n \\
0 ; & j > n 
\end{cases} \quad \ldots (5.1.6) \]

Hence by (5.1.5)
\[ z = \sum_{j=0}^{n} \frac{(1-p^{j+1})(1-p^{j+2}) \ldots (1-p^n)(1-p)(1-q)^j}{(1-p)^j (1-q)^j} \frac{j}{x^j} \]

\[ z = \sum_{j=0}^{n} \frac{(1-p)^{n-p} (1-q)^j}{(1-p)^{n-j} (1-q)^j (q)_{j,q}} \frac{j}{x^j} \quad \ldots (5.1.7) \]

\[ = \sum_{j=0}^{n} \frac{(1-p)^{n-p} (1-q)^j}{(1-p)^j (1-q)^j (1-p)^{n-j}} x^j (1-y)^n-j \quad \ldots (5.1.8) \]

The following theorem justifies that \( z \) is an analogue of \( n \).
THEOREM 5.1.1. \( z^{(n)} \) is \((p,q)\)-analytic and satisfies the three requirements of (5.1.4).

PROOF: \( z^{(n)} = \sum_{j=0}^{n} \frac{(1-p)^{n-j,p} (1-q)^j}{(1-p)^{n-j,p} (1-q)^j} (iy)^j x^{n-j} \)

and so since \( \mathcal{D} \) is a linear operator

\[ \mathcal{D}_{p,x} \left( z^{(n)} \right) = \sum_{j=0}^{n} \frac{(1-p)^{n-j,p} (1-q)^j}{(1-p)^{n-j,p} (1-q)^j} (iy)^j x^{n-j} \]

Hence by (5.1.3)

\[ \mathcal{D}_{p,x} \left( z^{(n)} \right) = \sum_{j=0}^{n-1} \frac{(1-p)^{n-j-1,p} (1-q)^j}{(1-p)^{n-j-1,p} (1-q)^j} (iy)^j x^{n-j-1} \]

\[ = \frac{(1-p)^n}{(1-p)} \sum_{j=0}^{n-1} \frac{(1-p)^{n-1-j,p} (1-q)^j}{(1-p)^{n-1-j,p} (1-q)^j} (iy)^j x^{n-1-j} \]

\[ = \frac{(1-p)^n}{(1-p)} (n-1) z \]

Similarly,

\[ \mathcal{D}_{q,y} \left( z^{(n)} \right) = \frac{(1-p)^n}{(1-p)} (n-1) z \]

and so \( z^{(n)} \) is \((p,q)\)-analytic and satisfies condition (i) of (5.1.4).
Now \( z^{(0)} = \frac{(1-p)^0 (1-q)^0}{(1-p)^0(1-q)^0(1-p)^0} (iy)^0 x = 1 \) and if
\[
(1-p)(1-q) \text{ and if } z = 0 \text{ then } x = y = 0 \text{ and so } 0 = 0, n > 0. \] This proves the theorem.

The form of \( z \) is remarkably similar to the function \( z \) since,
\[
z = (x+iy) = \sum_{j=0}^{n} \binom{n}{j} (iy)^j x
\]

Also, since
\[
\lim_{p \to 1} \lim_{q \to 1} \frac{(1-p)^n_p, (1-q)^j}{(1-p)^n-j, p(1-q)^j, q(1-p)^j} = \frac{n!}{(n-j)! j!} z^{(j)},
\]
it follows that
\[
\lim_{p \to 1} \lim_{q \to 1} z^{(n)} = z^n.
\]

