CHAPTER SIX
Chapter 6

Spectral study of some classes of self-complementary perfect graphs

This chapter is devoted to the spectral results of sc comparability graphs and sc chordal graphs. The following will be studied.

a) Some spectral properties of sc comparability graphs with respect to adjacency matrix are discussed.

b) Lower and upper bounds for the largest Laplacian eigenvalues of sc chordal graphs are obtained. Lower bound for the largest Laplacian eigenvalues of sc comparability graphs is also obtained.

c) Spectral properties of sc comparability graphs and sc chordal graphs with respect to Laplacian matrix are discussed.

The chapter grows through the following sections. In section 6.2, we show that no two non-isomorphic sc comparability graphs with \( n \) vertices \( (n < 13) \) are cospectral and there does not exist a hyper-energetic sc comparability graph for \( n < 13 \) vertices. In section 6.3, we obtain lower and upper bounds for the largest Laplacian eigenvalues for sc chordal graphs and sc comparability graphs. Same section deals with more spectral results with respect to Laplacian matrix for sc graphs, sc comparability graphs and sc chordal graphs.
Chapter 6

Spectral study of some classes of sc perfect graphs

6.1 Introduction

Finding pair of non-isomorphic graphs with the same spectrum, known as cospectral, is one of the earliest and continuing problems in spectral graph theory. Collatz and Sinogowitz [31] were the first to report that non-isomorphic graphs can have same set of eigenvalues. For various other results related to cospectral graphs we refer to [37], [38], [40] and [41]. Bearing this problem in mind we study the spectrum of sc graphs, sc comparability graphs and sc chordal graphs with respect to adjacency and Laplacian matrices of the graphs.

The concept of energy was introduced by Gutman [82], in connection to the so called total \( \pi \)-electron energy as \( E_n(G) = \sum_{i=1}^{n} |\mu_i| \), where \( \mu_1, \mu_2, \ldots, \mu_n \) are the eigenvalues of adjacency matrix of \( G \) [82].

6.2 Spectral results with respect to adjacency matrix

Sridharan and Balaji [160] studied sc chordal graphs with respect to adjacency matrix, showing that the least positive integer for which there exists non-isomorphic cospectral sc chordal graphs is 12. In this section, we consider sc comparability graphs only.

6.2.1 On the cospectrality of sc comparability graphs

In this section, we show that no two non-isomorphic sc comparability graphs with \( n \) vertices (\( n < 13 \)) have the same spectrum. Obviously, no two non-isomorphic sc comparability graphs with 4 or 5 vertices are cospectral, as there

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is only one sc comparability graph with 4 or 5 vertices shown in figure-3.11.

**Theorem 6.1.** No two non isomorphic sc comparability graphs with 8 vertices are cospectral.

**Proof.** Characteristic polynomials of the four non-isomorphic sc comparability graphs with 8 vertices shown in figure-3.12 are

- \( a. 1 - 4x - 6x^2 + 20x^3 + 11x^4 - 20x^5 - 14x^6 + x^8 \)
- \( b. 16x^2 + 40x^3 + 21x^4 - 16x^5 - 14x^6 + x^8 \)
- \( c. -8x - 8x^2 + 32x^3 + 37x^4 - 8x^5 - 14x^6 + x^8 \)
- \( d. 12x^2 + 40x^3 + 33x^4 - 8x^5 - 14x^6 + x^8 \)

We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc comparability graphs with 8 vertices have the same spectrum.

**Theorem 6.2.** No two non isomorphic sc comparability graphs with 9 vertices are cospectral.

**Proof.** Characteristic polynomials of the four non-isomorphic sc comparability graphs with 9 vertices shown in figure-3.13 are

- \( a. -5x + 16x^2 + 6x^3 - 40x^4 - 11x^5 + 32x^6 + 18x^7 - x^9 \)
- \( b. -32x^2 - 68x^3 - 25x^4 + 28x^5 + 18x^6 - x^9 \)
- \( c. -16x + 32x^2 + 44x^3 - 64x^4 - 61x^5 + 16x^6 + 18x^7 - x^9 \)
- \( d. 24x^2 + 8x^3 - 72x^4 - 57x^5 + 16x^6 + 18x^7 - x^9 \)

We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc comparability graphs with 9 vertices have the same spectrum.

**Theorem 6.3.** No two non isomorphic sc comparability graphs with 12 vertices are cospectral.

**Proof.** Characteristic polynomials of the fourteen non-isomorphic sc
comparability graphs with 12 vertices shown in figure-3.14 are

a. \[ 1 - 12x + 7x^2 + 80x^3 - 66x^4 + 172x^5 + 91x^6 + 192x^7 + 10x^8 - 92x^9 - 33x^{10} + x^{12} \]
b. \[ 16x^2 + 8x^3 - 115x^4 - 104x^5 + 187x^6 - 256x^7 + 16x^8 - 88x^9 - 33x^{10} + x^{12} \]
c. \[ 12x^2 + 8x^3 - 139x^4 + 96x^5 + 259x^6 + 344x^7 + 60x^8 - 80x^9 - 33x^{10} + x^{12} \]
d. \[ 81x^4 + 360x^5 + 607x^6 + 444x^7 + 72x^8 - 76x^9 - 33x^{10} + x^{12} \]
e. \[ -20x^2 + 128x^3 - 239x^4 + 12x^5 + 435x^6 + 464x^7 + 112x^8 - 68x^9 - 33x^{10} + x^{12} \]
f. \[ 16x^2 - 24x^3 + 151x^4 + 12x^5 + 419x^6 + 488x^7 + 140x^8 - 60x^9 - 33x^{10} + x^{12} \]
g. \[ -20x^2 - 160x^3 - 391x^4 - 272x^5 + 271x^6 + 484x^7 + 172x^8 - 52x^9 - 33x^{10} + x^{12} \]
h. \[ -36x^2 + 264x^3 - 671x^4 + 644x^5 + 43x^6 + 448x^7 + 184x^8 - 52x^9 + 33x^{10} + x^{12} \]
i. \[ 12x^2 - 8x^3 - 123x^4 - 52x^5 + 351x^6 + 492x^7 + 176x^8 - 48x^9 - 33x^{10} + x^{12} \]
j. \[ -3 + 55x^2 + 64x^3 - 202x^4 + 340x^5 + 135x^6 + 468x^7 + 198x^8 - 44x^9 - 33x^{10} + x^{12} \]
k. \[ -48x^2 - 179x^4 - 116x^5 + 295x^6 + 484x^7 + 196x^8 - 40x^9 - 33x^{10} + x^{12} \]
l. \[ 48x^2 + 80x^3 + 211x^4 - 436x^5 + 39x^6 + 452x^7 + 212x^8 - 40x^9 - 33x^{10} + x^{12} \]
m. \[ -72x^2 - 291x^4 - 360x^5 + 175x^6 + 468x^7 + 204x^8 - 40x^9 + 33x^{10} + x^{12} \]
n. \[ 45x^2 + 252x^3 + 535x^4 + 516x^5 + 180x^6 - 40x^7 - 33x^{10} + x^{12} \]

