CHAPTER TWO
Chapter 2

On self-complementary comparability and self-complementary chordal graphs

In this chapter, we deal with some results on sc comparability and sc chordal graphs. In particular,

a) We study sc comparability graphs and obtain its inclusion relation with some other classes of sc perfect graphs.

b) Sc comparability and sc chordal graphs are considered and a condition is achieved for a sc chordal graph to be comparability using bipartite transformation of sc chordal graph.

c) We also show that chordal completion number and interval graph completion number of a sc comparability graph is less than or equal to the total number of $C_4$ in a sc comparability graph.

The chapter is developed through the following sections. In section 2.2, inclusion relation among various classes of sc perfect graphs namely sc comparability, sc chordal and sc weakly chordal graphs is obtained. In section 2.3 we obtain a condition for a sc chordal graph to be comparability. Section 2.4 deals with the triangulation of sc comparability graphs.
Chapter 2  On sc comparability and sc chordal graphs

2.1 Introduction

Ghouila-Houri [73] in 1962 and Gilmore and Hoffmann [68] in 1964 characterized comparability graphs. The first paper considers only finite graphs while the second allows graphs to be infinite. The next major result is due to Gallai [62], which characterizes finite comparability graphs in terms of minimum list of graphs that are excluded as induced subgraphs using asteroidal triple (AT in short). The asteroidal triple was introduced by Lekkerkerker and Boland [97] to study interval graphs. In fact there is a close relation between asteroidal triple and comparability graphs. Later, Kelly [93] extended the idea of Gallai to infinite graphs. In 1979, Golumbic [71] gave algorithmic characterization of comparability graphs in terms of implication classes.

A weakly chordal comparability graph is a graph which is both weakly chordal and comparability. Weakly chordal comparability graph has been studied by Eschen et al. [50]. A chordal comparability graph is a graph which is both chordal and comparability. Hsu and Ma [90] and Ma and Spinrad [105] studied the recognition problem of chordal comparability graphs, however the characterization problem for this class has not been well focused. Therefore it is required to study a characterization of chordal comparability graphs. So we study the same and obtain a necessary and sufficient condition for a sc chordal graph to be comparability using bipartite transformation of sc chordal graphs.
2.2 Sc comparability graphs and some other classes of sc perfect graphs

"An independent set of three vertices such that there exist a path between the two avoiding the neighborhood of the third, forms asteroidal triple (AT)".

Below, we illustrate various instances of asteroidal triple.

\[ \text{Figure-2.1: Examples of AT: vertices } x, y \text{ and } z \text{ form AT in each graph} \]

The complement of AT is a triangle and referred as CAT in the chapter.

The following Theorem gives the relation between edges \( xy \) and \( \sigma(x)\sigma(y) \) in a sc graph.

**Theorem 2.1.** Let \( G \) be a sc graph then \( xy \in E(G) \) if and only if \( \sigma(x)\sigma(y) \notin E(G) \).

**Proof.** Note that \( xy \in E(G) \) if and only if \( \sigma(x)\sigma(y) \in E(\overline{G}) \) as \( \sigma \) is isomorphism of \( G \) onto \( \overline{G} \). Moreover, \( \sigma(x)\sigma(y) \in E(\overline{G}) \) if and only if \( \sigma(x)\sigma(y) \notin E(G) \). Hence, \( xy \in E(G) \) if and only if \( \sigma(x)\sigma(y) \notin E(G) \).

The following results can be obtained from the structure of sc graphs.

**Lemma 2.2.** Let \( G \) be a sc graph and \( H \) be an induced subgraph of \( G \). Then there exists an induced subgraph of \( G \) isomorphic to \( \overline{H} \).

**Proof.** Let \( \sigma \) be a cp of \( G \) and \( V_H \) be the vertex set of induced subgraph \( H \). Since \( \sigma \) is cp, \( <\sigma(V_H)> \) is an induced subgraph isomorphic to \( \overline{H} \) in \( G \). Hence
Next, we show the relation between AT and comparability graphs with the help of examples. We observe that for the graph in figure-2.1(a), vertices \( x, y \) and \( z \) form AT and graph is not comparability. Though for the graph in figure-2.2, vertices \( x, y \) and \( z \) form AT but graph is comparability. It is clear from these examples that

"AT free is not a necessary condition for a graph to be comparability."

Now, we relate AT and sc comparability graphs. For this we need the following result.

**Theorem 2.3[62].** \( CAT \) cannot be transitively oriented.

The following Lemma relates cycle \( C_n \) for \( n > 6 \) and AT.

**Lemma 2.4.** Every cycle \( C_n \) for \( n \geq 6 \), contains AT.

**Proof.** For \( n = 6 \), i.e., \( C_6 \), it can be clearly seen that every triplet of alternate vertices of \( C_6 \) forms AT. For example in figure-2.2, vertices \( x, y \) and \( z \) are alternate and form AT.

For \( n > 6 \), as number of vertices increases such triplets of alternate vertices also increase in \( C_n \), see figure-2.3. So there always exists AT in \( C_n, n > 6 \). Hence the Lemma.
Next Theorem characterizes sc comparability graph in terms of AT.

**Theorem 2.5.** Let $G$ be a sc comparability graph. Then it is AT-Free.

**Proof.** Let $G$ be a sc comparability graph with cp $\sigma$. Suppose $G$ contain AT and let $V' = \{x, y, z\}$ be the set of vertices of AT. Then by Lemma 2.2, $<\sigma(V')>$ induces a subgraph isomorphic to $CAT$ in $G$. By Theorem 2.3, $CAT$ can not be oriented transitively. So transitive orientation of $G$ is not possible. Hence $G$ is AT-Free. 

