CHAPTER-5
GENERALISED DUGDALE MODEL APPROACH

5.1 Introduction:

The elastic analysis of stress distribution in vicinity of a crack using complex variable method was given by Muskhelishvili [60]. Using his technique, Dugdale [23] proposed a 'strip yield model' giving elastic-plastic analysis for determining plastic zone size ahead of crack tips. The effect of partial closure on stress intensity factor of a Griffith crack opened by a parabolic distribution was investigated by Burniston and Gurley [16]. A photo-elastic study for determination of plastic zone size ahead of a crack tip contained in a thin sheet under uniaxial loading has been carried out by Mishra and Parida [58]. The model was modified by Harrop [37] for cases when plastic zones were closed by a cohesive normal parabolic stress distribution. The Dugdale model is extended by Theocaris [84] for the case of two unequal collinear straight cracks when the plastic zones developed at the four tips were separated.
Recently, Chen et al. [18] gave a Dugdale model for hardening materials under plane stress conditions. A Dugdal model is extended to cases combining mode I, II and III based on the Von Mises yield criterion by Nicholson [66]. A numerical study of crack-tip plasticity in glassy polymers has been carried out by Lai and Giessen [47].

5.2 Two Unequal Cracks With Coalesced Plastic Zones-The Generalised Dugdal Model Approach:

In this chapter, the Dugdale model is extended to the case when an infinite thin plate is weakened by two unequal cracks with coalesced plastic zones. The model is modified when the plastic zones developed, due to the remotely applied tension, are closed by linearly varying normal compressive stress distribution. Complex variable theory of elasticity has been used to obtain closed form expressions for plastic zone size and crack opening displacement.

Basic Formulae:

Stress components $\tau_{xx}$, $\tau_{yy}$ and $\tau_{xy}$ and displacement components $u_x$ and $u_y$ may be expressed in terms of two complex potentials $\phi(z)$ and $\psi(z)$, as developed by Muskhelishvili [60], as

$$\tau_{yy} - i \tau_{xy} = \phi(z) + \psi(z) - (z - \bar{z}) \phi'(z), \quad (5.2-1)$$

$$2\mu(u_{xx} + i u_{yx}) = \kappa \phi(z) - \psi(z) - (z - \bar{z}) \phi'(z), \quad (5.2-2)$$

where a bar over a function denotes its complex conjugate. A prime after a function denotes differentiation with respect to the argument, while a comma after a function signifies partial differentiation with respect to the subscript.
following it. The elastic constant $\mu$ denotes shear modulus and $\kappa = (3-4\nu)$ for the plane strain case and $\kappa = (3-\nu) / (1+\nu)$ for the plane stress case, $\nu$ being Poisson’s ratio.

Let us consider a homogeneous, isotropic elastic infinite plate in the $xy$ plane containing $n$ straight cracks $L_i$ ($i=1, 2, 3, ..., n$) lying on the real $x$-axis. Dual problems of linear relationship are obtained using (5.2-1), when the rims of the cracks are acted upon by stresses $\tau_{yy}^+$ and $\tau_{xy}^+$. The two Hilbert problems so obtained may be written as

$$\phi^+ (t) + \psi^+ (t) = \tau_{yy}^+ - i \tau_{xy}^+,$$  

(5.2-3) 

$$\phi^- (t) + \psi^- (t) = \tau_{yy}^- - i \tau_{xy}^-,$$  

(5.2-4) 

where $L = \sum_{i=1}^{n} L_i$, under the assumption $\lim_{y \to 0} [y \phi'(t+i y)] = 0$.

Superscripts $+$ and $-$ indicate the limiting value of the function when any point $t$ on the crack, other than end points, is approached from the positive $y$-plane ($y>0$) and the negative $y$-plane ($y<0$), respectively.