Note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc comparability graphs with 12 vertices has the same spectrum.

Theorem 6.4. There exist non-isomorphic cospectral sc comparability graphs with 13 vertices.

Proof. Consider the two sc graphs with 13 vertices given in figure-6.1, since their degree sequences are different they are non-isomorphic. These graphs are comparability graphs as there exist transitive orientation.
The characteristic polynomials associated with these sc comparability graphs are given below.

a. \[24x^2 + 248x^3 + 940x^4 + 1633x^5 + 1142x^6 - 229x^7 - 736x^8 - 250x^9 + 70x^{10} + 39x^{11} - x^{13}\]

b. \[24x^2 + 248x^3 + 940x^4 + 1633x^5 + 1142x^6 - 229x^7 - 736x^8 - 250x^9 + 70x^{10} + 39x^{11} - x^{13}\]

Since spectrums are same for these graphs, Thus they are cospectral.

Combining Theorems 6.1, 6.2, 6.3 and 6.4, we get the following result.

**Theorem 6.5.** The smallest positive integer for which there exist cospectral sc comparability graphs is 13.

**6.2.2 Energy of sc comparability graphs**

If for two graphs \(G_1\) and \(G_2\), the equality \(E_n(G_1) = E_n(G_2)\) is satisfied, then \(G_1\) and \(G_2\) are said to be *equi-energetic* [135]. Obviously "Cospectral graphs are equi-energetic but converse need not be true."

Since there exist only one sc comparability graph on 4 or 5 vertices, so there are no equi-energetic sc comparability graphs on 4 or 5 vertices. Therefore, we check for equi-energetic sc comparability graphs for 8 or more vertices.

**Theorem 6.6.** There exist non-isomorphic non-cospectral equi-energetic sc comparability graphs with 8 vertices.

**Proof.** Consider two sc graphs as shown in Figure-3.12(b) and Figure-3.12(d). These two graphs are non-isomorphic, as the graph in Figure 3.12(b) has four vertices of degree 2 while graph in Figure-3.12(d) does not have any vertex of degree 2. These graphs are comparability graphs as there exists transitive orientation. By Theorem 6.1, their spectrums are different so they are non-
cospectral. The energies of these graphs are 11.1231, so these graphs are equi-energetic. Hence there exist non-isomorphic non-cospectral equi-energetic sc comparability graphs with 8 vertices.

Thus, we have the following result.

**Theorem 6.7.** The smallest positive integer for which there exist non-isomorphic non-cospectral equi-energetic sc comparability graphs is 8.

The spectrum of complete graph $K_n$ is $\{n-1, -1, -1, \ldots, -1\}$. So the energy of complete graph $K_n$ is $2(n - 1)$. If the energy of a graph $G$ is greater than $2(n - 1)$, i.e., energy of complete graph, then graph is said to be *hyper-energetic* [82]. Gutman [82] conjectured in 1978 that there is no hyper-energetic graph. The same has been disproved by Walikar et al. [173].

Koolen and Moulton [94] gave the following upper bound of energy of graphs.

**Theorem 6.8[94].** For a graph $G$ with $n$ vertices and $m$ edges, the energy $E_n(G)$ satisfies the condition

$$E_n(G) \leq \frac{2m}{n} + \sqrt{(n-1) \left[ \frac{2m - \left(\frac{2m}{n}\right)^2}{n} \right]}.$$

Now, we obtain the following result.

**Theorem 6.9.** A sc graph $G$ with $n$ vertices ($n \leq 8$) cannot be hyper-energetic.

**Proof.** Let $G$ be sc graph with $n$ vertices. Then by Theorem 6.8, we have
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\[ E'_n(G) \leq \frac{2m}{n} + \sqrt{(n-1) \left( \frac{2m}{n} \right)^2}. \]

Since the graph is sc, so \( m = \frac{n(n-1)}{4} \).

Using this value of \( m \) in above inequality, we get

\[ E'_n(G) \leq \frac{n-1}{2} \left( 1 + \sqrt{(n+1)} \right). \]

If the graph is not hyper-energetic, then following condition holds.

\[ E'_n(G) \leq \frac{n-1}{2} \left( 1 + \sqrt{(n+1)} \right) \leq 2(n-1), \]

or \( \frac{n-1}{2} \left( 1 + \sqrt{(n+1)} \right) \leq 2(n-1), \)

or \( \left( 1 + \sqrt{(n+1)} \right) \leq 4, \)

or \( (n+1) \leq 9, \) or \( n \leq 8 \)

Hence no sc graph is hyper-energetic with \( n \leq 8 \). \hfill \Box

From the Theorem 6.9, it is clear that there does not exist hyper-energetic sc graph on 4, 5 and 8 vertices. Now we check for hyper-energetic sc comparability graphs for \( n > 8 \).

**Theorem 6.10.** No sc comparability graph with 9 vertices is hyper-energetic.

**Proof.** Energies of the four non-isomorphic sc comparability graphs with 9 vertices shown in Figure-3.13 are 13.0712, 12.5262, 14.7123, and 15.1231. Clearly none of the graphs have energy more than \( 2(n-1) = 16 \). Hence, no non-isomorphic sc comparability graph with 9 vertices is hyper-energetic. \hfill \Box
Theorem 6.11. No sc comparability graph with 12 vertices is hyper-energetic.

