CHAPTER-3
3.1 Introduction:

In this chapter, the integral transform technique to solve fundamental problems of elasticity theory has been discussed. It has been found that this method is of great help in solving the problems of punch and crack in an elastic medium. Much work has been done in this direction for which one may refer to Sneddon [80], but they are mainly concerned with bodies without having any initial stress. Using Hankel's transform, crack and punch problems for transversely isotropic bodies have been solved by Elliot [24]. Hara et. al. [36] have discussed an axisymmetric contact problem of a transversely isotropic layer indented by an annular rigid punch. Recently, Fan and Hwu [26] have solved the punch problems for an isotropic elastic half-plane by combining stork's formalism and the method of analytic continuation. In solving the boundary value problems in elasticity the following transforms are very frequently used:
(a) Fourier Transform:

The Fourier transform is defined by

\[ F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{ix\alpha} \, dx, \]

with its inverse transform

\[ f(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} F(\alpha) e^{-ix\alpha} \, d\alpha. \]

The Fourier transform is used for functions whose domain is the whole real line. If the domain is the positive real line the Fourier Cosine, Sine transforms defined as given below are used:

\[ F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(\alpha x) \, dx, \]

\[ F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(\alpha x) \, dx, \]

with the inversion formulas given respectively as:

\[ f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(\alpha) \cos(\alpha x) \, d\alpha, \]

\[ f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(\alpha) \sin(\alpha x) \, d\alpha. \]

(b) Hankel Transform:

Hankel transform of order \(r\) is defined in terms of \(J_r(\alpha x)\), the Bessel function of the first kind of order \(r\), as follows:

\[ H(x) = \int_{0}^{\infty} \alpha f(\alpha) J_r(\alpha x) \, d\alpha, \]

with its inversion formula given by
\[ f(\alpha) = \int_0^\infty x \ H(x) \ J_1(\alpha x) \, dx. \]

(c) Mellin Transform:

Mellin transform of a function \( f(x) \) whose domain is the positive real line is defined as follows:

\[ F(x) = M [f(\alpha); x] = \int_0^\infty \alpha^{x-1} \ f(\alpha) \, d\alpha, \]

with the inversion formula given by

\[ f(\alpha) = M^{-1} [F(x); \alpha] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \alpha^{-x} F(x) \, dx. \]

The integral on the right is taken over a line parallel to the imaginary axes at a distance \( \gamma \), in the x-plane and to the right of all the singularities of \( f(x) \).

In the foregoing section, deals with indentation of a semi-infinite initially stressed elastic medium which possess some initial stress.

3.2 Small Deformation of an Initially Stressed Body:

Various elastic bodies are found to possess initial stress which exists in the body by process of preparation or by the action of body forces. For example, if a sheet of metal rolled up into a cylinder and the edges welded together, the body so formed is in a state of initial stress and the unstrained state can not be attained without cutting the cylinder open. If such a body is further subjected to deforming forces then apart from the initial finite deformation it will have incremental deformation also. Trefftz [88], Neuber [65] and Green [30], [31] have discussed and given basic equations of such incremental deformation theory. Later on Kurashige [45], [46] discussed an axisymmetric circular crack problem and a two-
dimensional crack problem for an initially stressed neo-Hookean solid. The problem of opening of a crack of prescribed shape in an initially stressed body has been discussed by Ali and Ahmad [5].

In this chapter, the problem of indentation of a semi-infinite initially stressed elastic medium under the action of an axisymmetric rigid punch has been discussed. The medium has been supposed to be isotropic, homogeneous, incompressible and the punch pressing it normally. The problem has been considered within the framework of incremental deformation theory for neo-Hookean solid using Hankel transformation.