The solutions of (5.2-3) and (5.2-4) for complex potentials $\phi (z)$ and $\psi (z)$ may be written, as

$$\phi (z) = \phi_0 (z) + \{\tau_n (z)/X (z)\},$$  

(5.2-5) 

$$\psi (z) = \psi_0 (z) + \{\tau_n (z)/X (z)\},$$  

(5.2-6) 

where $\phi_0 (z) = \int_L^t \frac{p(t) X(t)}{t-z} dt + \int_L^t \frac{q(t) X(t)}{t-z} dt - \frac{1}{2} \sigma_\infty,$  

(5.2-7) 

$$\psi_0 (z) = \int_L^t \frac{p(t) X(t)}{t-z} dt - \int_L^t \frac{q(t) X(t)}{t-z} dt + \frac{1}{2} \sigma_\infty,$$  

(5.2-8)
\[ \sigma_\infty \text{ being the tension applied at infinity,} \]
\[ p(t) = \frac{1}{2} \left[ \tau_{yy} + \tau_{yy} \right] - \frac{i}{2} \left[ \tau_{xy} + \tau_{xy} \right], \quad (5.2-9) \]
\[ q(t) = \frac{1}{2} \left[ \tau_{yy} - \tau_{yy} \right] - \frac{i}{2} \left[ \tau_{xy} + \tau_{xy} \right] \quad (5.2-10) \]

and
\[ X(z) = \prod_{i=1}^{n} [(z - a_i)(z - b_i)]^{1/2}, \quad (5.2-11) \]

\[ a_i \text{ and } b_i \text{ are the end points of the crack } L_i \text{ and} \]
\[ P_n(z) = C_0 z^n + C_1 z^{n-1} + \cdots + C_n. \quad (5.2-12) \]

The constants \( C_i \) (\( i = 1, 2, 3, \ldots, n \)) are determined by the condition of single-valuedness of displacement at the rims of the crack and \( C_0 \) is determined from the boundary condition at infinity. The stress intensity factor, \( K = (K_1 - i K_2) \), at the crack tip \( z = z_1 \) may be calculated, as given by Cherepanov [19], from
\[ K = K_1 - i K_2 = 2\sqrt{2\pi} \lim_{z \to z_1} [(z-z_1)^{1/2} \phi(z)], \quad (5.2-13) \]
where \( \phi(z) \) is obtained from (5.2-5)

**Statement of the Problem:**

An infinite, homogeneous, isotropic elastic-perfectly plastic plate under the condition of plane stress occupies the \( xoy \) plane. The plate is cut along two hairline, unequal, collinear and straight cracks \( L_1 \) and \( L_2 \). These cracks lie on the \( ox \) axis of the \( xoy \) plane. The crack \( L_1 \), following the notations of Theocaris [84], occupies interval \([a_1, b_1]\) and \( L_2 \) occupies \([c_1, d_1]\) (Fig.1). The configuration thus obtained is subjected to uniform constant unidirectional tension, \( \sigma_\infty \), at infinite boundary. The tension acts
Fig. 1. Configuration
in a direction perpendicular to the rims of the cracks \( L_1 \) and \( L_2 \). Consequently the faces of the cracks \( L_1 \) and \( L_2 \) open forming the plastic zones ahead of the tips \( a_1, b_1, c_1 \) and \( d_1 \).

The prescribed tension at infinity is increased to the limit where the plastic zones developed at the tips \( b_1 \) and \( c_1 \) get coalesced. Thus there remain only three plastic zones \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \); \( \Gamma_1 \) occupies the interval \([b_1, c_1]\); \( \Gamma_2 \) occupies the interval \([a, a_1]\) and \( \Gamma_3 \) occupies the interval \([d_1, d]\). Each rim of the plastic zones in turn is subjected to a cohesive, normal stress linear distribution \( \tau_{yy} = t \sigma_{ye} \), where \( \sigma_{ye} \) is the yield point stress and \( t \) is the x-coordinate of any point on the x-axis and denotes a point on any of the plastic zones.

**Solution of the Problem:**

The solution of the problem stated in above is obtained by superimposing the solutions of two component problems contributing towards the stress singularity at the tips of the cracks. These two problems, termed as *problem I and problem II* are appropriately derived that stated in above.

**Problem I**

The configuration of the *Problem I* is an infinite, homogeneous, isotropic and elastic-perfectly plastic plate bounded by the xoy plane. The plate is cut along a hairline straight crack \( R_1 \) which occupies the interval \([a, d]\) on the ox-axis of the plate. The rims of the crack \( R_1 \) are stress free. At the infinite boundary uniform, constant and unidirectional tension \( \sigma_w \) is applied perpendicular to the rims of the crack. The complex
potentials $\phi^I(z)$ for this case may directly be written, using Muskhelishvili [60], as

$$\phi^I(z) = \frac{\sigma_\infty}{2iX(z)} \left( z - \frac{a + d}{2} \right) - \frac{1}{4} \sigma_\infty,$$  

(5.2-14)

where $X(z) = \sqrt{(z - a)(z - d)}$.  