Proof. Below we give the energies of 14 non-isomorphic sc comparability graphs with 12 vertices, shown in Figure-3.14.

a) 19.1128  b) 18.6082  c) 18.8042  d) 17.893  e) 19.1164
f) 19.161  g) 19.4695  h) 19.9059  i) 19.1242  j) 20.3072
k) 18.9625  l) 20.0305  m) 19.2901  n) 17.893

We note that none of the graphs have energy more than $2(n - 1) = 22$.

Hence no non-isomorphic sc comparability graph with 12 vertices is hyper-energetic.

Theorem 6.12. There exists a hyper-energetic sc comparability graph with 13 vertices.

Proof. Consider a sc graph on 13 vertices as shown in figure-6.2. It is comparability since there exists transitive orientation. Energy of this graph is 24.1439, which is greater than $E(K_{13}) = 2(13 - 1) = 24$. Thus, the graph is hyper-energetic. Hence the result.

Figure-6.2

Combining the above arguments, we have the following result.

Theorem 6.13. The smallest positive integer for which there exists a hyper-energetic sc comparability graph is 13.
6.2.3 Catalogue and some observations

Now we present the catalogue for equi-energetic, cospectral, hyper-energetic sc comparability graphs. We also include the details of sc comparability graphs with $\mu_n \geq -2$, which have been investigated in much detail in [2], [3] and [4].

<table>
<thead>
<tr>
<th>Vertices</th>
<th>Sc comp. graphs</th>
<th>Equi-energetic sc comparability graphs</th>
<th>Hyper-energetic sc comparability Graphs</th>
<th>Cospectral sc comparability graphs</th>
<th>sc comparability graphs for which $\mu_n \geq -2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>01</td>
<td>00</td>
<td>00</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>5</td>
<td>01</td>
<td>00</td>
<td>00</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>8</td>
<td>04</td>
<td>*02 Energy = 11.1231</td>
<td>00</td>
<td>00</td>
<td>01</td>
</tr>
<tr>
<td>9</td>
<td>04</td>
<td>0</td>
<td>00</td>
<td>00</td>
<td>00</td>
</tr>
<tr>
<td>12</td>
<td>14</td>
<td>*02 Energy = 17.893</td>
<td>00</td>
<td>00</td>
<td>00</td>
</tr>
<tr>
<td>13</td>
<td>31</td>
<td>*02 Energy = 21.8682 *02 Energy = 22.8333 *02 Energy = 22.8934</td>
<td>01 Energy = 24.1439</td>
<td>3 pairs</td>
<td>00</td>
</tr>
</tbody>
</table>

*non-cospectral equienergetic graphs
#cospectral equienergetic graphs

Table - 6.1

We observe that for $n = 13$, 3 pairs of graphs are cospectral graphs. There are no other graphs which are cospectral on $n = 4, 5, 8, 9, 12$ vertices.

There are 2 graphs on $n = 8$ and $n = 12$ and 3 pairs on $n = 13$, which are equi-energetic. But for $n = 8$ and $n = 12$, these equi-energetic graphs are non cospectral. Hence this verifies that cospectral graphs have same energy but converse need not be true.

From the behavior of equi-energetic graphs it can be observed that for $n = 4k + 1$, there do not exist equi-energetic sc comparability graphs, which are non-cospectral. Note that for $n = 13$ all the equienergetic graphs are cospectral.
From the final catalogue we observe that there exist sc comparability graphs with 4, 5, 8 vertices for which \( \mu_n \geq -2 \), but from \( n = 9 \) onwards there does not exist any sc comparability graph satisfying \( \mu_n \geq -2 \).

We conclude the section with the following conjectures.

**Conjecture 1.** For \( n = 4k + 1 \) vertices, no non-isomorphic non-cospectral sc comparability graph is equi-energetic.

**Conjecture 2.** For \( n > 8 \) vertices there does not exist any sc comparability graph for which \( \mu_n \geq -2 \).

### 6.3 Spectral result with respect to Laplacian matrix

Gutman et al. [80] and [81] discovered connection between photoelectron spectra of saturated hydrocarbons (alkanes) and the Laplacian eigenvalues of underlying molecular graphs. So it is significant and necessary to investigate the relations between the graph theoretic properties of \( G \) and its eigenvalues.

We shall assume that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \). It is well known that \( \lambda_n = 0 \) and the multiplicity of zero equals to the number of connected components in \( G \). Since sc graph is always connected so for a sc graph \( G \), multiplicity of zero is always 1. Matrix \( |L| = D + A \) is called the signless Laplacian matrix. We denote cospectral graphs with respect to Laplacian matrix as L-cospectral, equi-energetic as L-equie-energetic and characteristic polynomial of Laplacian matrix as LCP. In this section we extend our study and also consider sc graphs, sc chordal graphs apart from sc comparability graphs.
6.3.1 Some bounds on Laplacian eigenvalues

The triangle number \( t(u) \) of a vertex \( u \) in a graph \( G \) is the number of triangles in \( G \) containing \( u \), while \( t(G) \) is the total number of triangles in \( G \). \( N(u) \) is the neighbors of vertex \( u \). Till now, plenty of upper bounds on the largest Laplacian eigenvalue of graphs have been given [2], [100], [101], [112], [145], [157] and [178]. But lower bounds for largest Laplacian eigenvalue \( \lambda_1 \) were given by Grone and Merris [74] and Li and Pan [99]. Nikiforov [123] obtained lower bounds for graphs characterized by forbidden induced subgraphs.

Turan theorem determines the maximum number of edges that a simple graph on \( n \) vertices can have without containing a clique of size \( r + 1 \) [12]. In this section, first we obtain lower bound for sc chordal graph and sc comparability graphs, for this we require the following results.

**Theorem 6.14**[122]. Let \( G \) be any graph. Then \( \sum_{u \in V} t(u) = 3t(G) \)

**Theorem 6.15** [45] If \( G \) is graph with \( n \) vertices and \( m \) edges. Then

\[
\sum_{u \in V} d^2(u) \geq \frac{4m^2}{n}
\]

The following result relating \( \lambda_1 \), the largest Laplacian eigenvalue of graph and \( t(G) \) the number of triangles in \( G \) is due to Nikiforov [123],

**Theorem 6.16**[123]. If \( G \) is a graph with \( n \) vertices and \( m \) edges then

\[
6nt(G) \geq (n + \lambda_1) \sum_{u \in V} d^2(u) - 2mn\lambda_1.
\]

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\( ^2 \) Submitted for publication.
Now, we obtain bounds on largest Laplacian eigenvalue for sc chordal graphs in terms of \( n \).