Moreover,
\[
\binom{n}{x} = x, (iy)^{\binom{n}{x}} = (iy)^n \text{ and for } n = 0, 1,
\]
\[
\binom{n}{z} = z^n. \text{ However for } n \geq 2, z^{\binom{n}{z}} \neq z^n. \text{ For example,}
\]
\[
z^{(2)} = x + (1+p) 1xy - \frac{1+p}{1+q} 2 y.
\]
It is also interesting to note the similarity of \( z^{(n)} \) to the function \((x+iy)^n\) which by using the results

\[
(1+x)_{a,q} \equiv (1+x)^a = \frac{(1+x)^\infty q}{(1+q^a x)^\infty q}; \ a \text{ is any constant} \ldots \ (5.1.9)
\]

\[
(y+x)_{a,q} \equiv (y+x)^a = y^{(1+y/a)} a_{q} \text{ is given by} \ldots \ (5.1.10)
\]

\[
(x+iy)_{n,q} = (x+iy)(x+iqy)(x+iqy)\ldots(x+iqy) y \ldots \ (5.1.11)
\]

and on simplification,

\[
(x+iy)_{n,q} = \sum_{j=0}^{n} \binom{n}{j} q^{j(j-1)/2} x^{n-j} y^j \ldots \ (5.1.12)
\]

\section*{\textsection 2. PROPERTIES OF \( z^{(n)} \)}

(a) From the definition of \( z^{(n)} \) it follows that,

\[
\lim_{z \to 0} z^{(n)} = 0.
\]

(b) Since,

\[
\mathcal{F}[z^{(n)}] = \frac{n}{(1-p)} \frac{(n-1)}{z} \ldots
\]

it follows that
\[ \int_{z_0}^{z} \left( \frac{t}{(1-p)} \right) d(t;p,q) = \int_{z_0}^{z} t^{(n-1)} d(t;p,q), \]

and by Corollary 3.3.3,

\[ z^{(n)} - z_0^{(n)} = \frac{n}{(1-p)} \int_{z_0}^{z} t^{(n-1)} d(t;p,q). \]

Taking limits as \( z_0 \rightarrow 0 \), \( z_0 \in Q' \), and using property (a) above,

\[ z^{(n)} = \frac{n}{(1-p)} \int_{0}^{t} d(t;p,q), \]  \[ \ldots (5.2.1) \]

which is similar to the definition in (5.1.1) used by Duffin.

From the definition of the lattice \( Q' \) it is clear that any neighbourhood of the origin contains an infinite number of points of \( Q' \). The discrete line integral \( \int_{C}^{z} \) defined above as \( \lim_{z_0 \rightarrow 0} \int_{z_0}^{z} \) will of course be defined on a discrete curve with \( z_0 \in Q' \) and an infinite number of points.

\( (c) \) An important property of the convolution operator \( \ast \) is that like ordinary multiplication it preserves the additive law of indices for discrete powers of \( z \). This fact is illustrated
as follows:

If \( m, n \) are non-negative integers then by (4.4.1),

\[
\begin{align*}
(m) \cdot (n) \cdot z & = C_y [(x,0)^{(m)} (x,0)^{(n)}] \\
\end{align*}
\]

By the definition of \( z \), \( (x,0)^{(m)} = x \), and so,

\[
\begin{align*}
(m) \cdot (n) \cdot z & = C_y [x^{m+n}] \\
\end{align*}
\]

and by (5.1.5),

\[
\begin{align*}
(m) \cdot (n) \cdot z & = z \\
\end{align*}
\]

\text{..... (5.2.2)}

In contrast, the function \((x+iy)^{(n)}\) from (5.1.9) satisfies

\[
\begin{align*}
(x+iy)^{(n)} (x+iy)^{(m+n)} & = (x+iy)^{(m+n)} \\
\end{align*}
\]

The above three properties have similar analogies in the monodiffric theory. The remainder of this section however, reveals relationships between \( z \) and \( z \) which are not evident for the monodiffric power of \( z \) given in (5.1.2).

(d) From the definition of \( z \) it is clear that if \( \lambda \) is a scalar constant then,

\[
\begin{align*}
(\lambda) \cdot (n) \cdot z & = \lambda \cdot z \\
\end{align*}
\]

\text{..... (5.2.3)}
(e) The classical function $z_n$ satisfies $\lim_{n \to \infty} z_n = 0$ if and only if $|z| < 1$.

A less restrictive condition ensures that $\lim_{n \to \infty} z_n = 0$ and is now given.