Since the class of sc comparability graphs is a subclass of sc perfect graphs, which does not allow any induced odd cycle ($C_5$, $C_7$, $C_9$ etc.). Therefore a sc comparability graph does not contain any induced odd cycle. Thus, we have the following Theorem pertaining to induced cycles in a sc comparability graph.

**Theorem 2.6.** Let $G$ be a sc comparability graph. Then it contains only $C_3$ or $C_4$ as an induced cycle.

**Proof.** Let $G$ be a sc comparability graph. Then by Theorem 2.5, $G$ is AT-free and clearly $G$ is $C_5$-free. By Lemma 2.4, for $n \geq 6$ every cycle $C_n$ contains AT. So $G$ has no induced cycle isomorphic to $C_n$ ($n \geq 6$). Therefore, $G$ cannot contain any induced cycle of length greater than or equal to 5. Hence $G$ can contain only $C_3$ or $C_4$ as an induced cycle. 

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Remark 1: From Theorem 2.6, it is obvious that

"Forbidden induced cycles for sc comparability graphs are $C_n$, for $n \geq 5$.”

Recall that a sc graph $G$ is weakly chordal if and only if $G$ does not contain a chordless cycle with 5 or more vertices. There is no relation between comparability graph and weakly chordal graph, as it can be easily seen from figure-2.1(a) and figure-2.2. The graph in figure-2.1(a) is weakly chordal but not comparability while graph in figure-2.2 is comparability but not weakly chordal. In the case of sc graphs, we get a relation between sc comparability graphs and sc weakly chordal graphs by the following result.

Theorem 2.7. Every sc comparability graph is sc weakly chordal.

Proof. Let $G$ be a sc comparability graph. From Theorem 2.6, the only possible induced cycles in $G$ are $C_3$ or $C_4$. This clearly shows that $G$ is weakly chordal. Hence the Theorem.

Now, AT is not allowed in a sc comparability graph by Theorem 2.5. But it may happen that sc weakly chordal graph contains AT. The graph shown in figure-2.4 is weakly chordal as there is no induced cycle $C_n$, $n > 5$, but it contains AT, as any three vertices of $\{x,y,z,w\}$ form AT. Hence converse need not be true, i.e., sc weakly chordal graph need not be comparability.
The following result shows the relation between \( \text{AT} \) and \( \text{sc weakly chordal} \) graphs.

**Lemma 2.8.** Every \((\text{AT}, C_5)\)-free \( \text{sc} \) graph is \( \text{sc weakly chordal} \).

**Proof.** Let \( G \) be a \((\text{AT}, C_5)\)-free \( \text{sc} \) graph. By Lemma 2.4, it does not contain \( C_n, n \geq 6 \). Also, it is \( C_5 \)-free. Clearly it does not contain \( C_n, n \geq 4 \). Hence graph is weakly chordal. \( \square \)

A \( \text{sc weakly chordal} \) graph may contain \( \text{AT} \). The graph shown in figure-2.4 is weakly chordal but it contains \( \text{AT} \). Thus converse need not be true, i.e., \( \text{sc weakly chordal} \) graph need not be \( \text{AT-free} \).

As it is well known that chordal graphs do not contain an induced cycle \( C_n, n > 3 \). There are some \( \text{sc comparability} \) graphs which contain \( C_3 \), but not \( C_4 \). Hence these graphs are also chordal and those graphs which contain \( C_4 \) as induced cycle are weakly chordal, but not chordal. Below, we give examples for each case. In figure-2.5(a), graph is chordal and comparability as there is no induced cycle \( C_n, n > 3 \) and there exists a transitive orientation of the graph. Graph in figure-2.5(b) is not chordal but weakly chordal as there exists induced cycle \( v_2, v_4, v_6, v_8, v_2 \) of length 4 and it is comparability as there also exists transitive orientation of the graph, which can be checked easily.

![Figure-2.5: Examples of sc comparability graphs](image-url)
Thus we obtain the following intersection, which shows the inclusion relation among the various subclasses of sc perfect graphs like(sc comparability graphs, sc weakly chordal graphs and sc chordal graphs. The inclusion relation is shown in figure-2.6.

By Theorem 2.7, every sc comparability graph is sc weakly chordal. So, there is no need to study the class of sc weakly chordal comparability graphs. In the next section, we consider the class of sc comparability graphs that is also chordal.
2.3 Sc chordal comparability graphs

The following graphs will be used to characterize sc chordal graphs to be comparability and are referred in the later part of the chapter.

![Rising sun, S₁, co-rising sun and S₃, cc stands for central clique](image)

From figure-2.6, it is clear that there is no inclusion relation between sc comparability graphs and sc chordal graphs. Thus, sc graphs which are both chordal and comparability can be studied in the following two ways:

i) sc comparability graphs which are also chordal and

ii) sc chordal graphs which are also comparability.

For case i), the following result gives a condition for a sc comparability graph to be chordal.

**Theorem 2.9.** Let $G$ be a sc comparability graph with $n = 4k$ or $n = 4k + 1$

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vertices. Then $G$ is chordal if and only if $G$ has no induced subgraph isomorphic to $C_4$.

**Proof.** Let $G$ be a sc comparability graph. If $G$ is also chordal then obviously it does not have $C_4$ as an induced cycle. Conversely let $G$ has no induced subgraph isomorphic to $C_4$. Then using Theorem 2.6, $G$ has no induced cycle $C_n$ (for $n > 3$), so $G$ is also chordal. Hence the result.

It would be interesting to study the case ii), i.e, when a sc chordal graph is comparability. For this we need a result due to Földes and Hammer [59].

**Theorem 2.10[59].** Let $G$ be a graph. Then $G$ is split if and only if both $G$ and $\overline{G}$ are chordal.

We get the following Corollary for a sc graph from Theorem 2.10.