**Basic Equations:**

We have adopted the fundamental equations of incremental deformation theory constructed by Biot [12]-[15] and Kurashige [46]. In rectangular cartesian coordinates $x_i$ and $t$, the equations of motion for incremental deformation theory of elasticity and the expression of incremental boundary forces per unit area respectively:

\[
\frac{\partial S_{ij}}{\partial x_j} + S_{jk} \frac{\partial w_k}{\partial x_j} + S_{ik} \frac{\partial w_j}{\partial x_i} - e_{jk} \frac{\partial S_{ik}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (3.2-1)
\]

\[
\Delta f_i = (S_{ij} + S_{kj} w_k + S_{ij} e - S_{ik} e_{jk}) n_j, \quad (3.2-2)
\]

where

- $x_i$ = Cartesian coordinates,
- $n_i$ = Components of unit normal to boundary surface,
- $S_{ij}$ = Initial stress, corresponding to initial finite deformation referred to $x_i$,
- $\rho$ = Density in a finite deformation,
- $u_i$ = Incremental displacement (infinitesimal),
\( e_{ij} = \) Incremental strain,
\( w_{ij} = \) Incremental rotation,
\( e = \) Incremental volume expansion,
\( s_{ij} = \) Incremental stress referred to axis which are incrementally displaced with the medium,
\( \Delta f_i = \) Incremental boundary force per unit initial area.

In equations (3.2-1) and (3.2-2) the usual convention for summation over repeated indices is applied. The second, third and fourth terms of left-hand side in equation (3.2-1) represent the effect of initial stress.

The incremental strain-displacement relations can be written in the same forms as in classical elasticity, because the incremental displacement is infinitesimal. Hence we have,

\[
e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

(3.2-3)

\[
w_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right).
\]

(3.2-4)

Taking the material to be a so-called neo-Hookean solid, elastic potential per unit volume is expressed in the form [88],

\[
W = \frac{1}{2} \mu_0 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3),
\]

(3.2-5)

with

\[
\lambda_1 \lambda_2 \lambda_3 = 1,
\]

(3.2-6)

where \( \mu_0 \) is shear modulus in an unstrained state and \( \lambda_1 \) is extension ratio.

The stress-strain relations for initial stresses are given by

\[
S_{11} - S_{22} = \mu_0 (\lambda_1^2 - \lambda_2^2),
\]

(3.2-7a)
\[ S_{22} - S_{33} = \mu_0 (\lambda_2^2 - \lambda_3^2) \quad (3.2-7b) \]
\[ S_{33} - S_{11} = \mu_0 (\lambda_3^2 - \lambda_1^2) \quad (3.2-7b) \]

The total differentiation of equations (3.2-7) and consideration of incremental shear deformation given the following incremental stress-strain relations [12]:

\[ S_{11} - S_{22} = 2\mu_0 (\lambda_1^2 e_{11} - \lambda_2^2 e_{22}) \quad (3.2-8a) \]
\[ S_{22} - S_{33} = 2\mu_0 (\lambda_2^2 e_{22} - \lambda_3^2 e_{33}) \quad (3.2-8b) \]
\[ S_{33} - S_{11} = 2\mu_0 (\lambda_3^2 e_{33} - \lambda_1^2 e_{11}) \quad (3.2-8c) \]

and

\[ S_{12} = \mu_0 (\lambda_1^2 + \lambda_2^2) e_{12} \quad (3.2-9a) \]
\[ S_{23} = \mu_0 (\lambda_2^2 + \lambda_3^2) e_{23} \quad (3.2-9b) \]
\[ S_{31} = \mu_0 (\lambda_3^2 + \lambda_1^2) e_{31} \quad (3.2-9c) \]

### 3.3 Axisymmetric Incremental Deformation:

The cylindrical polar co-ordinates \((r, \theta, z)\) of a point in the initially deformed body are connected with rectangular co-ordinates by relations:

\[ r = \sqrt{x_1^2 + x_2^2}, \quad (3.3-1a) \]
\[ \theta = \tan^{-1}(x_2/x_1), \quad (3.3-1b) \]
\[ z = x_3, \quad (3.3-1c) \]