**Theorem 6.17.** Let \( G \) be a sc chordal graph then

\[
\lambda_1 \geq \frac{2n(n-1)}{(n-2)(n+1)} \text{ for } n = 4k,
\]

\[
\lambda_1 \geq \frac{n^2 - 3}{n+1} \text{ for } n = 4k + 1
\]

**Proof.** For \( n = 4k \)

By Theorem 218., for sc chordal graph \( \omega(G) = 2k \). It implies that \( G \) is \( K_{2k+1} \)-free. Therefore, \( N(u) \) induces \( K_{2k} \)-free graph for each \( u \) belongs to \( V \).

Then by Turan's Theorem

\[
t(u) \leq \frac{2k-2}{2(2k-1)} d^2(u).
\]

Summing the above inequality for all \( u \) belongs to \( V \), we get

\[
\sum_{u \in V} t(u) \leq \frac{2k-2}{2(2k-1)} \sum_{u \in V} d^2(u).
\]

Using Theorem 6.14, we get

\[
3t(G) \leq \frac{2k-2}{2(2k-1)} \sum_{u \in V} d^2(u),
\]

\[
\Rightarrow 6t(G) \leq \frac{2k-2}{(2k-1)} \sum_{u \in V} d^2(u).
\]

Using Theorem 6.16 and above inequality, we get

\[
n \left( \frac{2k-2}{2k-1} \right) \sum_{u \in V} d^2(u) \geq (n + \lambda_i) \sum_{u \in V} d^2(u) - 2mn\lambda_i
\]
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\[ 2mn\lambda_1 \geq \left( (n + \lambda_1) - n \left( \frac{2k-2}{2k-1} \right) \right) \sum_{u \in V} d^2(u). \]

In the view of Theorem 6.15,

\[ 2mn\lambda_1 \geq \left( (n + \lambda_1) - n \left( \frac{2k-2}{2k-1} \right) \right) \frac{4m^2}{n} \]

\[ \Rightarrow \lambda_1(n^2 - 2m) \geq \frac{2mn}{2k-1}. \]

Since \( G \) is sc, so \( m = \frac{n(n-1)}{4} \), so the inequality reduces to

\[ \lambda_1 \left( n^2 - \frac{n(n-1)}{2} \right) \geq \frac{n(n-1)}{2} \frac{n}{2k-1} \]

(6.1)

Putting, \( 2k = n/2 \), as \( n = 4k \) and solving, the lower bound on \( \lambda_1 \) is obtained as

\[ \lambda_1 \geq \frac{2n(n-1)}{(n-2)(n+1)}. \]

Hence completing the proof for \( n = 4k \).

For \( n = 4k + 1 \)

Again, by Theorem 2.17, for sc graph on \( n = 4k + 1 \) vertices \( \omega(G) = 2k + 1 \). It implies that \( G \) is \( K_{2k+2} \)-free. Therefore, \( N(u) \) induces \( K_{2k+1} \)-free graphs for each \( u \) belongs to \( V \). Thus, again by Turan’s Theorem,

\[ t(u) \leq \frac{2k+1-2}{2(2k+1-1)} d^2(u), \]

\[ t(u) \leq \frac{2k-1}{4k} d^2(u). \]

Summing the inequality for all \( u \) belongs to \( V \).
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\[
\sum_{u \in V} t(u) \leq \frac{2k - 1}{4k} \sum_{u \in V} d^2(u).
\]

\[
\Rightarrow 12t(G) \leq \frac{2k - 1}{k} \sum_{u \in V} d^2(u)
\]

Using Theorem 6.16, we get the following inequality

\[
\left( \frac{2k - 1}{2k} \right) \sum_{u \in V} d^2(u) \geq (n + \lambda_1) \sum_{u \in V} d^2(u) - 2mn\lambda_1.
\]

Now, solving the inequality as solved for \( n = 4k \), the following bound is obtained,

\[
\lambda_1 \geq \frac{(n^2 - 3)}{(n + 1)}.
\]

Hence the Theorem.

These bounds are always true for \( X \)-free sc chordal graph, sc strongly chordal graph and sc chordal comparability graphs as these graphs are also chordal. But for those sc comparability graphs which are not chordal, we need another bound to be obtained. For this we take the advantage of a result due to Sridharan and Balaji [161] which states that

**Theorem 6.18 [161].** Let \( G \) be sc graph, then

\[
\omega(G) \leq 2n \text{ if } n = 4k
\]

\[
\omega(G) \leq 2n + 1 \text{ if } n = 4k + 1.
\]

Since equality holds if and only if \( G \) is sc chordal, so for sc comparability graphs, which are not sc chordal we have

\[
\omega(G) < 2k \text{ if } n = 4k \text{ and } \omega(G) < 2k + 1 \text{ if } n = 4k + 1.
\]

This gives another bound for sc comparability graphs.
Theorem 6.19. Let $G$ be a sc comparability graph which is not chordal then

$$\lambda_1 \geq \frac{2n(n-1)}{(n-4)(n+1)} \quad \text{if } n = 4k,$$

$$\lambda_1 \geq \frac{2n(n-1)}{(n-3)(n+1)} \quad \text{if } n = 4k + 1.$$ 

Proof. For $n = 4k$

Since graph $G$ is not chordal so $\omega(G) < 2k$. This implies that $G$ is $K_{2k}$-free. So for each vertex $u$ belongs to $V$, $N(u)$ induces $K_{2k-1}$-free graph. Hence, by Turan’s Theorem

$$t(u) \leq \frac{2k - 3}{4(k - 1)} d^2(u)$$

$$\Rightarrow \sum_{u \in V} t(u) \leq \frac{2k - 3}{4(k - 1)} \sum_{u \in V} d^2(u)$$

$$\Rightarrow 12t(G) \leq \frac{2k - 3}{k - 1} \sum_{u \in V} d^2(u).$$

Now, solving as earlier cases we get the inequality

$$\lambda_1 \geq \frac{2n(n-1)}{(n-4)(n+1)}.$$ 

For $n = 4k + 1$

Since graph $G$ is not chordal so $\omega(G) < 2k + 1$. This implies that $G$ is $K_{2k+1}$-free. So for each vertex $u$ belongs to $V$, $N(u)$ induces $K_{2k}$-free graph. So this case is similar as Theorem 6.17 for $n = 4k$ up to inequality (6.1)

$$\lambda_1 \geq \frac{n(n-1)}{2} \geq \frac{2}{2k-1}.$$
Putting, \( 2k = (n-1)/2 \), as \( n = 4k + 1 \), we get

\[
\lambda_1 \geq \frac{2n(n-1)}{(n-3)(n+1)}.
\]