**Corollary 2.11.** Let $G$ be a sc graph. Then $G$ is split if and only if $G$ is chordal.

From the Corollary 2.11 it is clear that the class of sc split graphs is equivalent to the class of sc chordal graphs. The following Theorem due to Földes and Hammer [60], gives the condition for a split graph to be comparability in terms of forbidden induced subgraphs.

**Theorem 2.12[60].** Let $G$ be a split graph. Then $G$ is comparability if and only if $G$ is $(S_n, \overline{S_n})$-free and co-rising sun-free.

Using Corollary 2.11, the above Theorem can be modified and rephrased as follows.

**Corollary 2.13.** Let $G$ be a sc chordal graph. Then $G$ is comparability if and only if $G$ is $(S_n, \overline{S_n})$-free and co-rising sun-free.
Lemma 2.14. Let $G$ be a sc graph. Then $G$ is $S_3$-free if and only if $G$ is $\overline{S}_3$-free.

Proof. Let $G$ be a sc graph with cp $\sigma$ such that $G$ is $S_3$-free. We have to show that $G$ is $\overline{S}_3$-free. Suppose $G$ contains $\overline{S}_3$ and $V'$ be the set of vertices that induces $\overline{S}_3$ in $G$. Then by Lemma 2.2, $<\sigma(V')>$ induces a subgraph isomorphic to $S_3$ in $G$, a contradiction. Thus $G$ is $S_3$-free implies that $G$ is $\overline{S}_3$-free. Similarly $G$ is $\overline{S}_3$-free implies that $G$ is $S_3$-free. This proves the Lemma.

The following result is analogue to Lemma 2.14 for co-rising sun and the proof is on similar pattern as given in Lemma 2.14.

Lemma 2.15. Let $G$ be a sc graph. Then $G$ is co-rising sun-free if and only if $G$ is rising sun-free.

Now, we have the following result for a sc chordal graph to be comparability.

Theorem 2.16. Let $G$ be a sc chordal graph. Then $G$ is comparability if and only if $G$ is $S_3$-free and rising sun-free.

Proof. Proof follows from Corollary 2.13, Lemma 2.14 and Lemma 2.15.

The following result is due to Sridharan and Balaji [161].

Theorem 2.17[161]. Let $G$ be a sc graph with $n = 4k$ or $n = 4k + 1$ vertices. Then $G$ is chordal if and only if

$$\omega(G) = 2k \quad \text{for} \quad n = 4k \quad \text{and} \quad \omega(G) = 2k + 1 \quad \text{for} \quad n = 4k + 1.$$
even labeled vertices of star cp $\sigma^*$, denoted by $\text{Even}(\sigma^*)$, for a sc chordal graph with $n = 4k$ vertices.

**Theorem 2.18[161].** Let $G$ be a sc graph with $n = 4k$ vertices, $\omega(G) = 2k$ and star cp $\sigma^*$. Then $<\text{Even}(\sigma^*)>$ is a maximum clique of $G$.

Since for a sc chordal graph $G$ with $n = 4k$ vertices and star cp $\sigma^*$ has $\omega(G) = 2k$, so the collection of even labeled vertices of $\sigma^*$, $\text{Even}(\sigma^*)$ induces a maximum clique in a sc chordal graph, i.e., $\omega(G) = |\text{Even}(\sigma^*)| = 2k$.

Next, for the stability number of sc chordal graph we have the following result.

**Theorem 2.19.** Let $G$ be a sc chordal graph with $n = 4k$ or $n = 4k + 1$ vertices. Then $G$ is chordal if and only if

$$\omega(G) = 2k \text{ for } n = 4k \text{ and } \omega(G) = 2k + 1 \text{ for } n = 4k + 1.$$ 

**Proof.** For a graph $G$, the maximum clique number is equal to the stability number of its complement, i.e., $\omega(G) = \alpha(G)$. Since $G$ is sc graph, so $\omega(G) = \alpha(G)$. Thus $\omega(G) = \alpha(G)$. So, for a sc chordal graph $G$, $\alpha(G) = \omega(G) = 2k$ for $n = 4k$ vertices and $\alpha(G) = \omega(G) = 2k + 1$ for $n = 4k + 1$ vertices.

For a sc chordal graph with $n = 4k$ vertices and star cp $\sigma^*$, $\alpha(G) = 2k$ and clearly $\text{Odd}(\sigma^*) = \sigma^*(\text{Even}(\sigma^*))$ as if vertices $x, y \in \text{Even}(\sigma^*)$ then $\sigma^*(x)\sigma^*(y) \notin E$ by Theorem 2.1. So vertices $\sigma^*(x), \sigma^*(y) \in \text{Odd}(\sigma^*)$. Thus $<\text{Odd}(\sigma^*)>$ induces stable set in $G$, i.e., the collection of odd labeled vertices of star cp $\sigma^*$ induces a maximum stable set in a sc chordal graph, i.e., $\alpha(G) =$
For a sc chordal graph with \( n = 4k + 1 \) vertices, we have the following result.

**Theorem 2.20.** Let \( G \) be a sc chordal graph with \( n = 4k + 1 \), \( \alpha(G) = 2k + 1 \) and star cp \( \sigma^* \). Then \( <\text{Even}(\sigma^*) \cup v_0> \) is a maximum clique of \( G \), where \( v_0 \) is the fixed vertex.

