CHAPTER - I
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INTRODUCTION

1. SPECIAL FUNCTIONS AND ITS GROWTH

A special function is a real or complex valued function of one or more real or complex variables which is specified so completely that its numerical values could in principle be tabulated. Besides elementary functions such as $x^n$, $e^x$, $\log x$, and $\sin x$, "higher" functions, both transcendental (such as Bessel functions) and algebraic (such as various polynomials) come under the category of special functions. The study of special functions grew up with the calculus and is consequently one of the oldest branches of analysis. It flourished in the nineteenth century as part of the theory of complex variables. In the second half of the twentieth century it has received a new impetus from a connection with Lie groups and a connection with averages of elementary functions. The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were solved in the eighteenth and nineteenth centuries, the boundary-value problems of electromagnetism and heat in the nineteenth, and the eigenvalue problems of quantum mechanics in the twentieth.

Seventeenth-century England was the birthplace of special functions. John Wallis at Oxford took two first steps towards the theory of the gamma function long before Euler reached it. Wallis had also the first encounter with
elliptic integrals while using Cavalieri’s primitive forerunner of the calculus. [It is curious that two kinds of special functions encountered in the seventeenth century, Wallis’ elliptic integral and Newton’s elementary symmetric functions, belongs to the class of hypergeometric functions of several variables, which was not studied systematically nor even defined formally until the end of the nineteenth century]. A more sophisticated calculus, which made possible the real flowering of special functions, was developed by Newton at Cambridge and by Leibnitz in Germany during the period 1665-1685. Taylor’s theorem was found by Scottish mathematician Gregory in 1670, although it was not published until 1715 after rediscovery by Taylor.

In 1703 James Bernoulli solved a differential equation by an infinite series which would now be called the series representation of a Bessel function. Although Bessel functions were met by Euler and others in various mechanics problems, no systematic study of the functions was made until 1824, and the principal achievements in the eighteenth century were the gamma function and the theory of elliptic integrals. Euler found most of the major properties of the gamma functions around 1730. In 1772 Euler evaluated the Beta-function integral in terms of the gamma function. Only the duplication and multiplication theorems remained to be discovered by Legendre and Gauss, respectively, early in the next century. Other significant developments were the discovery of Vandermonde’s theorem in 1772 and the definition of Legendre polynomials and the discovery of their addition theorem by Laplace and
Legendre during 1782-1785. In a slightly different form the polynomials had already been met by Liouville in 1722.

The golden age of special functions, which was centered in nineteenth century German and France, was the result of developments in both mathematics and physics: the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) which required solutions of partial differential equations containing the Laplacian operator. The discovery of elliptic functions (the inverse of elliptic integrals) and their property of double periodicity was published by Abel in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century, specially in the hands of Jacobi and Weierstrass.

Another major development was the theory of hypergeometric series which began in a systematic way (although some important results had been found by Euler and Pfaff) with Gauss’s memoir on the \( _2F_1 \) series in 1812, a memoir which was a landmark also on the path towards rigor in mathematics. The \( _3F_2 \) series was studied by Clausen (1828) and the \( _1F_1 \) series by Kummer (1836). The functions which Bessel considered in his memoir of 1824 are \( _0F_1 \) series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics. Near the end of the century Appell (1880) introduced hypergeometric functions of two variables, and Lauricella generalized them to several variables in 1893.
The subject was considered to be part of pure mathematics in 1900, applied mathematics in 1950. In physical science special functions gained added importance as solutions of the Schrödinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907 Barnes used gamma function to develop a new theory of Gauss’s hypergeometric functions $2F_1$. Various generalizations of $2F_1$ were introduced by Horn, Kampe de Feriet, MacRobert, and Meijer. From another new viewpoint, that of a differential difference equation discussed much earlier for polynomials by Appell (1880), Truesdell (1948) made a partly successful effort at unification by fitting a number of special functions into a single framework.

2. ORTHOGONAL POLYNOMIALS

Orthogonal polynomials constitute an important class of special functions in general and hypergeometric functions in particular. The subject of orthogonal polynomials is a classical one whose origins can be traced to Legendre’s work on planetary motion with important applications to physics and to probability and statistics and other branches of mathematics, the subject flourished through the first third of this century. Perhaps as a secondary effect of the computer revolution and the heightened activity in approximation theory and numerical analysis, interest in orthogonal polynomials has revived in recent years.
The ordinary hypergeometric functions have been the subject of extensive researches by a number of eminent mathematicians. These functions play a pivotal role in mathematical analysis, physics, Engineering and allied sciences. Most of the special functions, which have various physical and technical applications and which are closely connected with orthogonal polynomial and problems of mechanical quadrature, can be expressed in terms of generalized hypergeometric functions. However, these functions suffer from a shortcoming that they do not unify various elliptic and associated functions. This drawback was overcome by E. Heine through the definition of a generalized basic hypergeometric series.

3. HISTORICAL DEVELOPMENTS OF FRACTIONAL CALCULUS

The fractional calculus, like many other mathematical disciplines and ideas, has its origin in the striving for extension of meaning. Well known examples are the extensions of the integers to the rational numbers, of the real numbers to the complex numbers, of the factorials of integers to the notation of the \( \Gamma \)-function. In differential and integral calculus the question of extension of meaning is: Can the derivatives \( \frac{d^n y}{dx^n} \) of integer order, \( n > 0 \), and the \( n \)-fold integrals, be extended to have a meaning where \( n \) is any number fractional, irrational or complex? The affirmative answer has led to the so-called fractional calculus, a misnomer for the theory of operators of integration and differentiation of arbitrary (fractional) order and their applications.
Thus the theory of fractional calculus is concerned with the nth derivative and n-fold integrals when n becomes an arbitrary parameter. One versed in the calculus finds that \( \frac{d^{-1}}{dx^{-1}} \) is nothing but an indefinite integral in disguise. But fractional orders of differentiation are more mysterious because they have no obvious geometric interpretation along the lines of the customary introduction to derivatives and integrals as slopes and areas. If one is prepared to dispense with a pictorial representation, however, will soon find that fractional order derivatives and integrals are just as tangible as those of integer order and that a new dimension in mathematics opens to him when n of the operator \( \frac{d^n}{dx^n} \) becomes an arbitrary parameter. It is not a sterile exercise in pure mathematics, many problems in the physical science can be expressed and solved succinctly by recourse to fractional calculus.

This is a discipline, "as old a mathematical analysis. It can be categorized, briefly, as applicable mathematical analysis. The properties and theory of fractional operators are proper objects of study in their own right. In recent decades, they have been found useful in various fields: rheology, quantitative biology, electrochemistry, scattering theory, diffusion, transport theory, probability, statistics, potential theory and elasticity etc. However, many mathematicians and scientists are unfamiliar with this topic and thus, while the theory has developed, its use has lagged behind."
These reasons made Prof. B. Ross, the author of the above cited phrase, backed by other famous fractional analysts: Prof. A. Erdelyi, I. Sneddon and A. Zygmund to organize in 1974 in New Haven the historical first international conference on fractional calculus. Ten years later, a second international conference on fractional calculus took place in Scotland, at Ross Priory – the Strathclyde University’s stunning property on the shores of Loch Lomond. Later on, in 1989, the third international conference was organized in Tokyo, on the occasion of the hundredth anniversary of Nihon University.