**Remark:** In fact Theorem 6.19 is true for the class of sc graphs which is not chordal.

Now, we obtain upper bound for the largest Laplacian eigenvalue of sc chordal graph. For this, we need the following results

**Lemma 6.20** [179]. Let \( G \) be a graph. Then

\[
\lambda_1(G) \leq \lambda_1(\bar{G}).
\]

Grone and Merris [74] proved the following lower bound for largest Laplacian eigenvalue, \( \lambda_1 \) in terms of largest degree, \( d_1 \).

**Lemma 6.21** [74]. Let \( G \) be a connected graph with \( n \) vertices and degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_n \). Then

\[
\lambda_1 \geq d_1 + 1.
\]

equality holds if and only if \( d_1 = n - 1 \).

For sc graph, we have following result.

**Lemma 6.22** [161]. Let \( G \) be a sc graph with degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_n \). Then

\[
d_i + d_{n+1-i} = n - 1 \text{ where } 1 \leq i \leq n
\]

From Lemma 6.22, \( d_1 = n - 1 \) never holds for a sc graph as \( d_i \neq 0 \). So, in the view of Lemma 6.21, we have the following result.

**Lemma 6.23** Let \( G \) be a sc graph with \( n \) vertices and degree
sequence $d_1 \geq d_2 \geq ... \geq d_n$. Then
\[ \lambda_i > d_i + 1 \] (6.2)

**Theorem 6.24[121].** Let $F = \{(x_1, x_2, ..., x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$. Then
\[ 1 - \frac{1}{\omega(G)} = \max_{x \in F} \langle x, Ax \rangle. \]

The following theorem gives the upper bound for largest Laplacian eigenvalues of sc chordal graphs.

**Theorem 6.25.** Let $G$ be a sc chordal graph on $n$ vertices with degree sequence $d_1 \geq d_2 \geq ... \geq d_n$. Then
\[ \lambda_1 \leq \sqrt{(n-1)(2n-1)} + d. \]

**Proof.** Let $(y_1, y_2, ..., y_n)$ be normalized eigenvector corresponding to $\lambda_1(\Delta)$, then
\[ \lambda_1(\Delta) = \sum_{y \in E} (y_i + y_j)^2 = \sum_{i \in V} y_i^2 + \sum_{y \in E} 2y_i y_j. \]
\[ \Rightarrow \lambda_1(\Delta) \leq d_1 \sum_{i \in V} y_i^2 + \sum_{y \in E} 2y_i y_j \]
Since $\sum_{i=1}^{n} y_i^2 = 1$, so
\[ \lambda_1(\Delta) \leq d_1 + \sum_{y \in E} 2y_i y_j \]
\[ \lambda_1(\Delta) - d_1 \leq \sum_{y \in E} 2y_i y_j. \]
By Lemma 6.20,
\[ \lambda_1(G) \leq \lambda_1(\Delta) \Rightarrow \lambda_1(\Delta) - d_1 \geq \lambda_1(G) - d_1 \text{ and } \lambda_1(G) - d_1 > 1. \]
\[ \lambda_i(|L|) - d_i > 1. \]

\[ \lambda_i(G) - d_i \leq \sum_{y \in E} 2y_i y_j \]

\[ (\lambda_i(G) - d_i) \leq \left( \sum_{y \in E} 2y_i y_j \right)^2. \]

Now, applying Cauchy inequality, we get

\[ (\lambda_i(G) - d_i) \leq \left( \sum_{y \in E} y_i^2 y_j^2 \right)^2 \leq 2m(2 \sum_{y \in E} y_i^2 y_j^2) \]

\[ = \frac{n(n-1)}{2} \left( 2 \sum_{y \in E} y_i^2 y_j^2 \right) \text{ as } m = \frac{n(n-1)}{4} \]

\[ \Rightarrow (\lambda_i(G) - d_i) \leq n(n-1) \left( \sum_{y \in E} y_i^2 y_j^2 \right) \]

\[ \Rightarrow \frac{(\lambda_i(G) - d_i)^2}{n(n-1)} \leq \left( \sum_{y \in E} y_i^2 y_j^2 \right). \quad (6.3) \]

Since \((y_1^2, y_2^2, ..., y_n^2) \geq 0\) and \(y_1^2 + y_2^2 + ... + y_n^2 = 1\), then by Theorem 6.24,

\[ 2 \sum_{y \in E} y_i^2 y_j^2 \leq 1 - \frac{1}{\omega(G)} = 1 - \frac{1}{2n}. \quad (6.4) \]

\[ \Rightarrow \sum_{y \in E} y_i^2 y_j^2 \leq \frac{1}{2} \left( 1 - \frac{1}{2n} \right). \]

In the view of (6.3) and (6.4)

\[ \frac{(\lambda_i(G) - d_i)^2}{n(n-1)} \leq \frac{1}{2} \left( 1 - \frac{1}{2n} \right) \]

\[ \Rightarrow 4(\lambda_i(G) - d_i) \leq (n-1)(2n-1) \]

\[ \Rightarrow (\lambda_i(G) - d_i) \leq \frac{(n-1)(2n-1)}{2} \]

\[ \Rightarrow \lambda_i \leq \frac{(n-1)(2n-1)}{2} + d_i. \]

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This proves the result. □

**Sum of two largest Laplacian eigenvalues of sc chordal graphs**

Li and Pan [99], proved the following lower bound for second largest Laplacian eigenvalue,

\[ \lambda_2 \geq d_2. \]  

(6.5)

Next, we give strict lower bound for the sum of two largest Laplacian eigenvalues of sc chordal graph.