**Proof.** Let \( G' \) be a sc chordal graph on \( 4k \) vertices with star cp \( \sigma^* \). Then, by Theorem 2.18, \( \text{Even}(\sigma^*) \) induces a maximum clique of size \( 2k \), i.e., \( \alpha(G') = 2k \). Now, we construct a sc chordal graph \( G \) on \( 4k + 1 \) vertices by adding a fixed vertex, \( v_0 \) to \( G' \). Since \( \alpha(G) = 2k + 1 \) for a sc chordal graph on \( n = 4k + 1 \) vertices, \( v_0 \) must be adjacent to all vertices in \( \text{Even}(\sigma^*) \) (i.e., \( 2k \) vertices). Moreover, \( v_0 \) can not be adjacent to any vertex of \( \text{Odd}(\sigma^*) \) as the degree of \( v_0 \) is always \( 2k \) in a sc graph with \( 4k + 1 \) vertices, so \( <\text{Even}(\sigma^*) \cup v_0> \) is a maximum clique of \( G \).

Now, similar result is obtain for \( \text{Odd}(\sigma^*) \cup v_0 \).

**Theorem 2.21.** Let \( G \) be a sc chordal graph with \( n = 4k + 1 \), \( \alpha(G) = 2k + 1 \) and star cp \( \sigma^* \). Then \( <\text{Odd}(\sigma^*) \cup v_0> \) is a maximum independent set of \( G \), where \( v_0 \) is the fixed vertex.

**Proof.** By the Theorem 2.20, \( <\text{Even}(\sigma^*) \cup v_0> \) is a maximum clique of \( G \). Since \( \sigma^*(\text{Even}(\sigma^*)) = \text{Odd}(\sigma^*) \), \( \sigma^*(v_0) = v_0 \), so for any vertex \( v \in \text{Even}(\sigma^*) \), \( \sigma^*(v)\sigma^*(v_0) \notin E \Rightarrow \sigma^*(v)v_0 \notin E \), i.e., the fixed vertex \( v_0 \) is not adjacent to any
vertex of Odd($\sigma^*$). So $<\text{Odd}(\sigma^*) \cup v_0>$ is a maximum independent set of $G$. □

Next, result deals with the behavior of the fixed vertex $v_0$, in a sc chordal graph with $n = 4k + 1$ vertices.

**Theorem 2.22.** Let $G$ be a sc chordal graph with $n = 4k + 1$ vertices and star cp $\sigma^*$. Then $\alpha(G) = \omega(G) = 2k + 1$ with exactly one vertex common in a maximum clique and a maximum independent set.

**Proof.** Let $G$ be a sc chordal graph on $4k + 1$ vertices and star cp $\sigma^*$. Then by

Theorem 2.20 and Theorem 2.21, $<\text{Even}(\sigma^*) \cup v_0>$ is a maximum clique and $<\text{Odd}(\sigma^*) \cup v_0>$ is a maximum independent set of $G$. Clearly $v_0$ is common in both maximum clique and maximum independent set. Now, for a vertex $u \in \text{Even}(\sigma^*)$, $\sigma^*(u) \in \text{Odd}(\sigma^*)$ so $u \notin \text{Odd}(\sigma^*)$. Similarly for a vertex $v \in \text{Odd}(\sigma^*)$, $\sigma^*(v) \in \text{Even}(\sigma^*)$, so $v \notin \text{Even}(\sigma^*)$. So no other vertex belongs to both maximum clique and maximum independent set. Hence exactly one vertex $v_0$ is common. □

From now onwards, for a sc chordal graph with $n = 4k + 1$, Even($\sigma^*$) is a set consisting of all even labelled vertices in star cp $\sigma^*$ and $v_0$, similarly Odd($\sigma^*$) is a set consisting of all odd labelled vertices in star cp $\sigma^*$ and $v_0$.

Now, since $\alpha(G) = |\text{Odd}(\sigma^*)| = 2k$ and $\omega(G) = |\text{Even}(\sigma^*)| = 2k$ for a sc chordal graph with $n = 4k$ vertices, we can partition the graph uniquely in two equal parts, i.e., into Even($\sigma^*$) and Odd($\sigma^*$), covering all the vertices. Again, for a sc chordal graph with $n = 4k + 1$ vertices, since $\alpha(G) = |\text{Odd}(\sigma^*)| = 2k + 1$
and \( \omega(G) = |\text{Even}(\sigma^*)| = 2k + 1 \), we can also partition the graph uniquely in two equal parts, i.e., into \( \text{Even}(\sigma^*) \) and \( \text{Odd}(\sigma^*) \). However, one vertex is common to both partitions.

In order to explore sc chordal graphs more with respect to rising sun and \( S_3 \), we require the vertices of a rising sun and \( S_3 \) which belong to \( \text{Even}(\sigma^*) \) and \( \text{Odd}(\sigma^*) \). Next result is a step towards the same.

**Theorem 2.23.** Let \( G \) be a sc chordal graph with \( n = 4k \) or \( n = 4k + 1 \) vertices and star cp \( \sigma^* \). Then central clique, \( cc \) of \( S_3 \) and \( cc \) of rising sun lies in \( \text{Even}(\sigma^*) \), i.e.,

\[
V(cc) \text{ of } S_3 \text{ lies in } \text{Even}(\sigma^*) ,
\]
\[
V(cc) \text{ of rising sun lies in } \text{Even}(\sigma^*) .
\]

**Proof.** We consider \( cc \) of each graph one by one.

**cc of \( S_3 \).** The vertices \( v_2, v_4 \) and \( v_6 \) form \( cc \) of \( S_3 \) (see figure-2.7). Clearly vertices either belong to \( \text{Even}(\sigma^*) \) or \( \text{Odd}(\sigma^*) \), so the following cases arise:

i) All the vertices belong to \( \text{Odd}(\sigma^*) \), which is not possible. Thus discarded.

ii) One vertex belongs to \( \text{Even}(\sigma^*) \) and two vertices belong to \( \text{Odd}(\sigma^*) \), this is not possible as two vertices that belong to \( \text{Odd}(\sigma^*) \) are not
adjacent, so discarded.