Going back to the history, it seems that Leibnitz was first to try to extend the meaning of a derivative $\frac{d^n y}{dx^n}$ of integer order $n$. Perhaps, it was naive toying with the symbols that prompted L’Hospital to ask: “What if $n$ be $\frac{1}{2}$?” Leibnitz, in 1695, replied, “It will lead to a paradox” but added prophetically. “From this apparent paradox, one day useful consequences will be drawn”. In 1819 the first mentioning of a derivative of arbitrary order appeared in a published article by Lacroix. Later on, Euler and Fourier gave meaning to derivatives of arbitrary order but still without examples and applications. So, the honour of devising the first application belonged to Abel in 1823. He applied fractional calculus techniques to the solution of an integral equation, related to the so-called isochrone (tautochrone) problem. Abel’s solution was so elegant that it probably attracted the attention of Liouville to make the first major attempt in giving a logical definition of the fractional
derivative, starting from two different points of view. In 1847, Riemann, while a student, wrote a paper which was eventually published posthumously and gave the definition, whose modifications is now known as the Riemann-Liouville integral. Like other mathematical ideas, the development of fractional calculus has passed through various errors, absurdities, controversies, etc. That sometimes made mathematicians distrusting in the general concept of fractional operators. Thus, it has taken 279 years, since L’ Hospital first raised the question, for a text to appear entirely devoted to the topic of fractional calculus: the book of Oldham and Spanier [143]. The intermediate period (1695 – 1974) is thoroughly described by Ross in his “chronological Bibliography of the Fractional Calculus with commentary” [157]. Among the many authors contributing to the topic the names of Lagrange, Laplace, De Morgan, Letnikov, Lourant, Hadamard, Heaviside, Hardy, Littlewood, Weyl, Post, Erdelyi, Kober, Widder, Osler, Sneddon, Mikolas, Al-Bassam can be mentioned. A more detailed exposition could be found in the historical remarks of Ross [158] and Mikolas [139] – [141] as well as in the surveys of Sneddon [164] – [165] and Al-Bassam [3] and the huge encyclopaedic book of Samko, Kilbas and Marichev [162].

4. OPERATORS OF FRACTIONAL INTEGRATION

Certain integral equations can be deduced from or transformed to differential equations. In order to make the transformation the following
Leibnitz’s rule, concerning the differentiation of an integral involving a parameter is very useful.

\[
\frac{d}{dx} \int_{A(x)}^{B(x)} f(x,t) \, dt = \int_{A}^{B} \frac{\partial}{\partial x} F(x,t) \, dt + F(x,B) \frac{dB}{dx} - F(x,A) \frac{dA}{dx}
\]  \hspace{1cm} (4.1)

The formula (4.1) is valid if both \( F \) and \( \frac{\partial F}{\partial x} \) are continuous functions of both \( x \) and \( t \) and if both \( A'(x) \) and \( B'(x) \) are continuous.

A useful application of the formula (4.1) is the derivation of the following formula:

\[
\int_{a}^{x} \cdots \int_{a}^{x} f(x) \, dx \, dx \ldots dx = \frac{1}{(n-1)!} \int_{a}^{x} (x-t)^{n-1} f(t) \, dt
\]  \hspace{1cm} (4.2)

Formula (4.2) is known as Cauchy’s Multiple Integral formula. It reduces \( n \)-fold integrals into a single integral. Fractional integration is an immediate generalization of repeated integration. If the function \( f(x) \) is integrable in any interval \((0, a)\) where \( a > 0 \), the first integral \( F_1(x) \) of \( f(x) \) is defined by the formula

\[
F_1(x) = \int_{0}^{x} f(t) \, dt,
\]

and the subsequent integrals by the recursion formula

\[
F_{r+1}(x) = \int_{0}^{x} F_{r}(t) \, dt, \quad r = 1, 2, 3, \ldots
\]

It can easily be proved by induction that for any positive integer \( n \)
Similarly an indefinite integral $F^*_n(x)$ is defined by the formulae

$$F^*_1(x) = -\int_x^\infty f(t) \, dt, \quad F^*_r(x) = -\int_x^\infty F^*_r(t) \, dt, \quad r = 1, 2, \ldots$$

and again it can be shown by induction that for any positive integer $n$,

$$F^*_n(x) = \frac{1}{n!} \int_x^\infty (t-x)^n f(t) \, dt.$$  \hspace{1cm} (4.4)

The Riemann-Liouville fractional integral is a generalization of the integral on the right-hand side of equation (4.3). The integral

$$R_\alpha \{ f(t); x \} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt$$  \hspace{1cm} (4.5)

is convergent for a wide class of function $f(t)$ if $\text{Re} \: \alpha > 0$. Integral (4.5) is called Riemann-Liouville fractional integral of order $\alpha$. Integrals of this kind occur in the solution of ordinary differential equations where they are called Euler transforms of the first kind. There are alternative notations for $R_\alpha \{ f(t); x \}$ such as $I^\alpha f(x)$ used by Marcel Riesz.

The Weyl fractional integral is a generalization of the integral on the right-hand side of equation (4.4). It is defined by the equation (Weyl)

$$W_\alpha \{ f(t); x \} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t) \, dt, \quad \text{Re} \: \alpha > 0.$$  \hspace{1cm} (4.6)

A pair of operators of fractional integration of a general kind have been introduced by Erdelyi and Kober (see Kober [132], Erdelyi and Kober [44],
Erdelyi [42-43], Kober [133] and it seems appropriate to call the operators $I_{\eta,\alpha}$, $K_{\eta,\alpha}$ defined below which are simple modifications of these operators, the Erdelyi-Kober operators. The operators $I_{\eta, \alpha}$ and $K_{\eta, \alpha}$ are defined by the formulae

\begin{equation}
I_{\eta, \alpha} f(x) = x^{-2\alpha-2\eta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x \left( x^2 - u^2 \right)^{\alpha-1} u^{2\eta+1} f(u) \, du
\end{equation}

\begin{equation}
K_{\eta, \alpha} f(x) = x^{2\eta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_x^\infty \left( u^2 - x^2 \right)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u) \, du
\end{equation}

where $\Re\alpha > 0$ and $\Re\eta > -\frac{1}{2}$.

The modified operator of the Hankel transform is defined by

\begin{equation}
S_{\eta, \alpha} f(x) = \left( \frac{2}{x} \right)^{\alpha} \int_0^\infty t^{1-\alpha} J_{2\eta+\alpha}(xt) f(t) \, dt.
\end{equation}

5. DEFINITIONS NOTATIONS AND RESULTS USED

Frequently occurring definitions, notations and results used in this thesis are as given under:

**THE GAMMA FUNCTION**: The gamma function is defined as
\[ \Gamma(z) = \begin{cases} \int_0^{\infty} t^{z-1} e^{-t} \, dt, & \text{Re}(z) > 0 \\ \frac{\Gamma(z+1)}{z}, & \text{Re}(z) < 0, z = 0, -1, -2, -3, \ldots \end{cases} \] (5.1)

POCHHAMMER'S SYMBOL AND THE FACTORIAL FUNCTION:

The Pochhammer symbol \((\lambda)_n\) is defined as

\[ (\lambda)_n = \begin{cases} \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & \text{if } n=1,2,3,\ldots \\ 1, & \text{if } n=0 \end{cases} \] (5.2)