**THEOREM 5.2.2.** $\lim_{n \to \infty} z_n = 0$ if and only if $\|z\| < 1$, where

$$\|z\| = \max \{|x|, |y|\}; z \in \mathbb{C}'.$$

**Proof:** (i) If $\|z\| < 1$ then $\max \{|x, y|\} < 1$ since $z$ is in the first quadrant of the complex plane.

The case $0 \leq y < x < 1$ is considered first, whereby $y = ax$ for some $a$ with $0 \leq a \leq 1$.

From (5.1.7),

$$z = \sum_{j=0}^{n} \frac{(1-p)^{n-p} (1-q)^j}{(1-p)^{n-j} p (1-q)^j} (iy)^x (1-p)^j$$

and hence
Now it can easily be shown that, \((1-p)^{\infty}, p < (1-p)_n, p < 1\) and 
\((1-q)^{\infty}, q < (1-q)_n, q < 1\) for all positive integers \(n\), and so,

\[
|z| < \frac{\sum_{j=0}^{n} (1-p)^{n} \frac{1}{1-p, j, q} (1-q)^{j}}{\sum_{j=0}^{n} (1-p)^{n-j} p, (1-q)^{j}, q (1-p)^{j}}.
\]

and since \(0 < x < 1\), it follows that

\[
\lim_{n \to \infty} |z| = 0 \quad \text{and hence,} \quad \lim_{n \to \infty} z = 0.
\]

Similarly if \(0 < x \leq y < 1\), the result follows by using the symmetric form of \(z\) given by (5.1.8).

(ii) If \(\lim_{n \to \infty} z = 0\), then
Assume that \( x \geq 1 \).

From the above then

\[
\lim_{n \to \infty} \sum_{j=0}^{n} \frac{(1-p)^{n-p} (1-q)^{j}}{(1-p)^{n-j,p} (1-q)^{j,q} (1-p)^{j}} \left( \frac{iy}{x} \right) = 0
\]

..... (5.2.4)

Now

\[
\lim_{n \to \infty} \frac{(1-p)^{n-p} (1-q)^{j}}{(1-p)^{n-j,p} (1-q)^{j,q} (1-p)^{j}} =
\]

\[
= \lim_{n \to \infty} \frac{n-j+1}{(1-q)^{j,q} (1-p)^{j}} \left( \frac{1}{1-p} \right)^{n-j+2} \ldots (1-p)^{n} (1-q)^{j}
\]

\[
= \frac{(1-q)^{j}}{(1-q)^{j,q} (1-p)^{j}}
\]

It follows that

\[
\lim_{n \to \infty} \sum_{j=0}^{n} \frac{(1-p)^{n-p} (1-q)^{j}}{(1-p)^{n-j,p} (1-q)^{j,q} (1-p)^{j}} \left( \frac{iy}{x} \right) =
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{(1-q)^{j,q}} \left\{ \frac{(1-q)iy}{(1-p)x} \right\}^{j},
\]
the series being convergent to \( e_q \left( \frac{(1-q)i\gamma}{(1-p)x} \right) \) if \( y < \frac{(1-p)x}{(1-q)} \) and divergent if \( y \geq \frac{(1-p)x}{(1-q)} \). In either case the series is non-zero. This contradicts (5.2.4) and so \( x \) must be less than unity.

Similarly it can be shown that \( y < 1 \) and hence,

\[
\lim_{n \to \infty} z^{(n)} = 0 \iff \|z\| < 1.
\]

This completes the proof of the theorem.

Methods similar to the above can be used to prove the following theorem and so the proof is omitted.

**THEOREM 5.2.3**

(a) \( |z|^{(n)} \to \infty \) if and only if \( \|z\| > 1 \).

(b) If \( \|z\| = 1 \), \( x \) and \( y \) not both unity, then

\[
z^{(n)} \to e_q \left( \frac{(1-q)i\gamma}{(1-p)} \right) \text{ or } e_q \left( \frac{(1-q)x}{(1-p)} \right) \text{ according as } x = 1 \text{ or } y = 1 \text{ respectively.}\]
(c) If \( x \) and \( y \) are both unity then in the limit \( z \) neither converges nor diverges to infinity, the limit being an oscillating series.