iii) Two vertices say \(v_2\) and \(v_4\) belong to \(\text{Even}(\sigma^*)\) and one vertex \(v_6\) belongs to \(\text{Odd}(\sigma^*)\). In this case vertex \(v_6\) must be adjacent to both the vertices in \(\text{Even}(\sigma^*)\), since \(v_6\) is adjacent to two more vertices in \(S_3\) namely \(v_1\) and \(v_3\) so these vertices lie in \(\text{Even}(\sigma^*)\). As a result four vertices \(v_1, v_2, v_4\) and \(v_5\) of \(S_3\) belong to \(\text{Even}(\sigma^*)\). Vertices \(v_1, v_2, v_4\) and \(v_5\) induce \(K_4\), as the vertices of \(\text{Even}(\sigma^*)\) are mutually adjacent. This is not possible, hence discarded.

iv) All the vertices \(v_2, v_4\) and \(v_6\) belong to \(\text{Even}(\sigma^*)\), then any vertex can be adjacent to at least two vertices in \(\text{Odd}(\sigma^*)\).

Therefore, \(V(\text{cc})\) of \(S_3\) lie in \(\text{Even}(\sigma^*)\).

**cc of rising sun.** The vertices \(v_2, v_4, v_5\) and \(v_7\) form \(cc\) of rising sun (see figure-2.7). The following cases are possible:

i) All the vertices belong to \(\text{Odd}(\sigma^*)\), which is not possible. Thus discarded.

ii) One vertex belongs to \(\text{Even}(\sigma^*)\) and three vertices belong to \(\text{Odd}(\sigma^*)\), this is not possible as three vertices that belong to \(\text{Odd}(\sigma^*)\) are not adjacent. So discarded. Similarly, if two vertices belong to \(\text{Even}(\sigma^*)\) and two vertices belong to \(\text{Odd}(\sigma^*)\), then two vertices that belong to \(\text{Odd}(\sigma^*)\) are not adjacent. This is not possible.

iii) Three vertices belong to \(\text{Even}(\sigma^*)\) and one vertex belongs to
Odd(\(\sigma^*\)). Then there are two subcases:

a) A vertex \(v_2\) of degree 5 belongs to Odd(\(\sigma^*\)), then \(v_2\) is adjacent to \(v_1\) and \(v_3\). So \(v_1\) and \(v_3\) can not lie in Odd(\(\sigma^*\)). Thus, \(v_1\) and \(v_3\) lie in Even(\(\sigma^*\)). This implies five vertices of rising sun lie in Even(\(\sigma^*\)). This is not possible as it induces \(K_5\).

b) A vertex \(v_4\) of degree 4 belongs to Odd(\(\sigma^*\)), then \(v_4\) is adjacent to all three vertices of \(cc\) in Even(\(\sigma^*\)). Moreover, vertex \(v_4\) is adjacent to vertex \(v_3\) and \(v_3\) can not belong to Odd(\(\sigma^*\)) so it belongs to Even(\(\sigma^*\)). This implies that vertex \(v_4\) is adjacent to four vertices of Even(\(\sigma^*\)) inducing \(K_5\), which is not possible. Thus discarded.

iv) All the vertices \(v_2, v_4, v_5\) and \(v_7\) belong to Even(\(\sigma^*\)).

Therefore, \(V(cc)\) of rising sun lie in Even(\(\sigma^*\)). This completes the proof.

Next result deals with the fixed vertex \(v_0\) in \(S_3\) and rising sun.

**Theorem 2.24.** Let \(G\) be a sc chordal graph on \(n = 4k + 1\) vertices and star cp \(\sigma^*\). Then for the fixed vertex \(v_0\)

i) \(v_0 \not\in S_3\)

ii) \(v_0 \not\in \text{rising sun}\).

**Proof.** Let \(v_0\) be the fixed vertex of \(G\), then by Theorem 2.22, \(v_0 \in \text{Even}(\sigma^*)\)
and \( v_0 \in \text{Odd}(\sigma^*) \).

i) If \( v_0 \in S_3 \), then \( v_0 = v_1 \) (or \( v_3 \) or \( v_5 \)) or \( v_0 = v_2 \) (or \( v_4 \) or \( v_6 \)), see figure-2.7.

Let \( v_0 = v_1 \) (or \( v_3 \) or \( v_5 \)). Now \( v_0 \in \text{Even}(\sigma^*) \) and \( v_2,v_4,v_6 \in \text{Even}(\sigma^*) \) by Theorem 2.23. Vertex \( v_0 \) is adjacent to \( v_2 \), \( v_4 \) and \( v_6 \), as vertices of \( \text{Even}(\sigma^*) \) are mutually adjacent inducing \( K_4 \), which is not possible. Thus \( v_0 \neq v_1 \) (or \( v_3 \) or \( v_5 \)).