**Theorem 6.26.** Let \( G \) be sc chordal graph, then

\[
\begin{align*}
\lambda_1 + \lambda_2 &> 6k^2 - 2k - \sum_{i=3}^{2k} d_i + 1 \quad \text{if } n = 4k, \\
\lambda_1 + \lambda_2 &> 6k2 - \sum_{i=3}^{2k} d_i + 1 \quad \text{if } n = 4k + 1.
\end{align*}
\]

**Proof.** Let \( G \) be sc chordal graph with degree sequence \( d_1 \geq d_2 \geq \ldots \geq d_n \).

Suppose the Laplacian eigenvalues are \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0 \).

Adding inequality (6.2) and (6.5), we get

\[ \lambda_1 + \lambda_2 > d_1 + d_2 + 1. \]  

(6.6)

By Corollary 3.9 , \( \sum_{i=1}^{2k} d_i = 6k^2 - 2k \) if \( n = 4k \) and \( \sum_{i=1}^{2k} d_i = 6k^2 \) if \( n = 4k + 1 \) for sc chordal graphs.

Now, for \( n = 4k \)

\[
\sum_{i=3}^{2k} d_i + d_1 + d_2 = 6k^2 - 2k
\]

\[
d_1 + d_2 = 6k^2 - 2k - \sum_{i=3}^{2k} d_i.
\]
Using inequality (6.6), we get
\[ \lambda_1 + \lambda_2 > 6k^2 - 2k - \sum_{i=3}^{2k} d_i + 1. \]  
(6.7)

Similarly, for \( n = 4k + 1 \),
\[ \lambda_1 + \lambda_2 > 6k^2 - \sum_{i=3}^{2k} d_i + 1. \] Hence the result. \( \square \)

**Theorem 6.27** \[118\]. If \( \bar{G} \) is complement of graph \( G \). Then
\[ \lambda_k(\bar{G}) = n - \lambda_{n-k}(G) \]

Now we obtain strict upper bound for smallest Laplacian eigenvalue of sc chordal graph.

**Theorem 6.28**. If \( G \) be a sc chordal graph. Then
\[ 0 < \lambda_{n-1} < \sum_{i=3}^{2k} d_i - (6k^2 - 2k + 1) + n + \lambda_2 \text{ for } n = 4k, \]
\[ 0 < \lambda_{n-1} < \sum_{i=3}^{2k} d_i - (6k^2 + 1) + n + \lambda_2 \text{ for } n = 4k + 1. \]

**Proof.** In Theorem 6.27, put \( k = 1 \), we get
\[ \lambda_1(\bar{G}) = n - \lambda_{n-1}(G). \]

For a sc graph \( G \), \( \bar{G} \cong G \). Thus, \( \lambda_1(G) = n - \lambda_{n-1}(G) \), i.e., \( \lambda_1 = n - \lambda_{n-1} \)
\[ \Rightarrow \lambda_1 + \lambda_{n-1} = n \]  
(6.8)

In the view of inequalities (6.7) and (6.8)
\[ n - \lambda_{n-1} + \lambda_2 > 6k^2 - 2k - \sum_{i=3}^{2k} d_i + 1 \]
\[ \lambda_{n-1} < \sum_{i=3}^{2k} d_i - (6k^2 - 2k + 1) + n + \lambda_2. \]  
(6.9)
Inequality (6.9) gives the upper bound for the smallest Laplacian eigenvalue (algebraic connectivity, different from zero) for sc chordal graphs on \( n = 4k \) vertices. Also \( \lambda_{n-1} > 0 \) for sc graphs as sc graphs are always connected. Thus, \( 0 < \lambda_{n-1} < \sum_{i=3}^{2k} d_i - (6k^2 - 2k + 1) + n + \lambda_2 \) for \( n = 4k \). (6.10)

Similarly, for \( n = 4k+1 \) vertices, we have

\[
0 < \lambda_{n-1} < \sum_{i=3}^{2k} d_i - (6k^2 + 1) + n + \lambda_2 \quad \text{for} \quad n = 4k + 1.
\] (6.11)

### 6.3.2 On the L-cospectrality of sc, sc comparability and sc chordal graphs

#### On L-cospectrality of sc graphs

Since, there exists only one sc graph on 4 vertices, no L-cospectral sc graph on \( n = 4 \) vertices exist. For \( n > 4 \), we have following results.

**Theorem 6.29.** No two non isomorphic sc graphs with 5 vertices are L-cospectral.

**Proof.** LCPs of the two non-isomorphic sc graphs (\( C_5 \), Bull) with 5 vertices are given below.

\[-15x + 40x^2 - 33x^3 + 10x^4 - x^5 \quad \text{and} \quad -25x + 50x^2 - 35x^3 + 10x^4 - x^5 \]

We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence, no pair of non-isomorphic sc graphs with 5 vertices has the same spectrum. □

**Theorem 6.30.** No two non isomorphic sc graphs with 8 vertices are L-cospectral.

**Proof.** All the 10 non-isomorphic sc graphs with 8 vertices and their LCPs are

\[
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\]
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Given below.

\[ a. -2688 x + 8480 x^2 - 10096 x^3 + 5876 x^4 - 1840 x^5 + 316 x^6 - 28 x^7 + x^8 \]
\[ b. -4608 x + 11904 x^2 - 12224 x^3 + 6480 x^4 - 1920 x^5 + 320 x^6 - 28 x^7 + x^8 \]
\[ c. -4800 x + 12080 x^2 - 12272 x^3 + 6484 x^4 - 1920 x^5 + 320 x^6 - 28 x^7 + x^8 \]
\[ d. -6272 x + 15008 x^2 - 14256 x^3 + 7076 x^4 - 2000 x^5 + 324 x^6 - 28 x^7 + x^8 \]
\[ e. -6336 x + 15024 x^2 - 14256 x^3 + 7076 x^4 - 2000 x^5 + 324 x^6 - 28 x^7 + x^8 \]
\[ f. -6528 x + 15200 x^2 - 14304 x^3 + 7080 x^4 - 2000 x^5 + 324 x^6 - 28 x^7 + x^8 \]
\[ g. -8704 x + 18816 x^2 - 16480 x^3 + 7688 x^4 - 2080 x^5 + 328 x^6 - 28 x^7 + x^8 \]
\[ h. -8192 x + 18432 x^2 - 16384 x^3 + 7680 x^4 - 2080 x^5 + 328 x^6 - 28 x^7 + x^8 \]
\[ i. -9216 x + 19200 x^2 - 16576 x^3 + 7696 x^4 - 2080 x^5 + 328 x^6 - 28 x^7 + x^8 \]
\[ j. -9408 x + 19376 x^2 - 16624 x^3 + 7700 x^4 - 2080 x^5 + 328 x^6 - 28 x^7 + x^8 \]

We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc graphs with 8 vertices has the same spectrum.

