SYNOPSIS

Numerical methods for solving scientific and engineering problems are gaining importance and the subject has become an essential part of the training of applied mathematicians, engineers and scientists. The advent of the high speed digital computer has made the use of numerical methods for solving differential equations not only feasible but also very desirable. One reason for this is that numerical methods can give the solution when ordinary analytical methods fail. A second and perhaps more important reason for the lively interest in numerical procedures is their interrelation with digital computers. The price one pays for the general applicability of the numerical scheme is arithmetic complexity. Numerical methods have almost unlimited breadth of application.

Since digital computers provide a nearly effortless way to perform the simple but long and tedious computations involved in problems solved by numerical methods, the advent of these marvellous servants has revived interest in what would otherwise be only a specialized field of applied mathematics. Furthermore, since computers operate essentially by repetitive arithmetic operations, the way computers are programmed to solve scientific problems is by the use of procedures which we know as numerical methods.

The thesis is concerned with development of polynomial and non-polynomial spline approximation methods and their convergence analysis for direct integration of certain classes of two-point boundary-value problems in ordinary and partial differential equations. The thesis consists of six chapters followed by references useful for the development and application of the methods discussed in this thesis. A brief description of the contents of each chapter is as follows:
Chapter-I: Introduction and Review of Spline Functions

In this chapter an introduction and review of spline functions is given. Our aim is to describe some parameters which are useful in subsequent chapters. We have explained boundary-value problems, an algorithm for the solution of tridiagonal and pentadiagonal linear systems. The cubic, quintic and sextic spline function approximations, which are useful for the solution of ordinary and partial differential equations, are given in detail. The definition of the cubic and quintic spline functions is extended to piecewise non-polynomial functions depending on a parameter $\omega$. For $\omega \to 0$ these (non-polynomial) functions reduce to ordinary cubic or quintic splines. The parameter may be chosen so as to tailor the (non-polynomial) spline function to improve the accuracy in the approximate solution of certain problems. Depending on the choice of the parameter, the parametric cubic spline function is known as cubic spline in compression, cubic spline in tension or adaptive cubic spline. Similarly, three of the splines derived from quintic spline are termed as 'parametric quintic spline-I', 'parametric quintic spline-II' and 'adaptive quintic spline'.

The applications of the cubic spline in compression, quintic spline, parametric quintic spline-I and sextic spline to solution of differential equations are given in subsequent chapters.

Chapter-II: Spline Methods for Solution of Singularly-Perturbed Boundary-Value Problems

In this chapter we obtain three methods $K_1$, $K_2$ and $K_3$ for the approximate solution of boundary-value problems with a small parameter affecting highest derivative of the differential equation. The class of singularly-perturbed boundary-value problem for second order ordinary differential equation has recently gained importance in the literature for two main reasons. Firstly, they occur frequently in many areas of science and engineering, for example, combustion, chemical reactor theory, nuclear engineering, control theory, elasticity, fluid mechanics etc. A few notable examples are boundary-layer problems, WKB theory, the modelling of steady
and unsteady viscous flow problems with large Reynold number and convective heat transport problems with large Peclet number. Secondly, the occurrence of sharp boundary-layers as ε, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical methods.

The method K_1 is based on a uniformly convergent uniform mesh difference scheme using cubic spline in compression for the solution of second order singularly-perturbed boundary-value problems. The main idea is to use the condition of continuity as a discretization equation. The advantage of our second and fourth order methods is higher accuracy without increasing the computational effort. The methods of Bickley and Kadalbajoo have been shown to be special cases of our method. The method K_2 is similar to the method K_1, but here we take non-uniform mesh.

In the method K_3 we consider the self-adjoint second order singularly perturbed problem. We have obtained a fourth order method based on quintic spline. To retain the band width of the coefficient matrix as five, we develop fourth order boundary equations. Convergence analysis of the method K_1 and K_3 are given. Numerical results are tabulated to show the superiority of our methods.

Chapter-III: Parametric Cubic Spline Solution of Second Order Boundary-Value Problems

In this chapter we obtain two methods M_1 and M_2 for uniform mesh. In the method M_1, by using cubic spline in compression we have developed second and fourth order methods for the solution of a linear second order two-point boundary value problem, a special case of which arises in the theory which describes the deflection of plates and a number of other scientific applications. Bickley scheme becomes a special case of our method. For a suitable choice of the parameters our method reduces to well known Numerov’s method and fourth order quartic spline method of Usmani.

In the method M_2, we consider the system of second order boundary-value problem which arises in the study of obstacle, unilateral, moving and free boundary-value problems. By using cubic spline in compression we get a new numerical method
for obtaining solution of such system of differential equations. Our method outperforms other collocation, finite difference and spline methods including the fourth order Numerov's method and thus represents an improvement over existing methods. Convergence of the methods $M_1$ and $M_2$ are established. We have also given the application of the system of second order boundary-value problems. The numerical experiments and comparison with other methods are given to show the advantage of our methods.

Chapter-IV: Numerical Solution of Third Order Boundary-Value Problems Using Quintic Spline

In this chapter we obtain the numerical solution of third order linear and non-linear boundary-value problems. By using the quintic spline, we have developed fourth order method for the solution of linear and non linear third order boundary-value problems. Our scheme leads to the five diagonal system. To restrict the band width of the coefficient matrix of the resulting system to five, we develop the boundary equations of $O(h^7)$. Truncation error is given. Numerical illustrations are given to confirm the theoretical analysis and the results compared with those obtained by known methods to demonstrate the efficiency of our method.

Chapter-V: Sextic Spline Solution of a System of Fourth-Order Boundary Value Problems

In this chapter we obtain the fourth order numerical method based on sextic spline for solving free boundary-value problems to obtain approximate solution to a class of contact, obstacle and unilateral boundary value problems of elasticity, like those describing the equilibrium configuration of an elastic beam stretched over an elastic obstacle. In the case of a known obstacle, these problems can be alternatively formulated as a system of non-linear boundary value problems. A variety of problems arising in elasticity, mechanics, optimization and control, fluid flow through porous media etc., the problem of equilibrium of linearly elastic bodies in contact with a rigid
frictionless foundation can be characterized by a class of variational inequalities. The location of free boundary becomes an intrinsic part of the solution and no special devices are needed to locate it. For the purpose of numerical illustrations, we consider example of an elastic beam lying over an elastic obstacle. We have developed the boundary formulas of order four to restrict the band width of the coefficient matrix of the linear system to five. We have briefly presented an application of the system of boundary-value problems. The numerical illustrations and comparison with other methods are given which shows the advantage of the present scheme.

Chapter-VI: Spline Methods for the Solution of Fourth-Order Parabolic Partial Differential Equations

In this chapter we have developed new three level methods based on parametric quintic spline-I for the numerical solution of fourth-order non-homogeneous parabolic partial differential equation that governs the behaviour of a vibrating beam. It has been shown that by suitably choosing the parameters, most of the previous known methods can be derived from our method. We also obtain two new high accuracy schemes of $O(k^4, h^8)$ and $O(k^4, h^8)$. Stability analysis has been carried out. Comparison of our method with some known methods shows the superiority of the present approach.