Again if \( v_0 = v_2 \) (or \( v_4 \) or \( v_6 \)). Since \( V(\sigma) \) lie in \( \text{Even}(\sigma^*) \), so vertices \( v_0 \), \( v_4 \) and \( v_6 \) are mutually adjacent. Moreover, \( v_0 \) is adjacent to \( v_1 \) and \( v_3 \), none of which can belong to \( \text{Even}(\sigma^*) \), as it induce a \( K_n, n > 3 \). So \( v_1,v_3 \in \text{Odd}(\sigma^*) \), also \( v_0 \in \text{Odd}(\sigma^*) \) so \( v_0 \) cannot be adjacent to \( v_1 \) and \( v_3 \), this implies that \( v_0 \neq v_2 \) (or \( v_4 \) or \( v_6 \)). This proves the first part of the Theorem.

ii) If \( v_0 \in \text{rising sun} \), then \( v_0 = v_1 \) (or \( v_3 \) or \( v_5 \)) or \( v_0 = v_2 \) (or \( v_7 \)) or \( v_0 = v_4 \) (or \( v_3 \)). Suppose \( v_0 = v_1 \) (or \( v_3 \) or \( v_5 \)) since \( V(\sigma) \) of rising sun lie in \( \text{Even}(\sigma^*) \) \( \Rightarrow \)

\( v_2,v_4,v_5,v_7 \in \text{Even}(\sigma^*) \). Also \( v_0 \in \text{Even}(\sigma^*) \), this induces \( K_5 \), which is not possible. Hence \( v_0 \neq v_1 \) (or \( v_3 \) or \( v_5 \)).

Again, if \( v_0 = v_2 \) (or \( v_7 \)) \( \Rightarrow \) \( v_0,v_4,v_5,v_7 \in \text{Even}(\sigma^*) \). \( v_0 \) is adjacent to \( v_1 \) and \( v_3 \), none of them belongs to \( \text{Even}(\sigma^*) \). So both the vertices belong to \( \text{Odd}(\sigma^*) \).

Moreover, \( v_0 \in \text{Odd}(\sigma^*) \) so \( v_0 \) cannot be adjacent to \( v_1 \) and \( v_3 \). Thus \( v_0 \neq v_2 \) (or \( v_7 \)).

Again, \( v_0 = v_4 \) (or \( v_5 \)) \( \Rightarrow \) \( v_0,v_2,v_5,v_7 \in \text{Even}(\sigma^*) \). Now, \( v_0 \) is adjacent to \( v_3 \), so \( v_3 \) belongs to \( \text{Odd}(\sigma^*) \). Moreover, \( v_0 \in \text{Odd}(\sigma^*) \) so \( v_0 \) cannot be adjacent to \( v_3 \).

Hence \( v_0 \neq v_2 \) (or \( v_7 \)). Hence the result. \( \Box \)
Next Theorem deals with the number of vertices of $S_3$ and rising sun belonging to $\text{Even}(\sigma^*)$ and $\text{Odd}(\sigma^*)$.

**Theorem 2.25.** For a sc chordal graph $G$ with $n = 4k$ or $n = 4k + 1$ vertices and star cp $\sigma^*$

i) Number of vertices of rising sun which lie in $\text{Even}(\sigma^*) = 4$,
Number of vertices of rising sun which lie in $\text{Odd}(\sigma^*) = 3$.

ii) Number of vertices of $S_3$ which lie in $\text{Even}(\sigma^*) = 3$,
Number of vertices of $S_3$ which lie in $\text{Odd}(\sigma^*) = 3$.

**Proof.** i) By the Theorem 2.23, $V(cc)$ of rising sun $\in \text{Even}(\sigma^*)$. This implies that four vertices of rising sun lie in $\text{Even}(\sigma^*)$. Now, if possible, one more vertex of rising sun belongs to $\text{Even}(\sigma^*)$, then it induces $K_5$ in $G$. This is not possible, as rising sun does not contain $K_5$ as an induced subgraph. So exactly four vertices of rising sun belong to $\text{Even}(\sigma^*)$. Since $G$ can be partitioned uniquely into two parts, so the remaining vertices of rising sun belong to $\text{Odd}(\sigma^*)$, which proves the part i).

ii) Part ii) can be proved with the help of similar arguments as given in part i).

Hence the result.

Now, we transform sc chordal graph $G$ into a new graph by partitioning $G$ into two parts: first part contains the vertices of $\text{Even}(\sigma^*)$ and second part contains the vertices of $\text{Odd}(\sigma^*)$ and define the adjacency between the vertices of $\text{Even}(\sigma^*)$ and $\text{Odd}(\sigma^*)$ in a new graph as follows.
a) For a vertex \( u \in \text{Even}(\sigma^*) \) and a vertex \( v \in \text{Odd}(\sigma^*) \), there is an edge \( uv \) in a new graph if and only if \( uv \in E \).

b) The vertices of \( \text{Even}(\sigma^*) \) and \( \text{Odd}(\sigma^*) \) are non adjacent in the new graph.

For example, consider the following sc chordal graph \( G \) which has star \( \sigma^* = (v_1, v_4, v_5, v_8) \). Vertex set of \( \text{Even}(\sigma^*) = \{v_2, v_4, v_6, v_8\} \) induces a \( K_4 \), vertex set of \( \text{Odd}(\sigma^*) = \{v_1, v_3, v_5, v_7\} \) induces a stable set. Now, we partition the graph \( G \) and define adjacency by rule a) and b) mentioned above, we get the following new graph \( B_G \) of the graph \( G \).

The new graph obtained is called bipartite transformation of \( G \). For the sake of simplicity we denote first partition of \( B_G \) as \( B_G(e) \) and second partition of \( B_G \) as \( B_G(o) \). Note that the vertices of \( B_G(e) \) are mutually adjacent in \( G \) and vertices in \( B_G(o) \) are mutually non adjacent in \( G \). Also note that only those edges of \( G \) are present in \( B_G \) for which one vertex lie in \( B_G(e) \) and other vertex lie in \( B_G(o) \). \( E(B_G) \) denotes the edges of \( B_G \).

Next result deals with the behaviour of \( v_0 \) in a \( B_G \) of sc chordal graph \( G \).
with \( n = 4k + 1 \) vertices.

**Theorem 2.26.** Let \( G \) be a sc chordal graph on \( n = 4k + 1 \) vertices and star cp \( \sigma^* \). Then the fixed vertex \( v_0 \) of sc chordal graph is isolated vertex in \( B_G \).