Since \((1)_n = n!\), \((\lambda)_n\) may be looked upon as a generalization of elementary factorial. In terms of gamma function, we have

\[ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \lambda \neq 0, -1, -2, \ldots \] (5.3)

The binomial coefficient may now be expressed as

\[ \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!} = \frac{(-1)^n (-\lambda)_n}{n!} \] (5.4)

Also, we have

\[ (\lambda)_n = \frac{(-1)^n}{(1-\lambda)_n}, n = 1,2,3,\ldots, \lambda \neq 0, \pm 1, \pm 2, \ldots \] (5.5)

Equation (5.3) also yields

\[ (\lambda)_{m+n} = (\lambda)_m (\lambda + m)_n \] (5.6)

which, in conjunction with (5.5), gives

\[ (\lambda)_{n-k} = \frac{(-1)^k (\lambda)_n}{(1-\lambda-n)_k}, 0 \leq k \leq n. \]

For \(\lambda = 1\), we have
\((-n)_k = \begin{cases} (-1)^k \frac{n!}{(n-k)!}, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \) \hspace{1cm} (5.7)

**LEGENDRE'S DUPLICATION FORMULA**: In view of the definition (5.2), we have

\[(\lambda)_{2n} = 2^{2n} \binom{\lambda}{2^n} \left(\frac{\lambda + 1}{2}\right)_n, \quad n = 0, 1, 2, \ldots \quad (5.8)\]

which follows also from Legendre's duplication formula for the Gamma function, viz.

\[\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2} \quad (5.9)\]

**GAUSS'S MULTIPLICATION THEOREM**: For every positive integer, we have

\[(\lambda)_m = m^m \prod_{j=0}^{m-1} \left(\frac{\lambda + j - 1}{m}\right)_n, \quad n = 0, 1, 2, \ldots \quad (5.10)\]

which reduces to (5.8) when \(m = 2\).

**THE BETA FUNCTION**: The Beta function \(B(\alpha, \beta)\) is a function of two complex variables \(\alpha\) and \(\beta\), defined by

\[B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt, & \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \\ \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}, & \text{Re}(\alpha) < 0, \text{Re}(\beta) < 0, \alpha, \beta \neq -1, -2, -3 \end{cases} \quad (5.11)\]
THE GENERALIZED HYPERGEOMETRIC FUNCTION: The generalized hypergeometric function is defined as

\[ _pF_q \left[ \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q \end{array} \bigg| z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!} \]  

(5.12)

where \( \beta_j \neq 0, -1, -2, -3, \ldots; j = 1, 2, \ldots, q. \)

The series in (5.12)

(i) Converges for \(|z| < \infty\) if \(p \leq q.\)

(ii) Converges for \(|z| < 1\) if \(p = q + 1,\) and

(iii) Diverges for all \(z, z \neq 0,\) if \(p > q + 1.\)

Furthermore, if we set

\[ \omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \]

then the \(_pF_q\) series, with \(p = q + 1,\) is

I. Absolute convergent for \(|z| = 1,\) if \(\text{Re} (\omega) > 0,\)

II. Conditionally convergent for \(|z| = 1, z \neq 1,\)

If \(-1 < \text{Re} (\omega) \leq 0,\) and

III. Diverges for \(|z| = 1\) of \(\text{Re} (\omega) \leq -1.\)

Further, a symbols of the type \(\Delta (k, \alpha)\) stands for the set of \(k\) parameters

\[ \frac{\alpha}{k}, \frac{\alpha + 1}{k}, \ldots, \frac{\alpha + k - 1}{k}. \]

Thus
in terms of hypergeometric function, we have

\[(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = {}_1F_0\left[\begin{array}{c} a \\ \frac{1}{z} \end{array} \right]\]

\[e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = {}_0F_0\left[\begin{array}{c} - \\ \frac{1}{z} \end{array} \right]\]

The linear transformations of the hypergeometric function, known as Euler’s transformations are as follows:

\[{}_{2}F_{1}\left[\begin{array}{c} a, b \\ c \end{array} ; z \right] = (1-z)^{-a} {}_{2}F_{1}\left[\begin{array}{c} a, c-b \\ c \end{array} ; \frac{z}{z-1} \right]\]

\[c \neq 0, -1, -2, \ldots, |\arg (1-z)| < \pi;\]

\[{}_{2}F_{1}\left[\begin{array}{c} a, b \\ c \end{array} ; z \right] = (1-z)^{-a-b} {}_{2}F_{1}\left[\begin{array}{c} c-a, c-b \\ c \end{array} ; z \right]\]

\[c \neq 0, -1, -2, \ldots, |\arg (1-z)| < \pi.\]

Appell function of two variables are defined as

\[F_{1}[a, b, b'; c; x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (b')_{n} x^m y^n}{m! n! (c)_{m+n}},\]

\[\max \{ |x|, |y| \} < 1;\]

\[F_{2}[a, b, b'; c, c'; x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (b')_{n} x^m y^n}{m! n! (c)_{m} (c')_{n}},\]

\[|x| + |y| < 1;\]
\[ F_3[a, a', b, b'; c; x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_m (a')_n (b)_m (b')_n)}{m! n! (c)_m (c')_n} x^m y^n, \]
max \{ |x|, |y| \} < 1; \tag{5.20} \\

\[ F_4[a, b; c, c'; x, y] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{m! n! (c)_m (c')_n} x^m y^n, \]
\[ \sqrt{|x|} + \sqrt{|y|} < 1. \tag{5.21} \]

Lauricella (1893) further generalized the four Appell functions \( F_1, F_2, F_3, F_4 \) to functions of \( n \) variables.

\[ F_A^{(a)}[a, b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n] \]
\[ = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}, \]
\[ |x_1| + \ldots + |x_n| < 1; \tag{5.22} \]

\[ F_B^{(a)}[a_1, \ldots, a_n, b_1, \ldots, b_n; c; x_1, \ldots, x_n] \]
\[ = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a_1)_{m_1} \ldots (a_n)_{m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1 + \ldots + m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}, \]
\[ \max \{ |x_1|, |x_2|, \ldots, |x_n| \} < 1; \tag{5.23} \]

\[ F_C^{(a)}[a, b; c_1, \ldots, c_n; x_1, \ldots, x_n] \]
\[ = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{(a_1)_{m_1 + \ldots + m_n} (b)_{m_1 + \ldots + m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!}, \]
\[ \sqrt{|x_1|} + \ldots + \sqrt{|x_n|} < 1 \tag{5.24} \]
\[ F_D^{(c)} \left[ a, b_1, \ldots, b_n; c; x_1, \ldots, x_n \right] \]

\[ = \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_1} \ldots (b)_{m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c)_{m_1+\ldots+m_n} m_1! \ldots m_n!}, \]

\[ \text{max} \{ |x_1|, \ldots, |x_n| \} < 1 \quad (5.25) \]

Clearly, we have

\[ F_A^{(2)} = F_2, \quad F_B^{(2)} = F_3, \quad F_C^{(2)} = F_4, \quad F_D^{(2)} = F_1, \]

\[ F_A^{(1)} = F_B^{(1)} = F_C^{(1)} = F_D^{(1)} = \text{2}_1 \text{F}_1, \]

Also, we have

\[ F_D^{(3)} \left[ a, b_1, b_2, b_3; c; x, y, z \right] \]

\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{k+m+n} (b_1)_k (b_2)_m (b_3)_n x^k y^m z^n}{(c)_{k+m+n} k! m! n!}. \quad (5.26) \]

Just as the Gaussian \( _2F_1 \) function was generalized to \( _pF_q \) by increasing the number of numerator and denominator parameters, the four Appell functions were unified and generalized by Kampe' de Feriet (1921) who defined a general hypergeometric function of two variables.