(5.2-15)

The superscript $I$ denotes that the function refers to Problem I. Thus the problem is completely solved.

**Problem II**

A stress free infinite, homogeneous, isotropic and elastic-perfectly plastic plate is occupying the xoy plane. The plate is cut along a straight crack $R_i (i = 1, 2, 3)$ which lies on ox-axis in the interval $[a, d]$ and is subjected to tensile stress $\tau_{yy} = t\sigma_{yy}$ along its parts $\Gamma_i (i = 1, 2, 3)$.  

$\Gamma_1, \Gamma_2$ and $\Gamma_3$ occupy the intervals $[b_i, c_i], [a, a_i]$ and $[d_i, d]$ respectively on the ox-axis. The remaining parts of the rims of $R_i$ are stress free.

The solution of the dual Hilbert problems, obtained using the condition on the rims of the cracks, is as follows:

$$[\phi^\Pi(t)]^+ + [\psi^\Pi(t)]^- = t\sigma_{yy},$$  

(5.2-16a)

$$[\phi^\Pi(t)]^- + [\psi^\Pi(t)]^+ = t\sigma_{yy},$$  

(5.2-16b)

on $\Gamma = \bigcup_{i=1}^3 \Gamma_i$. The solution, using the Muskhelishvili [60] complex variable technique, may be written as

$$\phi^\Pi(z) = \frac{\sigma_{yy}}{2\pi iX(z)} \int_{\Gamma_i} \frac{t X(t)}{(t - z)} \, dt + \frac{C_0 z + C_1}{X(z)},$$  

(5.2-17)
where superscript II denotes that the potential refers to Problem II. Here X(z) is same as defined in equation (5.2-15). The constants \( C_0 \) and \( C_1 \) are determined by the boundary conditions at infinity and single-valuedness of displacements at the rims of crack. After these computations the final \( \phi^\gamma(z) \) may be written as

\[
\phi^\gamma(z) = -\frac{\sigma_{ys}}{2\pi X(z)} \left\{ \left[ z - \frac{(a + d)}{2} \right] N_2 - \left\{ \frac{X^2(z) + (d - a)^2}{4} \right\} N_0 - z X(z)N_3 \right\},
\]

(5.2-18)

where

\[
N_0 = \sin^{-1}\left(\frac{-2a + (a + d)}{(d - a)}\right) + \sin^{-1}\left(\frac{-2c + (a + d)}{(d - a)}\right) - \sin^{-1}\left(\frac{-2b + (a + d)}{(d - a)}\right) - \sin^{-1}\left(\frac{-2d + (a + d)}{(d - a)}\right) + \pi,
\]

(5.2-19)

\[
N_2 = X(a) - X(b) + X(c) - X(d) + \frac{(a + d)}{2} N_0,
\]

(5.2-20)

\[
N_3 = \ln\left[ \frac{2X(z)X(a)}{(a - z)} + \frac{2X^2(z)}{(a - z)} + (a + d - 2z) \right] - \ln\left[ - (d - a) \right]
\]

+ \ln\left[ \frac{2X(z)X(c)}{(c - z)} + \frac{2X^2(z)}{(c - z)} + (a + d - 2z) \right] - \ln\left[ \frac{2X(z)X(b)}{(b - z)} - \frac{2X^2(z)}{(b - z)} \right]

+ (a + d - 2z) - \ln[(d - a)] - \ln\left[ \frac{2X(z)X(d)}{(d - z)} + \frac{2X^2(z)}{(d - z)} + (a + d - 2z) \right].
\]

(5.2-21)

The Problem II is completely solved.
Plastic Zones:

Lengths of plastic zones at the tips \( z = d \), of crack \( L_2 \) and \( z = a \), of crack \( L_1 \), are calculated by superimposing the solutions of Problem I and Problem II. Note that the direction of applied load is to be reversed in Problem II. The stress intensity factors \( (K^I)^1 \) and \( (K^I)^{II} \) for Problem I and Problem II respectively, are superimposed. \( (K^I)^1 \) is obtained using equation (5.2-13) and replacing \( \phi(z) \) by \( \phi^I(z) \) for the Problem I. For Problem II, \( (K^I)^{II} \) is obtained replacing \( \phi(z) \) by \( \phi^{II}(z) \) from equation (5.2-18) into equation (5.2-13). The condition that stress intensity factors at the tip \( z = a \) of \( R_1 \) must balance each other, yields the equation

\[
\pi \sigma_\infty - \sigma_y e \left(N_2 + \frac{(d-a)}{2} N_0\right) = 0. \quad (5.2-22)
\]