It is assumed that the only non-zero components of initial stress are \(S_{rr}, S_{\theta\theta}\) and \(S_{zz}\) which are uniform throughout the body and the body is in the state of symmetrical incremental strain with respect to z-axis. The equations of motion (3.2-1) reduce, in the cylindrical polar coordinates, to:
\[
\frac{\partial s_{\tau}}{\partial \tau} + \frac{s_{\sigma} - s_{\theta\theta}}{r} + \frac{\partial s_{\tau z}}{\partial z} - (S_{\tau} - S_{\tau z}) \frac{\partial w_{\tau z}}{\partial z} = \rho \frac{\partial^2 u_{\tau}}{\partial t^2}, \quad (3.3-2a)
\]

\[
\frac{\partial s_{\tau z}}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r s_{\tau}) - (S_{\tau z} - S_{\tau z}) \frac{1}{r} \frac{\partial}{\partial r} (r w_{\tau z}) = \rho \frac{\partial^2 u_{\tau}}{\partial t^2}. \quad (3.3-2b)
\]

The incremental displacements \(u_{\tau}\) and \(u_{\tau z}\) in terms of potential function \(\phi(r, z)\) are given by:

\[
u_{\tau} = -\frac{\partial^2 \phi}{\partial \tau \partial z}, \quad u_{\tau z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right), \quad (3.3-3)
\]

where \(\phi\) is a scalar function \(r\) and \(z\). Further it is easily checked that the condition for incompressibility

\[e = e_{\tau} + e_{\theta\theta} + e_{\tau z} = 0, \quad (3.3-4)\]

is satisfied. The non-zero components of incremental strain, rotation and stress can be expressed in terms of the function \(\phi\) as follows:

\[
e_{\tau} = -\frac{\partial^2 \phi}{\partial \tau^2 \partial z}, \quad e_{\theta\theta} = -\frac{1}{r} \frac{\partial^2 \phi}{\partial \tau \partial \theta}, \quad e_{\tau z} = \frac{1}{r} \frac{\partial}{\partial \tau} \left( r \frac{\partial \phi}{\partial \tau} \right), \quad (3.3-5a)
\]

\[
e_{\tau r} = \frac{1}{2} \frac{\partial}{\partial \tau} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial \tau} \right) - \frac{\partial^2 \phi}{\partial z^2} \right], \quad (3.3-5b)
\]

\[
w_{\tau z} = -\frac{1}{2} \frac{\partial}{\partial \tau} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial \tau} \right) + \frac{\partial^2 \phi}{\partial z^2} \right], \quad (3.3-5c)
\]

\[
s_{\tau} - s = -\mu_0 \lambda_r^2 \frac{\partial}{\partial \tau} \left( \frac{1}{r} \frac{\partial^2 \phi}{\partial \tau \partial \theta} \right), \quad (3.3-6a)
\]

\[
s_{\theta\theta} - s = \mu_0 \lambda_r^2 \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial^2 \phi}{\partial \tau \partial \theta} \right), \quad (3.3-6b)
\]

\[
s_{\tau z} - s = \mu_0 (\lambda_r^2 + 2 \lambda_z^2) \frac{1}{r} \frac{\partial}{\partial \tau} \left( r \frac{\partial^2 \phi}{\partial \tau \partial \theta} \right), \quad (3.3-6c)
\]

49
\[ s_{zz} = \frac{1}{2} \mu_0 (\lambda_r^2 + \lambda_z^2) \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \frac{\partial^2 \phi}{\partial z^2} \right], \]  

(3.3-6d)

where

\[ s = \frac{1}{2} (s_{rr} + s_{\theta \theta}). \]  

(3.3-7)

Substitution form expressions (3.3-5) and (3.3-6) into equation (3.3-2) gives the following equations:

\[ \frac{\partial s}{\partial r} - \frac{1}{2} \frac{\partial}{\partial z} \left\{ \mu_0 (\lambda_r^2 + \lambda_z^2) + (S_{rr} - S_{zz}) \right\} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \left\{ \mu_0 (\lambda_r^2 + \lambda_z^2) - (S_{rr} - S_{zz}) \right\} \frac{\partial^2 \phi}{\partial z^2} \right\} + \left\{ \mu_0 \lambda_r^2 + (S_{rr} - S_{zz}) \right\} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \left\{ \mu_0 \lambda_z^2 \right\} \frac{\partial^2 \phi}{\partial z^2} \right\} - (S_{rr} - S_{zz}) \left\{ \mu_0 \lambda_r^2 + (S_{rr} - S_{zz}) \right\} \frac{\partial^2 \phi}{\partial z^2} \right\} + \left\{ \mu_0 \lambda_z^2 \right\} \frac{\partial^2 \phi}{\partial z^2} \right\} = - \rho \frac{\partial^3 \phi}{\partial z \partial t^2}, \]  