**Theorem 6.31.** There exist non-isomorphic L-cospectral sc graphs with 9 vertices.

**Proof.** Consider the following two graphs of sc graphs shown in figure 6.4. The LCPs associated with these sc graphs are given below.
Since spectrums are same for these graphs, they are L-cospectral. Hence the result.

In the view of Theorem 6.31, we have the following result.

**Theorem 6.32.** The smallest positive integer for which there exist non-isomorphic L-cospectral sc graphs is 9.

**On L-cospectrality of sc comparability graphs**

Since, there exists only one sc comparability graph on 4 and 5 vertices, obviously, no L-cospectral sc comparability graph on $n = 4$ or $n = 5$ vertices exist. For $n > 5$, we have following results.

**Corollary 6.33.** No two non-isomorphic sc comparability graphs with 8 vertices are L-cospectral.

**Proof.** Follows from Theorem 6.30.

**Theorem 6.34.** No two non-isomorphic sc comparability graphs with 9 vertices are L-cospectral.

**Proof.** LCPs of the 4 non-isomorphic sc comparability graphs with 8 vertices, shown in figure 3.12, are
We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence, no pair of non-isomorphic sc graphs with 8 vertices has the same spectrum.

**Theorem 6.35.** There exist non-isomorphic L-cospectral sc comparability graphs with 12 vertices.

**Proof.** Consider the two non-isomorphic sc comparability graphs shown in figure-3.14(g) and 3.14(h). The characteristic polynomials associated with these sc comparability graphs are given below.

\[
\begin{align*}
\text{a. } & -20349 x + 66924 x^2 - 85358 x^3 + 55224 x^4 - 20159 x^5 + 4320 x^6 - 538 x^7 + 36 x^8 - x^9 \\
\text{b. } & -35280 x + 94752 x^2 - 104488 x^3 + 61704 x^4 - 21329 x^5 + 4428 x^6 - 542 x^7 + 36 x^8 - x^9 \\
\text{c. } & -76320 x + 169704 x^2 - 157528 x^3 + 80388 x^4 - 24797 x^5 + 4752 x^6 - 554 x^7 + 36 x^8 - x^9 \\
\text{d. } & -72000 x + 165600 x^2 - 156100 x^3 + 80172 x^4 - 24785 x^5 + 4752 x^6 - 554 x^7 + 36 x^8 - x^9
\end{align*}
\]

Since spectrums are same for these graphs, they are L-cospectral. Hence the result.

Thus, from the above discussion we have following result.

**Theorem 6.36.** The smallest positive integer for which there exist non-isomorphic L-cospectral sc comparability graphs is 12.

**On L-cospectrality of sc chordal graphs**

Since, there exists only one sc chordal graph on 4 and 5 vertices, no L-cospectral sc chordal graph on \( n = 4 \) or \( n = 5 \) vertices exist. For \( n > 5 \), we have following results.

**Corollary 6.37.** No two non-isomorphic sc chordal graphs with 8 vertices are
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L-cospectral.

Proof. Follows from Theorem 6.30.

Theorem 6.38. No two non-isomorphic sc chordal graphs with 9 vertices are L-cospectral.

Proof. All the 3 non-isomorphic sc chordal graphs with 9 vertices and their LCPs are given below.

![Graphs](a) ![Graphs](b) ![Graphs](c)

Figure-6.5

a. \(-20349 x + 66924 x^2 - 85358 x^3 + 55224 x^4 - 20159 x^5 + 4320 x^6 - 538 x^7 + 36 x^8 - x^9\)
b. \(-35280 x + 94752 x^2 - 104488 x^3 + 61704 x^4 - 21329 x^5 + 4428 x^6 - 542 x^7 + 36 x^8 - x^9\)
c. \(-36288 x + 95904 x^2 - 104940 x^3 + 61776 x^4 - 21333 x^5 + 4428 x^6 - 542 x^7 + 36 x^8 - x^9\)

We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc graphs with 9 vertices has the same spectrum.
Theorem 6.39. No two non-isomorphic sc chordal graphs with 12 vertices are L-cospectral.

Proof. All 16 non-isomorphic sc chordal graphs with 12 vertices and their LCPs are shown below.
We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc graphs with 12 vertices has the same Laplacian spectrum.

**Theorem 6.40.** No two sc non-isomorphic chordal graphs with 13 vertices are L-cospectral.

**Proof.** All 16 non-isomorphic sc chordal graphs with 13 vertices and their LCPs are shown in figure-6.7.
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Figure 6.7
We note that none of the above polynomials can be obtained from the other polynomials by multiplying by a real number. Hence no pair of non-isomorphic sc graphs with 13 vertices has the same Laplacian spectrum.

We conclude with the following result.

**Theorem 6.41.** There do not exist $L$-cospectral sc chordal graphs up to 13 vertices.
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In view of above results, we report the following conjecture.

**Conjecture 3.** There do not exist L-cospectral sc chordal graphs on \( n = 4k \) or \( n = 4k + 1 \) vertices.

### 6.3.3 Laplacian Energy of sc, sc comparability and sc chordal graphs

Let \( G \) be a graph on \( n \) vertices and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of Laplacian matrix of \( G \). Gutman [79] defined the Laplacian energy of \( G \) as

\[
LE(G) = \sum_{i=1}^{n} \sqrt{\lambda_i - \frac{2m}{n}}.
\]

Let \( G_1 \) and \( G_2 \), be two graphs such that \( LE(G_1) = LE(G_2) \), then we call \( G_1 \) and \( G_2 \) L-equi-energetic.