Similarly calculating stress intensity factor \( (K^I)^d \) at tip \( z = d \) for Problem I and \( (K^{II})^d \) at tip \( z = d \) for Problem II and equating them to each other, the following non-linear equation is obtained

\[
\pi \sigma_\infty - \sigma_y e \left(N_2 - \frac{(d-a)}{2} N_0\right) = 0. \quad (5.2-23)
\]

The unknowns \( d \) and \( a \) are calculated by approximately solving non-linear equations (5.2-22) and (5.2-23) for prescribed \( a_1, d_1 \) and \( \sigma_\infty / \sigma_y \). Once \( d \) and \( a \) are calculated the plastic zone lengths at each tip \( a_1 \) and \( d_1 \) of cracks \( L_1 \) and \( L_2 \) are calculated by evaluating \( |a_1 - a| \) and \( |d - d_1| \), respectively.

Crack Opening Displacement:

To determine the shape of the crack after it has been opened due to
prescribed loads at infinity, the crack face opening displacement is calculated. The displacement in the y-direction in the plane $y = 0$ is given by Harrop [37] as:

$$u_y = \frac{4}{E} \text{Im} \left[ \int \phi(z) \, dz \right]$$

where $E$ is Young’s Modulus, $\text{Im} \left[ \right]$ denotes the imaginary part of the quantity in the bracket.

The component of displacement, $u_y$, in y-direction gives opening of crack faces in Mode I deformation. To calculate this crack opening displacement component, the non-singular terms of potentials $\phi^I(z)$ and $\phi^{II}(z)$ are superimposed thus we get the potential

$$\phi(z) = \frac{-i\sigma ye}{2\pi} N_0 X(z)$$

Displacement, $u_y$, is now obtained using equation (5.2-24) and substituting the value of $\phi(z)$ from (5.2-25) and then integrating:

$$u_y = \frac{2\sigma ye}{\pi E} N_0 \left[ \frac{-2z + (a+d)}{4} X(z) \frac{(d-a)^2}{8} \sin^{-1} \frac{-2z+(a+d)}{(d-a)} \right].$$

The crack opening displacement at four tips $a_1, b_1$ of crack $L_1$ and $c_1$ and $d_1$ of crack $L_2$ may be written using equation (5.2-26) as

$$(u_y)_{z=a_1} = \frac{2\sigma ye}{E\pi} N_0 \left[ \frac{-2a_1 + (a+d)}{4} X(a_1) \frac{(d-a)^2}{8} \sin^{-1} \frac{-2a_1+(a+d)}{(d-a)} \right],$$

$$(u_y)_{z=d_1} = \frac{2\sigma ye}{E\pi} N_0 \left[ \frac{-2d_1 + (a+d)}{4} X(d_1) \frac{(d-a)^2}{8} \sin^{-1} \frac{-2d_1+(a+d)}{(d-a)} \right].$$

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Discussions:

An illustrative numerical example is considered to study the variation of load ratio required for the closure of the prescribed plastic zone. The variation of load required to close the plastic zone at the exterior tips of the two cracks has been studied as a function of the increase in plastic zone length. The lengths of the two cracks are unequal and in the ratio of 1:1.2. The interior distance between the two cracks forms the coalesced plastic zone. It is termed the interior plastic zone.

Fig. 2. depicts the behaviour of load ratio as the size of exterior plastic zone at the tip $a_1$ is increased. The graphs are plotted for different inter-crack distances. It is observed that as the distance between the two cracks increases, their effect on each other diminishes and more load is required for closing them.

Load variation at the tip $d$ of the crack $L_2$ is plotted in Fig. 3. It is also observed in this case that as the inter-crack distance is increased, more load is required for closure of the plastic zone. It is worth mentioning that bigger inter-crack distance means larger interior plastic zone in this cases and closure of the plastic zone when crack length is not-too big affects the load required to close the plastic zone at the exterior tip also.
Fig. 2. Variation of normalised load ratio at exterior tip 'a' of crack versus plastic zone.
Fig. 3. Variation of normalised load ratio at exterior tip 'd' of crack versus plastic zone.