The notation introduced by Kampe' de Feriet for his double hypergeometric function of superior order was subsequently abbreviated in 1941 by Burchnall and Chaundy [21]. We recall here the definition of a more general double hypergeometric function [than the one defined by Kampe' de Feriet] in a slightly modified notation [see, for example Srivastava and Panda [169]].
where, for convergence

(i) \( p + q < 1 + m + 1, p + k < 1 + n + 1, |x| < \infty, |y| < \infty, \) or

(ii) \( p + q = 1 + m + 1, p + k = 1 + n + 1, \) and

\[
\max\{|x|, |y|\} < 1, \text{if } p > 1
\]
\[
\max\{|x|, |y|\} < 1, \text{if } p \leq 1
\] (5.28)

Although the double hypergeometric function defined by (5.27) reduces to the Kampe' de F'eriet function in the special case:

\[ q = k \text{ and } m = n, \]

yet it is usually referred to in the literature as the Kampe’ de F’eriet function.

Similarly, a general triple hypergeometric series \( F^{(3)} [x, y, z] \) [cf. Srivastava [166], p. 428] is defined as:

\[
F^{(3)} [x, y, z] = F^{(3)} \left[ \begin{array}{c}
(a); (b); (b^*); (d); (d^*); \\
(e); (g); (g^*); (h); (h^*)
\end{array} \right]
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda(m, n, p) \frac{x^m y^n z^p}{m! n! p!},
\] (5.29)

where, for convenience
\[ \Lambda(m, n, p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p} \prod_{j=1}^{B}(b_j)_{m+n} \prod_{j=1}^{B'}(b'_j)_{n+p} \prod_{j=1}^{B''}(b''_j)_{p+m} \prod_{j=1}^{D}(d_j)_{m} \prod_{j=1}^{D'}(d'_j)_{n} \prod_{j=1}^{D''}(d''_j)_{p}}{\prod_{j=1}^{E}(e_j)_{m+n+p} \prod_{j=1}^{G}(g_j)_{m+n} \prod_{j=1}^{G'}(g'_j)_{n+p} \prod_{j=1}^{G''}(g''_j)_{p+m} \prod_{j=1}^{H}(h_j)_{m} \prod_{j=1}^{H'}(h'_j)_{n} \prod_{j=1}^{H''}(h''_j)_{p}} \]

(5.30)

where (a) abbreviates, the array of parameters \( a_1, a_2, \ldots, a_A \), with similar interpretations for (b), (b'), (b''), et cetera. The triple hypergeometric series in (5.29) converges absolutely when

\[
\begin{align*}
1 + E + G + G' + H - A - B - B' - C &> 0 \\
1 + E + G + G'' + H' - A - B' - B'' - C^0 &> 0 \\
1 + E + G' + G'' + H'' - A - B' - B'' - C'^0 &> 0
\end{align*}
\]

(5.31)

where the equalities hold true for suitable constrained values of \(|x|, |y|, |z|\).

In this thesis we also need to extend the definition of Kampe’ de F’eriet’s double hypergeometric function to generalized triple hypergeometric function. We define and denote the extended form as follows:

\[
\begin{align*}
&\left[ \left( (a), \alpha, \beta, \gamma; (b), \delta, \varepsilon; (c), \eta, \theta; (d), \phi, \rho; \right.ight. \\
&\left. \left( (e), \xi; (f), \psi; (g), \chi; \\
&\left. (n), \sigma; (p), \Delta; (q), \nabla \right) \right] \\
&x, y, z
\end{align*}
\]

\[
\begin{align*}
&= \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{\alpha k + \beta r + \gamma s} \prod_{j=1}^{B}(b_j)_{\delta k + \varepsilon r + \zeta s} \prod_{j=1}^{C}(c_j)_{\eta k + \theta r + \zeta s} \prod_{j=1}^{D}(d_j)_{\phi k + \rho r + \varphi s}}{\prod_{j=1}^{E}(e_j)_{\xi k + \psi r + \chi s} \prod_{j=1}^{F}(f_j)_{\varphi k + \psi r + \chi s} \prod_{j=1}^{G}(g_j)_{\zeta k + \sigma r + \tau s} \prod_{j=1}^{H}(h_j)_{\Delta k + \nabla r + \omega s}} \\
&\times \frac{\prod_{j=1}^{E}(e_j)_{\xi k} \prod_{j=1}^{F}(f_j)_{\varphi r} \prod_{j=1}^{G}(g_j)_{\zeta s}}{\prod_{j=1}^{N}(n_j)_{\alpha k} \prod_{j=1}^{P}(p_j)_{\delta r} \prod_{j=1}^{Q}(q_j)_{\sigma s}} x^{k} y^{r} z^{s} \quad (5.32)
\end{align*}
\]
For $\alpha = \beta = \gamma = \delta = \epsilon = \eta = \theta = \phi = \rho = \xi = \psi = \chi = \lambda = \mu = \nu = \zeta = \tau = t$

$= u = v = w = \sigma = \Delta = \nabla = 1$, (5.32) reduces to the three variable version of the general form of the Kampe’ de Féret’s double hypergeometric function.

6. FRACTIONAL DERIVATIVES AND HYPERGEOMETRIC FUNCTIONS :

The simplest approach to a definition of a fractional derivatives commences with the formula

$$\frac{d^\alpha}{dz^\alpha}(e^{az}) = D_z^\alpha(e^{az}) = a^\alpha e^{az} \quad (6.1)$$

where $\alpha$ is an arbitrary (real or complex) number.

For a function expressible as

$$f(z) = \sum_{n=0}^{\infty} C_n \exp(a_n z) \quad (6.2)$$

Liouville defines the fractional derivative of order $\alpha$ by

$$D_z^\alpha \{ f(z) \} = \sum_{n=0}^{\infty} C_n a_n^\alpha \exp(a_n z) \quad (6.3)$$

In 1731 Euler extended the derivative formula

$$D_z^\alpha \{ z^\lambda \} = \lambda (\lambda - 1) \ldots \ldots (\lambda - n + 1) z^{\lambda - n} \quad (n = 0, 1, 2, \ldots)$$

$$= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - n + 1)} z^{\lambda - n} \quad (6.4)$$

to the general form :

$$D_z^\mu \{ z^\lambda \} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \mu + 1)} z^{\lambda - \mu} \quad (6.5)$$
where $\mu$ is an arbitrary complex number.

Another approach to fractional calculus begins with Cauchy's iterated integral:

$$
D_z^{-n} \{ f(z) \} = \int_0^z \cdots \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_1) \, dt_1 \, dt_2 \cdots dt_n
$$

$$
= \frac{1}{(n-1)!} \int_0^z f(t) (z-t)^{m-1} \, dt, \quad n=1,2,3, \ldots
$$

(6.6)

writing $\mu$ for $-n$, we get

$$
D_z^{-\mu} \{ f(z) \} = \frac{1}{\Gamma(-\mu)} \int_0^z f(t) (z-t)^{-\mu-1} \, dt, \quad \text{Re} \, (\mu) < 0,
$$

(6.7)

where the path of integration is along a line from 0 to $z$ in the complex $t$-plane and $\text{Re} \, (\mu) < 0$.