**On L-equi-energetic sc graphs**

Since there exist only one sc graph on 4 vertices, we start with \( n = 5 \) vertices.

**Theorem 6.42.** No two non isomorphic sc graphs with 5 vertices are L-equi-energetic.

**Proof.** There exist only two sc graphs (Bull, \( C_5 \)) on \( n = 5 \) vertices and the Laplacian energies of Bull and \( C_5 \) are 7.84162 and 6.47212. Clearly they are not equal. Thus, no two non-isomorphic sc graphs with 5 vertices are L-equi-energetic. \( \square \)

**Theorem 6.43.** No two non isomorphic sc graphs with 8 vertices are L-equi-energetic.

**Proof.** Laplacian energies of all 10 non isomorphic sc graphs shown in figure-6.3(a-j), are 18.8098, 17.6569, 17.798, 15.3137, 14.9282, 15.391, 12.391, 11.6569, 9.5, 13.6568. We note that no two sc graphs have equal energy. Thus,
no two non-isomorphic sc graphs with 8 vertices are L-equi-energetic.

**Theorem 6.44.** There exist non-isomorphic non L-cospectral, L-equi-energetic
sc graphs with 9 vertices.

**Proof.** Consider the following two sc graphs in figure-6.8.

![Figure-6.8](image)

These two graphs are non-isomorphic. They are non L-cospectral as the LCPs
of both graphs are different as given below.

\[-9\cdot x + 18\cdot x^2 - 169\cdot x^3 + 3994\cdot x^4 - 26999\cdot x^5 + 4860\cdot x^6 - 556\cdot x^7 + 36\cdot x^8 - x^9\]

\[-1\cdot x + 976\cdot x^2 - 2099\cdot x^3 + 874\cdot x^4 - 263\cdot x^5 + 496\cdot x^6 - 556\cdot x^7 + 36\cdot x^8 - x^9\]

The Laplacian energies of graphs are 16. Thus, these graphs are non-
isomorphic non L-cospectral and L-equi-energetic.

Thus, we have the following result.

**Theorem 6.45.** The smallest positive integer for which there exists non-
isomorphic non L-cospectral L-equi-energetic sc graph is 9.

**On L-equi-energetic of sc comparability and sc chordal graphs**

Since their exist only one sc comparability and one sc chordal graph on 4 or 5
vertices, so there are no L-equienergetic sc comparability graphs and L-equi-
energetic sc chordal graphs on 4 or 5 vertices. Thus, we get the following
result.
Theorem 6.46. No two non isomorphic sc comparability graphs and sc chordal graphs with 8 vertices are L-equi-energetic.

Proof. Follows from Theorem 6.43.

Theorem 6.47. No two non isomorphic sc comparability graphs and sc chordal graphs with 9 vertices are L-equi-energetic.

Proof. Laplacian energies of the four non-isomorphic sc comparability graphs with 9 vertices, shown in Figure-3.13(a-d), are 23.8299, 23.4031, 16.5164, and 15.4031. We note that no two graphs have equal energy. Thus, no two non-isomorphic sc comparability graphs with 9 vertices are L-equi-energetic. Similarly, Laplacian energies of the three non-isomorphic sc chordal graphs with 9 vertices, shown in Figure-6.5(a-c) are 23.8299, 23.4031 and 23.4891. Clearly no two graphs have same energy. This proves the Theorem.

Theorem 6.48. No two non isomorphic sc comparability graphs and sc chordal graphs with 12 vertices are L-equi-energetic.

Proof. Laplacian energies of the fourteen non-isomorphic sc comparability graphs with 12 vertices, shown in Figure-3.14(a-h) are 40.8058, 39.6529, 33.6529, 38.4852, 35.0746, 30.4852, 27.0746, 27.0746, 24.4852, 22.5919, 19.8284, 20.6264, 19.8284 and 18.4852. We note that no two graphs have equal energy. Thus, no two non-isomorphic sc comparability graphs with 12 vertices are L-equi-energetic. Similarly, Laplacian energies of the 16 non-isomorphic sc chordal graphs with 12 vertices, shown in Figure-6.6(a-p) are 40.8058, 39.6529, 39.794, 39.2992, 39.3588, 39.1925, 39.3045, 40.7009,

Clearly, no two graphs have same energy. Hence the result. □

**Theorem 6.49.** No two non isomorphic sc comparability graphs and sc chordal graphs with 13 vertices are L-equie-energetic.

**Proof.** Laplacian energies of the 31 non-isomorphic sc comparability graphs with 13 vertices (graphs are not shown) are 47.827, 47.4001, 41.8633, 39.4001, 40.1111, 40.2605, 35.0591, 40.1418, 39.4272, 33.2973, 32.0159, 40.0432, 34.2947, 39.7527, 46.8251, 38.9249, 38.8251, 34.0966, 29.1397, 33.1397, 32.4315, 32.6227, 30.8251, 30.9249, 27.4757, 27.2, 27.1562, 27.442, 24.6227, 25.6307 and 22.8251. We note that no two graphs have equal energy. Thus, no two non-isomorphic sc comparability graphs with 13 vertices are L-equie-energetic. Similarly, Laplacian energies of the 16 non-isomorphic sc chordal graphs with 13 vertices, shown in Figure-6.7(a-p), are 47.827, 47.4001, 47.4861, 47.2234, 47.2631, 47.1621, 47.2377, 47.7527, 47.7718, 47.2123, 47.184, 47.0664, 46.8251, 46.9451, 47.0102, and 47.0476. Clearly no two graphs have same energy. This proves the result. □

**Theorem 6.50.** There do not exist non-isomorphic non L-cospectral L-equie-energetic sc comparability and sc chordal graphs up to 13 vertices.

In view of above Theorem 6.50, we propose the following conjecture.

**Conjecture 4.** There do not exist non-isomorphic non L-cospectral L-equie-energetic sc comparability and sc chordal graphs on $n = 4k$ or $n = 4k + 1$ vertices.