(3.3-8a)

and further substitution from the first of equation (3.3-8) into the second and use of stress-strain relations:

\[ S_{rr} - S_{\theta \theta} = \mu_0 (\lambda_{r}^2 - \lambda_{\theta}^2) = 0, \]  

(3.3-9a)

\[ S_{rr} - S_{zz} = \mu_0 (\lambda_{r}^2 - \lambda_{z}^2) = 0, \]  

(3.3-9b)

which are reduced from equation (3.2-7), the function \( \phi \) is given by the simple partial differential equation and an expression for stress \( s \) as follows:
\[
\begin{align*}
\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \right\} & \left\{ K^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} - \rho \frac{\partial^2 \phi}{\partial t^2} \right\} = 0 , \\
& \quad (3.3-10)
\end{align*}
\]

\[
\begin{align*}
s = \mu \lambda_z^2 \frac{\partial^3 \phi}{\partial z^2} - \rho \frac{\partial^3 \phi}{\partial z \partial t^2} , \\
& \quad (3.3-11)
\end{align*}
\]

where
\[
K = \frac{\lambda_r}{\lambda_z} . \\
& \quad (3.3-12)
\]

### 3.4 A Punch Problem for Initially Stressed neo-Hookean Solid:

#### Formulation of the Problem:

It is supposed that the semi-infinite medium \( z > 0 \) is initially deformed and the components \( S_{zz} \), in addition to \( S_{\theta\theta} \), is also zero so that
\[
S_r = \mu \left( \lambda_r^2 - \lambda_z^2 \right) = -P . \\
& \quad (3.4-1)
\]

The Hankel transformation and its inversion transform of order zero are defined respectively by relations:
\[
\begin{align*}
\tilde{\phi} (\xi) &= \int_0^\infty \phi (r) J_0 (r \xi) \, dr , \\
& \quad (3.4-2a) \\
\phi (r) &= \int_0^\infty \tilde{\phi} (\xi) J_0 (r \xi) \, d\xi , \\
& \quad (3.4-2b)
\end{align*}
\]

from which the following formulae for differentiated function are obtained:
\[
\begin{align*}
1 \frac{\partial}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) &= -\int_0^\infty \tilde{\phi} \xi^3 J_0 (r \xi) \, d\xi , \\
& \quad (3.4-3a) \\
\frac{\partial}{\partial r} \phi &= -\int_0^\infty \tilde{\phi} \xi^2 J_1 (r \xi) \, d\xi , \\
& \quad (3.4-3b)
\end{align*}
\]

where \( J_0 \) and \( J_1 \) are Bessel functions of order 0 and 1, respectively.

Using these formulae, the incremental displacements \( u_r, u_z \), the
incremental rotation \( \omega_{rz} \) and the components of incremental stress \( s_{zz}, s_{rr} \)

\( s \) are expressed in Hankel inversion as follows:

\[
\begin{align*}
\omega_r &= \int_0^\infty \frac{1}{\xi} \frac{\partial \phi}{\partial z} \xi^2 J_1(\xi \rho) \, d\xi, \\
\omega_z &= -\int_0^\infty \frac{\phi}{\xi} \xi^3 J_0(\xi \rho) \, d\xi, \\
\omega_{rz} &= -\frac{1}{2} \int_0^\infty (\xi^4 \frac{\partial^2 \phi}{\partial z^2} - \xi^2 \frac{\partial^2 \phi}{\partial z^2}) J_1(\xi \rho) \, d\xi,
\end{align*}
\]  

\( (3.4-4a) \)

\( (3.4-4b) \)