Equation (6.7) defines the Riemann-Liouville fractional integral of order $-\mu$; it is usually denoted by $\Gamma^{-\mu} f(z)$.

In case $m - 1 < \text{Re} \, (\mu) < m$ ($m = 1, 2, 3, \ldots$), it is customary to write (6.7) in the form:

$$
D_z^{-\mu} \{ f(z) \} = \frac{d^m}{dz^m} D_z^{-m} \{ f(z) \}
$$

$$
= \frac{d^m}{dz^m} \frac{1}{\Gamma(-\mu+m)} \int_0^z f(t) (z-t)^{-\mu+m-1} \, dt.
$$

(6.8)

The representation (6.7) is consistent with (6.5) when $f(z) = z^\lambda$, since
starting from the iterated integral

\[ D^m_z \{ f(z) \} = \frac{1}{\Gamma(-\mu)} \int \int \cdots \int f(t) \, dt_1 \, dt_2 \cdots dt_n \]

\[ = \frac{1}{(n-1)!} \int f(t) (t-z)^{m-1} \, dt, \quad n=1, 2, 3, \ldots \ldots \quad (6.10) \]

and replacing \( n \) by \(-\mu\), we are led to the definitions

\[ \omega D^m_z \{ f(z) \} = \frac{1}{\Gamma(-\mu)} \int f(t) (t-z)^{m-1} \, dt, \quad \text{Re} (\mu) < 0, \quad (6.11) \]

\[ \omega D^m_z \{ f(z) \} = \frac{d^m}{dz^m} \left( \omega D^{m-1}_z \{ f(z) \} \right) \]

\[ = \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(-\mu+m)} \int f(t) (t-z)^{-\mu-m-1} \, dt \right\} \quad (6.12) \]

\( m - 1 < \text{Re} (\mu) < m \quad (m = 1, 2, 3, \ldots \ldots) \).

Equation (6.11) defines the Weyl fractional integral or order \(-\mu\); it is usually denoted by \( K^{-\mu} f(z) \).

There is yet another approach based upon the generalization of Cauchy's integral formula:

\[ D^n_z \{ f(z) \} = \frac{n!}{2\pi i} \oint \frac{f(t)}{(t-z)^{n+1}} \, dt, \quad (6.13) \]
which is employed by Nekrossov.

The literature contains many examples of the use of fractional derivatives in the theory of hypergeometric functions, in solving ordinary and partial differential equations and integral equations, as well as in other contexts. Although other methods of solution are usually available, the fractional derivative approach to these problems often suggests methods that are not so obvious in a classical formulation.

Fractional derivative operator plays the role of augmenting parameters in the hypergeometric functions involved. Applying this operator on identities involving infinite series a large number of generating functions for a variety of special functions have been obtained by numerous mathematicians.

The following theorem on term-by-term fractional differentiation embodies, in an explicit form, the definition of a fractional derivative of an analytic function:

**THEOREM 1.** If a function \( f(z) \), analytic in the disc \( |z| < p \), has the power series expansion

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < p,
\]

provided that \( \text{Re} (\lambda) > 0, \text{Re} (\mu) < 0 \) and \( |z| < p \).

Yet another theorem is stated as follows:

\[
D_z^\mu \left \{ z^{\lambda-1} f(z) \right \} = \sum_{n=0}^{\infty} a_n D_z^\mu \left \{ z^{\lambda+n-1} \right \}
= \frac{\Gamma (\lambda)}{\Gamma (\lambda - \mu)} z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{a_n (\lambda)_n}{(\lambda - \mu)_n} z^n
\]

provided that \( \text{Re} (\lambda) > 0, \text{Re} (\mu) < 0 \) and \( |z| < p \).
THEOREM 2. Under the hypotheses surrounding equation (5.14),

\[
D_z \left\{ z^{\lambda-1} f(z) \right\} = \sum_{n=0}^{\infty} a_n D_z^{\mu} \left\{ z^{\lambda+n-1} \right\}
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)} z^{\lambda-\mu-1} \sum_{n=0}^{\infty} \frac{a_n (\lambda)_n}{(\lambda-\mu)_n} z^n,
\]  

(6.16)

provided that \( \text{Re} (\lambda) > 0 \) and \( |z| < p \).

The following fractional derivative formulas are useful in deriving generating functions:

\[
D_z^{\lambda-n} \left\{ z^{\lambda-1} (1-az)^{-\alpha} (1-bz)^{-\beta} (1-cz)^{-\gamma} \right\}
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-n-1} F_D^{(3)} \left[ \lambda, \alpha, \beta, \gamma; \mu; az, bz, cz \right]
\]

(6.17)

\[
\text{Re} (\lambda) > 0, |az| < 1, |bz| < 1, |cz| < 1;
\]

\[
D_y^{\lambda-n} \left\{ y^{\lambda-1} (1-y)^{-\alpha} F \left[ \alpha, \beta; \frac{x}{1-y} \right] \right\}
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma(\mu)} y^{\mu-n} F \left[ \alpha, \beta, \lambda; \gamma, \mu; x, y \right],
\]

(6.18)

\[
\text{Re} (\lambda) > 0, |x| + |y| < 1.
\]

In particular (6.17) with \( c = 0 \) yields

\[
D_z^{\lambda-n} \left\{ z^{\lambda-1} (1-az)^{-\alpha} (1-bz)^{-\beta} \right\}
\]

\[
= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-n} F \left[ \lambda, \alpha, \beta; \mu; az, bz \right],
\]

(6.19)

\[
\text{Re} (\lambda) > 0, |az| < 1, |bz| < 1.
\]

on the other hand, in its special case when \( a = 1 \) and \( b = c = 0 \), (6.17) reduces immediately to
SISTER CELINE'S POLYNOMIALS OF TWO VARIABLES

SUGGESTED BY APPELL’S FUNCTIONS: In 1947 Sister Mary Celine Fasenmyer concentrated her studies on polynomials generated by

\[(1 - t)^{-1} pF_q \left[ \begin{array}{c} a_1, \ldots, a_p ; -4xt \\ b_1, \ldots, b_q ; (1 - t)^x \end{array} \right] = \sum_{n=0}^{\infty} f_n \left[ \begin{array}{c} a_1, \ldots, a_p ; x \\ b_1, \ldots, b_q ; x \end{array} \right] t^n \]  

(7.1)

which yield

\[ f_n \left[ \begin{array}{c} a_1, \ldots, a_p ; x \\ b_1, \ldots, b_q ; x \end{array} \right] = p_{+2}F_{q+2} \left[ \begin{array}{c} -n, n + 1, a_1, \ldots, a_p ; x \\ 1, \frac{1}{2}, b_1, \ldots, b_q \end{array} \right] \]  

(7.2)