3. THE DISCRETE POWER \( (z-z_0)^n \) — A discrete analogue of the function \( (z-z_0)^n \) proves useful in the next chapter and is now derived.

Consider the function \( (x-x_0)^n \) defined in (5.1.10). Since,

\[
\begin{align*}
\nabla_p, x \ (x-x_0)^n, p &= \frac{(x-x_0)^n, p - (px-x_0)^n, p}{(1-p)x} \\
&= \frac{(x-x_0)^n, p - p (x-\frac{x_0}{p})^n, p}{(1-p)x} \\
&= \frac{(x-x_0)^{n-1}, p \ [x-p \ x_0^{-p} (x-\frac{x_0}{p})]}{(1-p)x} \\
&= \frac{(1-p)}{(1-p)} \ (x-x_0)^{n-1}, p \\
&= \frac{(1-p)}{(1-p)} \ (x-x_0)^{n-1}, p \quad \ldots (5.3.1)
\end{align*}
\]

It follows that \( (x-x_0)^n, p \) is the \( p \)-analogue of the function \( (x-x_0)^n \). Similarly \( (y-y_0)^j, q \) is the \( q \)-analogue of \( (y-y_0)^j \).

A function \( (z-z_0)^n \), \( z \in \mathbb{Q} \), is called a \( (p,q) \)-analytic analogue of \( (z-z_0)^n \) if it satisfies,
The following theorem gives the form of the function $F^{(n)}(z-z_0)$.

**Theorem 5.3.1.** The function $F^{(n)}(z-z_0)$ defined by

$$F^{(n)}(z-z_0) = \sum_{j=0}^{n} \frac{(1-p)^n (1-q)^j}{(1-p)^{n-j,p} (1-q)^{j,q}(1-p)^j} (x-x_0)^{n-j,p} i (y-y_0)^{j,q}$$

is $(p,q)$-analytic and satisfies (i), (ii) and (iii) above.

**Proof:**

$$F_{p,x}^{(n)}(z-z_0) = \sum_{j=0}^{n} \frac{(1-p)^n (1-q)^j}{(1-p)^{n-j,p} (1-q)^{j,q}(1-p)^j} (x-x_0)^{n-j,p} i (y-y_0)^{j,q}$$

$$= \sum_{j=0}^{n} \frac{(1-p)^n (1-q)^j}{(1-p)^{n-j,p} (1-q)^{j,q}(1-p)^j} \sum_{j=0}^{n} \frac{(1-p)^n (1-q)^j}{(1-p)^{n-j,p} (1-q)^{j,q}(1-p)^j} i (y-y_0)^{j,q}$$
By (5.3.1) and since \( \mathcal{A}_{p,x} (x-x_0)_0 = 0 \) it follows that,

\[
\mathcal{A}_{p,x} (z-z_0)^{(n)} = \sum_{j=0}^{n-1} \frac{(1-p)^n}{(1-p)^{n-j}p^{(1-q)^j}q^{(1-p)^j}} (1-p)^{n-j} \frac{(x-x_0)_{n-j-1,p}}{i(y-y_0)_{j,q}} ^j
\]

\[
= \frac{(1-p)^n}{(1-p)} \frac{(1-p)^{n-1,p}}{j=0} \frac{(1-p)^{n-1,j}p^{(1-q)^j}}{q^{(1-p)^j}} (x-x_0)_{n-1-j,p} ^j \frac{y}{i(y-y_0)_{j,q}} ^j
\]

\[
= \frac{(1-p)^n}{(1-p)} (z-z_0)^{(n-1)}
\]

Similarly,

\[
\mathcal{A}_{q,y} (z-z_0)^{(n)} = \frac{(1-p)^n}{(1-p)} (z-z_0)^{(n-1)}
\]

and so \( (z-z_0)^{(n)} \) is \((p,q)\)-analytic and satisfies condition (i) above. Conditions (ii) and (iii) are trivially satisfied, completing the proof of the theorem.