**Proof.** Let \( G \) be a sc chordal graph on \( n = 4k + 1 \) vertices and star cp \( \sigma^* \). Let vertex set of \( G \) be \( \{v_0, v_1, \ldots, v_{4k}\} \), then Even(\( \sigma^* \)) = \( B_G(e) \) induces maximum clique of size \( 2k + 1 \) in \( G \) and Odd(\( \sigma^* \)) = \( B_G(o) \) induces maximum independent set of size \( 2k + 1 \) in \( G \). By Theorem 2.22, \( v_0 \) is common in both set of vertices.

Let \( v_0, v_1, v_3, \ldots, v_{4k-1} \) are the vertices of Odd(\( \sigma^* \)) and \( v_0, v_2, v_4,\ldots,v_{4k} \) are the vertices of Even(\( \sigma^* \)). Now \( v_0v_1,v_0v_3,v_0v_5,v_0v_7,\ldots,v_0v_{4k-1} \notin E(B_G) \) as \( v_0,v_1,v_3,v_5,v_7,\ldots,v_{4k-1} \) is independent set. Moreover \( v_0v_2,v_0v_4,v_0v_6,v_0v_8,\ldots,v_0v_{4k} \notin E(B_G) \), by the definition of \( B_G \) (rule b)). So \( v_0 \) is isolated vertex in \( B_G \). \( \square \)

**Remark:** So while constructing \( B_G \) of \( G \) on \( n = 4k + 1 \) vertices, we do not take \( v_0 \) into consideration.

Next, we obtain a result that relates \( S_3 \), rising sun in \( G \) and its corresponding subgraph in \( B_G \).

**Theorem 2.27.** Let \( G \) be a sc chordal graph with \( n = 4k \) or \( n = 4k + 1 \) vertices and star cp \( \sigma^* \). Then \( G \) has an induced subgraph isomorphic to \( S_3 \) if and only if \( B_G \) has \( C_6 \) as an induced subgraph.

**Proof.** Let \( G \) be a sc chordal on \( n = 4k \) or \( n = 4k + 1 \) vertices and \( B_G \) be its bipartite transformation. Assume that \( G \) has induced subgraph isomorphic to \( S_3 \). Then by Theorem 2.23, \( V(cc) \) lie in Even(\( \sigma^* \)) \( \Rightarrow v_2,v_4,v_6 \in B_G(e) \), See figure-
2.9. By Theorem 2.25, $v_1, v_3, v_5 \in \text{Odd}(\sigma^*) \Rightarrow v_1, v_3, v_5 \in B_G(o)$.

Now the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1 \in E(B_G)$. Clearly it induces $C_6$ in $B_G$. To prove converse, let vertices $v_1, v_2, v_3, v_4, v_5$ and $v_6$ induce $C_6$ in $B_G$, then there is only one possibility that three vertices of $C_6$ belong to $B_G(e)$ and the remaining three vertices belong to $B_G(o)$. Since the vertices that lie in $B_G(e)$ are mutually adjacent in $G$. So vertices $v_2, v_4$ and $v_6$ induce $K_3$ in $G$. Again these vertices are alternate in cycle. Hence vertices $v_1, v_2, v_3, v_4, v_5$ and $v_6$ induce a graph isomorphic to $S_3$, see figure-2.10. Hence the result.

**Figure-2.9**

**Theorem 2.28.** Let $G$ be a sc chordal graph with $n = 4k$ or $n = 4k + 1$ vertices and star cp $\sigma^*$. Then $G$ has an induced subgraph isomorphic to rising sun if and only if $B_G$ has $P_7$ as an induced subgraph.

**Proof.** Let $G$ be a sc chordal graph on $n_0 = 4k$ or $n_0 = 4k + 1$ vertices and $B_G$ be its bipartite transformation. Suppose that $G$ has rising sun as an induced
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subgraph, then by Theorem 2.23 and Theorem 2.25, vertices \( v_2, v_4, v_5, v_7 \in B_G(e) \) and vertices \( v_1, v_3, v_6 \in B_G(o) \). Now from the rising sun vertex \( v_5 \) is adjacent to \( v_6, v_6 \) is adjacent to \( v_7, v_7 \) is adjacent to \( v_1, v_1 \) is adjacent to \( v_2, v_2 \) is adjacent to \( v_3 \) and \( v_3 \) is adjacent to \( v_4 \), this adjacency relation induces \( P_7 \) in \( B_G \).

\[ \text{Figure-2.11} \]

Conversely, let there exists \( P_7 \) as an induced subgraph in \( B_G \), then there are two possibilities.

i) 4 vertices of \( P_7 \) lie in \( B_G(e) \) and 3 vertices of \( P_7 \) lie in \( B_G(o) \) as shown in figure-2.12.

\[ \text{Figure-2.12} \]

Since all four vertices of \( B_G(e) \) induce a \( K_4 \) in \( G \). Thus, in this case the vertices of \( P_7 \) in \( B_G \) induce rising sun in \( G \).

ii) 3 vertices of \( P_7 \) lie in \( B_G(e) \) and 4 vertices of \( P_7 \) lie in \( B_G(o) \) as shown in figure-2.13.
Since the vertices $v_2, v_4$ and $v_6$ are mutually adjacent and vertices $v_1, v_3, v_5$ and $v_7$ are mutually non adjacent in $G$, so we get the following subgraph induced by $P_7$ of $B_G$ in $G$.

Clearly it is a co-rising sun. By Lemma 2.2, $<\sigma^*(V(\text{co-rising sun})>)$ is an induced subgraph isomorphic to rising sun. Therefore, existence of induced $P_7$ in $B_G$ such that three vertices of $P_7$ are in $B_G(e)$ and four vertices of $P_7$ are in $B_G(o)$ implies that there exists an induced rising sun in $G$. So, it is clear from both the cases that $P_7$ in $B_G$ always contributes an induced rising sun in $G$. Hence the Theorem.