Her polynomials include as special cases the Legendre polynomials \( P_n (1-2x) \), some Jacobi polynomials, Rice’s \( H_n (\xi, \rho, \nu) \), Bateman’s \( Z_n (x) \), \( F_n (z) \) and Pasternak’s \( F_{n}^{m} (z) \) which is a generalization of Bateman’s \( F_n (x) \). The simple Bessel polynomial \( y_n (x) \) is also included. She also obtained a few results of interest for some of the simpler of her polynomials for instance she derived the result

\[ f_n \left[ \begin{array}{c} a_1, \ldots, a_p ; x \\ b_1, \ldots, b_q ; x \end{array} \right] = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{y^2} e^{-y} f_n \left[ \begin{array}{c} a_1, \ldots, a_p ; x, y \end{array} \right] dy \]  

(7.3)

which includes
\[ P_n (1 - 2x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-\frac{1}{2}} e^{-y} f_n (-; xy) \, dy \] (7.4)

and

\[ H_n (\xi, p, v) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-\frac{1}{2}} e^{-y} f_n (\xi, p, vy) \, dy \] (7.5)

using \( p = 1, q = 1, a_1 = \frac{1}{2}, b_1 = 1 \), we find that Sister Celine’s (7.3) becomes

\[ f_n \left( \frac{1}{2}; 1; x \right) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} y^{-\frac{1}{2}} e^{-y} f_n (-; 1; xy) \, dy \] (7.6)

As she points out for Bateman’s \( Z_n(x) \)

\[ f_n \left( \frac{1}{2}; 1; x \right) = z_n (x) \] (7.7)

and in terms of the simple Laguerre polynomial

\[ f_n (-; 1; \frac{1}{2}) = L_n (x) L_n (-x) \] (7.8)

By combining (7.6), (7.7) and (7.8), sister Celine obtained

\[ Z_n (x^2) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \exp \left( -\frac{\alpha^2}{4} \right) = L_n (\alpha x) L_n (-\alpha x) d\alpha . \] (7.9)

In chapter II of the present thesis four new classes of polynomials suggested by the Four Appell’s Functions have been introduced which are two variable generalization of Sister Celine’s polynomials. Certain double generating functions, integral relations, fractional derivatives, interconnections and certain Burchnall and Chaundy type expansions of these polynomials have been obtained.
8. A NOTE ON A THREE VARIABLES ANALOGUE OF BESSEL POLYNOMIALS: In 1949 Krall and Frink [136] initiated a study of simple Bessel polynomial

\[ Y_n(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \left( \frac{x}{2} \right)^k \]  

(8.1)

and generalized Bessel polynomial

\[ Y_n(a, b, x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \left( \frac{x}{b} \right)^k \]  

(8.2)

These polynomials were introduced by them in connection with the solution of the wave equation in spherical coordinates. They are the polynomial solutions of the differential equation.

\[ x^2 y''(x) + (ax + b) y'(x) = n(n + a - 1) y(x) \]  

(8.3)

where \( n \) is a positive integer and \( a \) and \( b \) are arbitrary parameters. These polynomials are orthogonal on the unit circle with respect to the weight function

\[ \rho(x, \alpha) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha + n - 1)} \left( \frac{2}{x} \right)^n. \]  

(8.4)

Several authors including Agarwal [2], Al-Salam [5], Brafman [17], Burchnall [22], Carlitz [27], Chatterjea [36], Dickinson [38], Eweida [48], Grosswald [57], Rainville [154] and Toscano [178] have contributed to the study of the Bessel polynomials.

Recently in the year 2000, Khan and Ahmad [116] studied two variables analogue \( Y_n^{(a, b)}(x, y) \) of the Bessel polynomials \( Y_n^{(a)}(x) \) defined by
Chapter III concerns with a study of a three variables analogue \( Y^{(a, \beta, \gamma)}(x, y, z; a, b, c) \) of (8.2). Certain integral representations, a Schlafl's contour integral, a fractional integral, Laplace transformations, some generating functions and double and triple generating functions have been obtained for \( Y^{(a, \beta, \gamma)}(x, y, z; a, b, c) \).

9. A STUDY OF THREE VARIABLE ANALOGUES OF CERTAIN FRACTIONAL INTEGRAL OPERATORS: The fractional calculus has been investigated by many mathematicians [156]. In their works the Riemann–Liouville operator (R – L) defined by

\[
R^{a}_{0, x} f = \frac{1}{\Gamma(a)} \int_{0}^{x} (x - t)^{a} f(t) dt
\]  

was the most central, while Erdelyi and Kober defined their operator (E – K) in connection with the Hankel transform [132] as

\[
I^{\alpha, \eta}_{0, x} f = \frac{x^{x-\alpha-\eta}}{\Gamma(\alpha)} \int_{0}^{x} (x - t)^{\alpha-1} t^{\eta} f(t) dt
\]  

Weyl and another Erdelyi – Kober fractional operators are defined as follows:

\[
W^{a}_{x, \infty} f = \frac{1}{\Gamma(a)} \int_{x}^{\infty} (t - x)^{a-1} f(t) dt
\]  

and
\[ K^{\gamma, \alpha}_{x, \infty} f = \frac{X^{\gamma}}{\Gamma(\alpha)} \int_{x}^{\infty} (x-t)^{\alpha-1} t^{-\gamma-\alpha} f(t) \, dt \]  

respectively.

In 1978, M. Saigo [159] defined a certain integral operator involving the Gauss hypergeometric function as follows:

Let \( \alpha > \beta \) and \( \eta \) be real numbers. The fractional integral operator \( I^{\alpha, \beta, \eta}_{x} \),

which acts on certain functions \( f(x) \) on the interval \((0, \infty)\) is defined by

\[ I^{\alpha, \beta, \eta}_{x} f = \frac{X^{\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} F((\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x}) f(t) \, dt \]  

where \( \Gamma \) is the gamma function, \( F \) denotes the Gauss hypergeometric series

\[ 2F_{1}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n, \quad |z| < |\]  

and its analytic continuation into \( |\arg(1-z)| < \pi \), and \( (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \).

Such an integral was first treated by Love [137] as an integral equation. However, if one regards the integral as an operator with a slight change, it will contain as special cases both \( R - L \) and \( E - K \) owing to reduction formulas for the Gauss function by restricting the parameters. The more interesting fact is that for this operator two kinds of product rules may be made up by virtue of Erdelyi's formulas [41], which were first proved by using the method of fractional integration by parts in the \( R - L \) sense. From the rules, of course, the ones for \( R - L \) and \( E - K \) are deduced. Moreover this operator is representable by products of \( R - L \)'s, from which it is possible to obtain the integrability and
estimations of Hardy – Littlewood type [59]. Saigo [159] also defined an integral operator on the interval \((x, \infty)\) as an extension of operators of Weyl and another Erdelyi – Kober operators as follows:

Under the same assumptions in defining (9.5), the integral operator \(J_x^{\alpha, \beta, \eta}\) is defined by

\[
J_x^{\alpha, \beta, \eta} f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t - x)^{\alpha-1} t^{-\alpha} F\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t}\right) f(t) dt
\]

(9.7)

Later on in 1988, Saigo and Raina [161] obtained the generalized fractional integrals and derivatives introduced by Saigo [159], [160] of the system \(S_q^n(x)\), where the general system of polynomials

\[
S_q^n(x) = \sum_{r=0}^{\lfloor n\rfloor} \frac{(-n)_{qr}}{r!} A_{n,r} x^r
\]

were defined by Srivastava [171], where \(q > 0\) and \(n \geq 0\) are integers, and \(A_{n,r}\) are arbitrary sequence of real or complex numbers.