\( (3.4-5) \)

\[
\begin{align*}
s_{zz} &= s - \mu_0 \lambda_z^2 \left( 2 + K^2 \right) \int_0^\infty \xi^3 \frac{\partial \phi}{\partial z} J_1(\xi \rho) \, d\xi, \\
s_{rr} &= \frac{1}{2} \mu_0 \lambda_z^2 \left( 1 + K^2 \right) \int_0^\infty \xi^3 \xi \frac{\partial^2 \phi}{\partial z^2} J_1(\xi \rho) \, d\xi, \\
s &= \frac{1}{2} (s_{rr} + s_{t\theta}) = \frac{1}{2} \mu_0 \lambda_z^2 \int_0^\infty \xi \frac{\partial^2 \phi}{\partial z^2} J_0(\xi \rho) \, d\xi.
\end{align*}
\]  

\( (3.4-6a) \)

\( (3.4-6b) \)

\( (3.4-6c) \)

writing the first of equations (3.3-2) and (3.3-3) in the form

\[
\frac{\partial}{\partial r} \left( r^2 s_{rr} \right) = r^2 \left\{ \frac{s_{rr} - s_{t\theta}}{r} - \frac{\partial s_{zz}}{\partial z} - \mu_0 \lambda_z^2 \left( 1 - K^2 \right) \frac{\partial w_{rz}}{\partial z} \right\}, \quad (3.4-7)
\]

substituting from expressions (3.4-5) and (3.4-6) into the above equation and integrating it with respect to \( r \) lead to the following expression for \( s_{rr} \):

\[
\begin{align*}
s_{rr} &= \mu_0 \lambda_z^2 \left[ \int_0^\infty \left( K^2 \xi^3 \frac{\partial \phi}{\partial z} + \xi \frac{\partial^2 \phi}{\partial z^2} \right) J_1(\xi \rho) \, d\xi \\
&- \frac{2K^2}{r} \int_0^\infty \xi^2 \frac{\partial \phi}{\partial z} J_1(\xi \rho) \, d\xi \right].
\end{align*}
\]  

\( (3.4-8) \)
Now equation (3.3-10), by Hankel's transform, reduces to the ordinary differential equation:

\[
\left( \frac{d^2}{dz^2} - \xi^2 \right) \left( \frac{d^2}{dz^2} - K^2 \xi^2 \right) \phi = 0.
\]  

(3.4-9)

**Boundary Conditions:**

The rigid punch is in the form of a solid of revolution which has the equation \( z = f(r) \), referred to the tip of the punch as origin and it has a radius of contact ‘\( a \)’ with the medium. If the pressure \( p(r) \) is assumed to be applied in the plane \( z = 0 \), and the contact is free from friction, the boundary conditions are:

\[
\begin{align*}
\sigma_{rr} (r,0) &= p(r), \quad (0 \leq r \leq \infty) \quad (3.4-10a) \\
\sigma_{rz} &= 0, \quad (0 \leq r \leq \infty) \quad (3.4-10b) \\
u_r (r,0) &= D - f(r), \quad (0 \leq r \leq a) \quad (3.4-11a) \\
\sigma_{zz} &= 0, \quad (r > a) \quad (3.4-11b)
\end{align*}
\]

where \( D \) is a parameter whose physical significance is that it is the depth to which the tip of the punch penetrates the elastic half-space and \( f(0) = 0 \).

**Solution of the Problem:**

The solution of the differential equation (3.4-9) is given by

\[
\bar{\phi} = A(\xi) e^{-\xi z} + B(\xi) e^{-K^2 \xi z}.
\]  

(3.4-12)

where \( A(\xi) \) and \( B(\xi) \) are integral constants. Applying boundary conditions (3.4-10) to the equation (3.4-12) gives,

\[
A(\xi) = \frac{1+K^2}{1-K^2} \frac{-\bar{p}(\xi)}{\xi^2},
\]  

(3.4-13a)
\[
\begin{align*}
B (\xi) &= \frac{-2}{1 - K^2} \frac{\mathbf{P}(\xi)}{\xi^2}. \quad (3.4-13b)
\end{align*}
\]

The boundary conditions (3.4-11) to equation (3.4-12) gives the following dual integral equations:

\[
\begin{align*}
\int_0^r \mathbf{p}(\xi) J_0 (r\xi) \, d\xi &= D - f (r), \quad (0 \leq r \leq a) \quad (3.4-14) \\
\int_0^r \mathbf{p}(\xi) J_1 (r\xi) \, d\xi &= 0, \quad (r > a). \quad (3.4-15)
\end{align*}
\]