Combining last two Theorems, the following result is immediate.

**Theorem 2.29.** Let $G$ be sc chordal graph with $n = 4k$ or $n = 4k + 1$ vertices and star cp $\sigma^*$. Then $G$ has an induced subgraph isomorphic to $S_3$ or rising sun if and only if $B_G$ has $C_6$ or $P_7$ as an induced subgraph respectively.
Finally, we have the following Theorem that relates sc chordal and sc comparability via bipartite transformation of sc chordal graph.

**Theorem 2.30.** Let $G$ be a sc chordal graph with $n = 4k$ or $n = 4k + 1$ vertices and star cp $\sigma^*$. Then $G$ is comparability if and only if $B_G$ has no induced subgraph isomorphic to $C_5$ or $P_7$.

**Proof.** Proof follows from Theorem 2.16 and Theorem 2.29. $\square$

### 2.4 Sc comparability graphs and triangulation

A triangulation of a graph $G$ is a graph $H$ on the same vertex set as $G$ that contains all edges of $G$ and is chordal. A minimal triangulation of $G$ is a triangulation $H$ such that the set $E(H) - E(G)$ is minimal w. r. t. inclusion.

Let $G$ be a graph. A set $F$ of edges is called a chord cover of $G$ if $F \cap E = \emptyset$ and if every induced $C_k$ of $G$ with $k \geq 4$ has a chord in $F$. A minimal chord cover of $G$ is a chord cover that is minimal w. r. t. inclusion.

Let $G$ be a graph. The minimum fill or chordal completion number $ccn(G)$ is the minimum number of the edges that must be added to $G$ to obtain a chordal graph, i.e.,

$$ccn(G) = \min \{|E(H) - E(G)| : H \text{ is chordal}, E(G) \subseteq E(H)\}.$$  

The interval graph completion number $icn(G)$ is the minimum number of edges that must be added to $G$ to obtain an interval graph, i.e.,

$$icn(G) = \min \{|E(H) - E(G)| : H \text{ is interval}, E(G) \subseteq E(H)\}.$$  

It has already been shown that sc comparability graph contains only $C_3$ or $C_4$ as a induced cycle so while triangulating the sc comparability graphs we
have to cover only $C_4$'s. Mohring [119] showed the following result for AT-free graphs.

**Theorem 2.31 [119].** Let $G$ be a AT-free graph and let $F$ be a minimal chord cover of $G$. Then $H = (V, E \cup F)$ is again AT-free.

Lekkerkerker and Boland [97] demonstrated the importance of asteroidal triple in the following Theorem.

**Theorem 2.32 [97].** A graph is interval graph if and only if it is chordal and AT-Free.

Next result is obtained for minimum triangulation of sc comparability graph, which states that.

**Theorem 2.33.** Let $G$ be a sc comparability graph. Then minimum triangulation of $G$ is interval.

**Proof.** Suppose $G$ is a sc comparability graph. By Theorem 2.5, $G$ is AT-free. Let $F$ be a minimal chord cover, which contains all the chords covering $C_4$'s for triangulating $G$. Then by Theorem 2.31, $H = (V, E \cup F)$ is again AT-free. Hence the resulting graph $H$ is AT-free and chordal, since all the $C_4$'s are covered. Thus by Theorem 2.32, $H$ is interval graph.

The following Theorem due to Mohring [119] gives the relation between $ccn(G)$ and $icn(G)$ for AT-free graphs.

**Theorem 2.34 [119].** Let $G$ be a AT-free graph. Then $ccn(G) = icn(G)$.

From the above Theorem we get the Corollary for sc comparability graphs.
**Corollary 2.35.** Let $G$ be a sc comparability graph. Then $ccn(G) = icn(G)$.

**Proof.** Since sc comparability graph is AT-free, so by Theorem 2.34, $ccn(G) = icn(G)$.

Next we give the result which gives the relation among $ccn(G)$, $icn(G)$ and number of $C_4$.

**Theorem 2.36.** Let $G$ be sc comparability graph. Then $ccn(G) = icn(G) \leq$ number of $C_4$ in $G$.

**Proof.** Let $G$ be sc comparability graph so by Theorem 2.6 only allowed induced cycles are $C_3$ or $C_4$. If $G$ contains only $C_3$ as an induced cycle then there is nothing to prove.

Suppose $G$ contains $C_4$, i.e., it is weakly chordal, so chords are needed to cover only $C_4$’s to make $G$ chordal. If every pair of $C_4$’s meets at edge or meets at vertex or is disjoint, see figure-2.15 (a)-(c), then one chord is required to cover each $C_4$. In this case, the total number of chords required is equals to the number of $C_4$, i.e., $ccn(G) = number of C_4$ in $G$.

Again, if any pair of $C_4$’s meets at non edge, see figure-2.15(d)-(e), then one chord is required to cover the pair of $C_4$’s. So the total number of chords required is less than the number of $C_4$ in $G$. i.e., $ccn(G) < number of C_4$ in $G$.

Thus, $ccn(G) \leq$ number of $C_4$ in $G$. 

![Diagram](image-url)
Again by Corollary 2.35, for sc comparability graph $G$, $ccn(G) = icn(G)$.

Hence $ccn(G) = icn(G) \leq \text{number of } C_4 \text{ in } G$.

**Remark 2.** In particular, if $G$ contains only $C_3$, i.e., $G$ is sc chordal then number of $C_4 = 0$. Hence $ccn(G) = icn(G) = \text{number of } C_4 = 0$. 