The three variable analogues of Saigo’s operators (9.5) and (9.7) as follows:

I. Let \(c > 0\), \(c' > 0\), \(c'' > 0\), \(a, b, b', b''\) be real numbers. A three variable analogue of fractional integral operator \(I_{0,x}^{\alpha, \beta, \eta}\) due to M. Saigo is defined as

\[
I_{0,x}^{\alpha, \beta, \eta} f(x,y,z) = \frac{x^{-a} y^{-a} z^{-a}}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int_0^x \int_0^y \int_0^z (x-u)^{c-1} (y-v)^{c'-1} (z-w)^{c''-1}
\]
where \( F_{A}^{(3)} \) is a Lauricella function of three variables defined by

\[
F_{A}^{(3)} \left[ a, b, b', b''; c, c', c''; \right] \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{n+r+s} (b)_{n} (b')_{r} (b'')_{s}}{n! \cdot r! \cdot s!} \frac{(c)_{n} (c')_{r} (c'')_{s}}{(c + x)^{n} (c' + y)^{r} (c'' + z)^{s}}
\]

SPECIAL CASES:

(i) For \( a = b = b' = b'' = 0, c = \alpha, c' = \beta, c'' = \gamma \), (9.8) reduces to

\[
R_{0,0,0,0;0,\alpha;0,\beta;0,\gamma} \left[ x, y, z \right] = \frac{1}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma)} \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} (x-u)^{\alpha-1} (y-v)^{\beta-1} (z-w)^{\gamma-1} f(w, v, w) dw dv du
\]

(ii) For \( a = c = \alpha, b = -\eta, b' = b'' = 0, c' = \beta, c'' = \gamma \), (9.8) becomes

\[
E_{0,0,0,0;0,\alpha;0,\beta;0,\gamma} \left[ x, y, z \right] = \frac{x^{-\alpha} y^{-\alpha} z^{-\alpha}}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\gamma)} \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} (x-u)^{\alpha-1} (y-v)^{\beta-1} (z-w)^{\gamma-1} u^{\eta} f(u, v, w) dw dv du
\]

(iii) For \( a = c = \alpha, b = b'' = 0, b' = -\eta, c' = \beta, c'' = \gamma \), (9.8) gives
\[ I_{\alpha,\beta,\gamma}^{\alpha,0,-\eta,0;0,0,0} f(x, y, z) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\beta-1}(z-w)^{\gamma-1} f(u, v, w) dw dv du \]

(9.11)

(iv) For \( a = c = \alpha, b = b' = 0, b'' = -\eta, c' = \beta, c'' = \gamma \), (9.8) yields

\[ I_{\alpha,\beta,\gamma}^{\alpha,0,0,-\eta;0,0,0} f(x, y, z) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\beta-1}(z-w)^{\gamma-1} w^n f(u, v, w) dw dv du \]

(9.12)

Here (9.10), (9.11) and (9.12) may be regarded as three variable analogues of Erdelyi Kober fractional integral operator.

Under the same conditions of (9.8), a three variable analogues of \( J_{x,\infty}^{\alpha,\beta,\gamma} \) is as defined below:

\[ J_{x,\infty}^{\alpha,\beta,\gamma} f(x, y, z) = \frac{1}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{c-1}(v-y)^{c'-1}(w-z)^{c''-1} \]

\[ \frac{u^{-c}v^{-c'}w^{-c''}}{\Gamma_{A}^{(3)}} \left[ a, b, b'; 1 - \frac{x}{u}, 1 - \frac{y}{v}, 1 - \frac{z}{w} \right] f(u, v, w) dw dv du \]

(9.13)

**SPECIAL CASES:**

(i) For \( a = b = b' = b'' = 0, c = \alpha, c' = \beta, c'' = \gamma \), (9.13) reduces to
\[ f(x, y, z) = \int f(x, y, z) \]

\[ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_{x}^{\infty} \int_{y}^{\infty} \int_{z}^{\infty} (u-x)^{a-1}(v-y)^{b-1}(w-z)^{c-1} f(u, v, w) \, dw \, dv \, du \]

\[ (9.14) \]

We may consider (9.14) as a three variable analogue of Weyl fractional integral operator \( L^{\alpha}_{x, \infty} \).

(ii) For \( a = c = \alpha, b = -\eta, b' = b'' = 0, c' = \beta, c'' = \gamma \), (9.13) reduces to

\[ = \frac{x^\eta}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_{x}^{\infty} \int_{y}^{\infty} \int_{z}^{\infty} (u-x)^{a-1}(v-y)^{b-1}(w-z)^{c-1} u^{-\alpha-n} v^{-\beta} w^{-\gamma} f(u, v, w) \, dw \, dv \, du \]

\[ (9.15) \]

(iii) For \( a = c = \alpha, b = b'' = 0, b' = -\eta, c' = \beta, c'' = \gamma \), (9.13) becomes

\[ = \frac{y^\eta}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_{x}^{\infty} \int_{y}^{\infty} \int_{z}^{\infty} (u-x)^{a-1}(v-y)^{b-1}(w-z)^{c-1} u^{-\alpha} v^{-\alpha-n} w^{-\gamma} f(u, v, w) \, dw \, dv \, du \]

\[ (9.16) \]

(iv) For \( a = c = \alpha, b = b' = 0, b'' = -\eta, c' = \beta, c'' = \gamma \), (9.13) gives

\[ = \frac{z^\eta}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_{x}^{\infty} \int_{y}^{\infty} \int_{z}^{\infty} (u-x)^{a-1}(v-y)^{b-1}(w-z)^{c-1} u^{-\alpha} v^{-\alpha} w^{-\alpha-n} f(u, v, w) \, dw \, dv \, du \]

\[ (9.17) \]
We may consider (9.15), (9.16) and (9.17) as three variable analogues of Erdelyi–Kober fractional integral operator $K_{x,\infty}^{\eta,\alpha}$.

II. Let $c > 0$, $a$, $a'$, $a''$, $b$, $b'$, $b''$ be real numbers. Then a second three variable analogue of $I_{0,x}^{\alpha,\beta,y}$ is as follows:

$$
2\int_{0,x;0,y;0,z}^{a,a',a'';b,b',b'';c} f(x, y, z) = \frac{x^{-a} y^{-a'} z^{-a''}}{\Gamma(c)^3} \int_0^x \int_0^y \int_0^z (x-u)^{c-1} (y-v)^{c-1} (z-w)^{c-1} f(u, v, w) dw dv du
$$

(9.18)

Where

$$
F_{(3)}^{a,a',a'',b,b',b'';c} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_n (a')_r (a'')_s (b)_n (b')_r (b'')_s}{n! r! s! (c)_{n+r+s}} x^n y^r z^s
$$

SPECIAL CASES:

(i) For $a = a' = a'' = 0$, $c = \alpha$, (9.18) reduces to

$$
2\int_{0,0;0,y;0,z}^{a,a';a'';b,b';b'';c} f(x, y, z) = 2R_{0,x;0,y;0,z}^{\alpha,\alpha,\alpha} f(x, y, z)
$$

$$
= \frac{1}{\Gamma(\alpha)^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} f(u, v, w) dw dv du
$$

(9.19)

Here (9.19) may be regarded as a three variable analogue of Riemann–Liouville fractional integral operator $R_{0,x}^{\alpha,\alpha,\alpha}$.