By writing the finite series involved in the reverse order \( (z-z_0)^{(n)} \) can also be written as
(z-z_o)^{(n)} = \sum_{j=0}^{n-j} \frac{(1-p)^{n-p}(1-q)^{n-j}}{(1-p)^j(1-q)^{n-j}(1-p)^{n-j}(x-x_o)^j(p)i(y-y_o)^{n-j}}  

(\tau.3.2)

It is clear that (z-z_o)^{(n)} is consistent with definition of z to which it reduces when z_o = 0.

Also,

\lim_{p \to 1} \lim_{q \to 1} (z-z_o)^{(n)} = (z-z_o)^n.

It may be noted that unlike the case of z, the operator * does not in general preserve the property of addition of indices for discrete powers (z-z_o)^{(n)}. However the following holds:

**THEOREM 5.3.2.** If z_o is purely real then

(z-z_o)^{(m)} * (z-p z_o)^{(n)} = (z-z_o)^{(m+n)}

and if z_o is purely imaginary then

(z-z_o)^{(m)} * (z-q z_o)^{(n)} = (z-z_o)^{(m+n)}.
PROOF: If \( z_o = x_o \), \( x_o \) real, then \( (z-z_o)^{(n)} \) reduces to

\[
(z-x_o)^{(n)} = \sum_{j=0}^{n} \frac{(1-p)^{n-p} (1-q)^j}{(1-p)^{n-j} p^{(1-q)} j, q^{(1-p)} j} (x-x_o)^{n-j, p} (iy)
\]

and by (5.3.1),

\[
(z-x_o)^{(n)} = \sum_{j=0}^{n} \frac{(1-q)^j}{(1-q)^j, q} (iy)^j \mathcal{C}_{p, x} [(x-x_o)^{n, p}]
\]

\[
= \mathcal{C}_y [(x-x_o)^{n, p}].
\]

Hence,

\[
(z-x_o)^{(m)} * (z-p \ x_o)^{(n)} = \mathcal{C}_y [(x-x_o)^{m, p} (x-p \ x_o)^{n, p}]
\]

\[
= \mathcal{C}_y [(x-x_o)^{m+n, p}]
\]

\[
= (z-x_o)^{(m+n)}
\]

Similarly if \( z_o = iy_o \),

\[
(z-iy_o)^{(m)} * (z-i q \ y_o)^{(n)} = (z-iy_o)^{(m+n)}
\]

This completes the theorem.
Analogues of more general powers can be obtained. When the exponent is a negative integer or in fact an arbitrary scalar constant, infinite series are involved. Examples of this type are discussed in the next chapter.

§4. DISCRETE POLYNOMIALS — A 'discrete polynomial' is defined to be a finite sum of the form

\[ \sum_{j=0}^{n} a_j z^{(j)} \quad \ldots \quad (5.4.1) \]

where \( z \) is given by (5.1.7) and \( a_j \) are scalar constants. By Corollary 2.3.1, discrete polynomials are of course \((p,q)\)-analytic. If \( a_n \neq 0 \), the above polynomial is said to be of degree \( n \).

The coefficients \( a_j \) can be represented as follows:

If \( p_n(z) \) is given by

\[ p_n(z) = \sum_{j=0}^{n} a_j z^{(j)} \], then

\[ \sum_{j=m}^{n} [p_n(z)] = \sum_{j=m}^{n} \frac{a_j z^{(j)}}{(1-p)^m(1-p^{j-1}) \cdots (1-p^{j-m+1})} \quad (j-m) \]
Since \( \delta_{0,j} \begin{cases} 0, & j > 0 \\ 1, & j = 0, \end{cases} \) it follows that,

\[
\sum_{j=0}^{m} \left[ p_n(0) \right] = \frac{(1-p)^{m+1}}{(1-p)^m} a_m,
\]

and so,

\[
a_m = \frac{(1-p)^m}{(1-p)^{m+1}} \sum_{j=0}^{m} \left[ p_n(0) \right] z^j.
\]

Hence \( p_n(z) \) has a Maclaurin type representation,

\[
p_n(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)^{j+1}} \sum_{j=0}^{m} \left[ p_n(0) \right] z^j.
\]  \( \ldots (5.4.2). \)

This is similar to the \( q \)-Taylor series defined by Jackson [75] for functions of a continuous variable.