Taking \( \mathbf{p}(\xi) = \Psi (a\xi) \), we have from equations (3.4-14) and (3.4-15),

\[
\begin{align*}
\int_0^a \Psi (\xi) J_0 (x\xi) \, d\xi &= D_1 - f_1 (x), \quad (0 \leq x \leq 1) \quad (3.4-16) \\
\int_a^\infty \Psi (\xi) J_0 (x\xi) \, d\xi &= 0, \quad (x \geq 1) \quad (3.4-17)
\end{align*}
\]

in which \( D_1 = aD, f_1 (x) = af(ax) \) and \( a\xi = \zeta \).

The solution of (3.4-17) is given as follows [80]:

\[
\Psi (\zeta) = \frac{1}{\sqrt{2\pi}} \int_0^1 g(t) \cos (\zeta t) \, dt. \quad (3.4-18)
\]

The equation (3.4-16) is equivalent to the Abel's integral equation:

\[
\sqrt{2\pi} \int_0^r \frac{g(t) \, dt}{\sqrt{t^2 - r^2}} = D_1 - f_1 (x), \quad (r > 0) \quad (3.4-19)
\]

where the unknown function \( g(t) \) is given by

\[
g(t) = \sqrt{2\pi} \left[ D_1 - t \int_0^1 \frac{f_1' (x)}{\sqrt{t^2 - x^2}} \, dx \right] \quad (3.4-20)
\]

and \( D_1 = \int_0^1 f_1' (x) \, dx \quad (3.4-21) \).
or \[ D = a \int_0^a \frac{f'(r)}{\sqrt{a^2 - r^2}} \, dr. \] (3.4-22)

From equation (3.4-18), \( \Psi(\zeta) \) and hence \( \Psi(a\zeta) \) is known. Thus \( p(\xi) \) being known, the components of stress and strain can be found out and the problem is completely solved.

**Special Cases:**

(i) **Flat-ended Circular Cylindrical Punch**

Let us consider the case in which the semi-infinite elastic medium is deformed by the normal indentation of the boundary by a flat-ended circular cylinder of radius 'a'. Since in the case the profile of the punch is not smooth at \( r = a \), we must regard \( D \) as one of the data of problem. In this case we take \( f_1(x) = 0 \). From equation (3.4-20), we get

\[ g(t) = \sqrt{2\pi} a D. \] (3.4-23)

Therefore we have

\[ p(x) = \frac{2D}{\pi} \frac{\sin a\xi}{\xi^2}. \] (3.4-24)

Hence

\[ \phi = \frac{2D}{\pi(1-K^2)} \left[ (1+K^2) e^{-\xi^2} - 2e^{-K^2\xi^2} \right] \frac{\sin a\xi}{\xi^4}. \] (3.4-25)

Thus the non-vanishing components of incremental displacement and stresses are found in terms of the Hankel inversion, Sneddon [80], as follows:

\[ u_z = \frac{-2D}{\pi(1-K^2)} \int_0^\infty \left[ (1+K^2) e^{-\xi^2} - 2e^{-K^2\xi^2} \right] \frac{\sin a\xi}{\xi} J_0(t\xi) \, d\xi, \] (3.4-26)
\[ u_r = \frac{-2D}{\pi(1-K^2)} \int_0^\infty \left[ (1+K^2) e^{-r \xi} - 2Ke^{-K\xi} \right] \frac{\sin \alpha}{\xi} J_1(\tau \xi) d\xi, \quad (3.4-27) \]

\[ s_{zz} = \frac{2D}{\pi(1-K^2)} \mu_0 \lambda_z^2 \int_0^\infty \left[ (1+K^2) e^{-\xi \tau^2} - 4K^2 e^{-K^2\xi^2} \right] \sin \alpha \xi J_0(\tau \xi) d\xi, \quad (3.4-28) \]

\[ s_{\tau} = \frac{2D}{\pi(1-K^2)} \mu_0 \lambda_z^2 \left\{ \int_0^\infty \left[ (1+K^2) e^{-\xi \tau^2} - 4K^2 e^{-K^2\xi^2} \right] \sin \alpha \xi J_0(\tau \xi) d\xi \right\}, \quad (3.4-29) \]

\[ s_{zz} = \frac{2D}{\pi(1-K^2)} \mu_0 \lambda_z^2 \int_0^\infty (1+K^2)^2 \left( e^{-\xi^2 \tau^2} - e^{-K^2\xi^2} \right) \sin \alpha \xi J_1(\tau \xi) d\xi. \quad (3.4-30) \]

(ii) **Conical Punch:**

Here the boundary of the semi-infinite elastic medium is deformed by a conical punch whose axis is normal to the indented plane. It is assumed that the axis of the cone coincides with the z-axis and that the vertex points downwards into the interior of the medium. In this case we take

\[ f(r) = r \tan \alpha, \]

where the semi-vertical angle \( \beta \) \( = (\pi/2) - \alpha \) of the conical punch is supposed to be large, so that the conical punch is not very much pointed. Therefore from (3.4-22), we have

\[ D = \frac{1}{2} \pi \varepsilon, \quad \text{where } \varepsilon = \tan \alpha. \]

From equation (3.4-20), we obtain the expression
Thus we get
\[
p(t) = \frac{g(t)}{\sqrt{2\pi}} \frac{2D}{\pi a} \frac{1 - \cos a \xi}{\xi^3}.
\]

Hence
\[
\phi = \frac{2D}{\pi a(1 + K^2)} \left[ (1 + K^2) e^{-\xi^2} - 2e^{-K\xi} \right] \frac{(1 - \cos a \xi)}{\xi^4}.
\]

Hence the non-vanishing components of incremental displacement and stresses are found in terms of Hankel's transformation as follows:

\[
u_x = \frac{-2D}{\pi a(1 - K^2)} \int_0^\infty \left[ (1 + K^2) e^{-\xi^2} - 2e^{-K\xi} \right] \frac{(1 - \cos a \xi)}{\xi^2} J_0(r\xi) \, d\xi,
\]

\[
u_r = \frac{-2D}{\pi a(1 - K^2)} \int_0^\infty \left[ (1 + K^2) e^{-\xi^2} - 2K e^{-K\xi} \right] \frac{(1 - \cos a \xi)}{\xi^2} J_1(r\xi) \, d\xi,
\]

\[
s_{zz} = \frac{2D}{\pi a(1 - K^2)} \mu_0 \lambda_z^2 \left[ \int_0^\infty (1 + K^2)^2 e^{-\xi^2} - 4K^2 e^{-K\xi} \right] \frac{(1 - \cos a \xi)}{\xi} J_0(r\xi) \, d\xi,
\]

\[
s_{rr} = \frac{2D}{\pi a(1 - K^2)} \mu_0 \lambda_z^2 \left[ \int_0^\infty (1 + K^2)^2 e^{-\xi^2} - 4K^3 e^{-K\xi} \right] \frac{(1 - \cos a \xi)}{\xi} J_1(r\xi) \, d\xi,
\]

\[
s_{zz} = \frac{2K^2}{r} \left[ \int_0^\infty (1 + K^2) e^{-\xi^2} - 2K e^{-K\xi} \right] \frac{(1 - \cos a \xi)}{\xi^2} J_1(r\xi) \, d\xi.
\]

**Limiting Case:**

The case of non-initial stresses and displacements will be obtained by making $K \to 1$. The results so obtained agree with those already obtained by Sneddon [80].
Numerical Results:

For a flat-ended circular cylindrical punch, variations of $s_{zz}$, $s_{\pi}$ and $u_z$ with various parameters as shown in (Fig. 1 to 4). Attention has been paid to investigate the influences of the initial stress $P$. 
Fig. 1. The variation of the normal component of incremental stress $s_{zz}$ with $r$. 
Fig. 2. The variation of the radial component of incremental stress $s_{zz}$ with $z$. 
Fig. 3. The variation of the radial components of incremental stress $s_{rr}$ with $r$.  

$z = 0.1a$

$P/\mu_0 \times$

1 0.8
2 0.6
3 0.4
4 0.0

$S_{rr}/P_0$ vs $r/a$
Fig. 4. The Variation of the Component of Incremental Displacement $u_z$ with $r$