(ii) For $a = c = \alpha$, $a' = a'' = 0$, $b = -\eta$, (9.18) reduces to
\[ 2 \mathcal{I}_{\alpha,0,0,-\eta;a',b';\alpha}^{x} f(x, y, z) = \frac{x^{\alpha-\eta}}{(\Gamma(\alpha))^3} \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} u^{\eta} f(u, v, w) \, dw \, dv \, du \]

\text{(9.20)}

(iii) For \( a = a'' = 0, a' = \alpha, b' = -\eta \), (9.18) gives

\[ 2 \mathcal{I}_{0,0,0,0,a',b';\alpha}^{x} f(x, y, z) = \frac{x^{\alpha-\eta}}{(\Gamma(\alpha))^3} \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} v^{\eta} f(u, v, w) \, dw \, dv \, du \]

\text{(9.21)}

(iv) For \( a = a' = 0, a'' = \alpha, b'' = -\eta \), (9.18) becomes

\[ 2 \mathcal{I}_{0,0,0,0,a',b';\alpha}^{x} f(x, y, z) = \frac{x^{\alpha-\eta}}{(\Gamma(\alpha))^3} \int_{0}^{x} \int_{0}^{y} \int_{0}^{z} (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} w^{\eta} f(u, v, w) \, dw \, dv \, du \]

\text{(9.22)}

Here (9.20), (9.21) and (9.22) may be thought of as the second three variable analogues of Erdelyi – Kober fractional integral operator \( \mathcal{E}_{a,\alpha,0,0}^{x} \).

Under the same conditions of (9.18), a second three variable analogues of \( \mathcal{J}_{x,\alpha,0,0}^{a,\alpha,0,0} \) is as defined below:

\[ 2 \mathcal{J}_{x,\alpha,0,0}^{a,\alpha,0,0} f(x, y, z) = \frac{1}{(\Gamma(c))^3} \int_{x}^{\infty} \int_{y}^{\infty} \int_{z}^{\infty} (u-x)^{c-1} (v-y)^{c-1} (w-z)^{c-1} u^{\alpha-a} v^{\alpha-a'} w^{\alpha-a''} f(u, v, w) \, dw \, dv \, du \]

\text{(9.23)}
SPECIAL CASES:

(i) For \( a = a' = a'' = 0, c = \alpha \), (9.23) reduces to

\[
2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z) = 2^1_{x, y, z, \infty} f(x, y, z)
\]

\[
= \frac{1}{[\Gamma(\alpha)]^3} \int_0^\infty \int_0^\infty \int_0^\infty (u - x)^{\alpha - 1} (v - y)^{\alpha - 1} (w - z)^{\alpha - 1} f(u, v, w) dw \, dv \, du
\]

(9.24)

It can be considered as a three variable analogue of Weyl fractional integral operator \( L_{x, \infty}^\alpha \).

(ii) For \( a' = a'' = 0, a = c = \alpha, b = -\eta \), (9.23) becomes

\[
2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z) = 2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z)
\]

\[
= \frac{x^\eta}{[\Gamma(\alpha)]^3} \int_0^\infty \int_0^\infty \int_0^\infty (u - x)^{\alpha - 1} (v - y)^{\alpha - 1} (w - z)^{\alpha - 1} u^{-\alpha - \eta} f(u, v, w) dw \, dv \, du
\]

(9.25)

(iii) For \( a = a'' = 0, a' = c = \alpha, b' = -\eta \), (9.23) gives

\[
2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z) = 2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z)
\]

\[
= \frac{y^\eta}{[\Gamma(\alpha)]^3} \int_0^\infty \int_0^\infty \int_0^\infty (u - x)^{\alpha - 1} (v - y)^{\alpha - 1} (w - z)^{\alpha - 1} v^{-\alpha - \eta} f(u, v, w) dw \, dv \, du
\]

(9.26)

(iv) For \( a = a' = 0, a'' = c = \alpha, b'' = -\eta \), (9.23) yields

\[
2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z) = 2^1_{x, y, \infty; y, \infty; z, \infty} f(x, y, z)
\]

\[
= \frac{z^\eta}{[\Gamma(\alpha)]^3} \int_0^\infty \int_0^\infty \int_0^\infty (u - x)^{\alpha - 1} (v - y)^{\alpha - 1} (w - z)^{\alpha - 1} w^{-\alpha - \eta} f(u, v, w) dw \, dv \, du
\]

(9.27)
Here (9.25), (9.26) and (9.27) may be taken as the second three variable analogues of Erdelyi – Kober fractional integral operator $K_{\alpha,\beta}^{(n)}$.

III. Let $c > 0$, $c' > 0$, $c'' > 0$, $a$, $b$ be real numbers. Then a third three variable analogue of $I_{0,x}^{\alpha,\beta,\eta}$ is defined below:

$$
J_{0,x}^{\alpha,\beta,\eta}[a, b; c', c'' \mid x, y, z] = \frac{x^{-a} y^{-c'} z^{-c''}}{\Gamma(c') \Gamma(c'') \Gamma(c)} \int_0^x \int_0^y \int_0^z (x-u)^{-1} (y-v)^{-1} (z-w)^{-1} f(u,v,w) \, dw \, dv \, du
$$

(9.28)

Where

$$
F_{0,x}^{(3)}[a, b; c, c', c'' \mid x, y, z] = \sum_{n=0}^\infty \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(a)_{n+r+s} (b)_{n+r+s}}{n! \, r! \, s!} (c)_n (c')_r (c'')_s x^n y^r z^s
$$

Under the same conditions of (9.28), a third three variable analogues of $I_{0,x}^{\alpha,\beta,\eta}$ is as given below:

IV. Let $c > 0$, $a$, $b$, $b'$, $b''$ be real numbers. A fourth three variable analogue of fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ due to M. Saigo is defined as:

$$
J_{0,x}^{\alpha,\beta,\eta}[a, b, b'; c; x, y, z] = \frac{x^{-a} y^{-b'} z^{-b''}}{\Gamma(c)^3} \int_0^x \int_0^y \int_0^z (x-u)^{-c'} (y-v)^{-1} (z-w)^{-1} f(u,v,w) \, dw \, dv \, du
$$

(9.30)

Where

$$
F_{0,x}^{(3)}[a, b, b'; c, c', c'' \mid x, y, z] = \sum_{n=0}^\infty \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{(a)_{n+r+s} (b)_{n+r+s}}{r! \, s!} (c)_n (c')_r (c'')_s x^n y^r z^s
$$
SPECIAL CASES:

(i) For $a = b = b' = b'' = 0$, $c = \alpha$, (9.30) reduces

\[ 4 \mathcal{I}^{\alpha,0,0,0;0,0}_{0,x;0,y;0,z} f(x,y,z) = 2 \mathcal{R}^\alpha_{0,x;0,y;0,z} f(x,y,z) \]

\[ = \frac{1}{\Gamma(\alpha)^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} \ f(u,v,w) \ dw \ dv \ du \]

(9.31)

Which is (9.19) i.e. a three variable analogue of Riemann – Liouville fractional integral operator $\mathcal{R}^\alpha_{0,x}$.

(ii) For $a = c = \alpha$, $b = -\eta$, $b' = b'' = 0$, (9.30) becomes

\[ 4 \mathcal{I}^\alpha_{0,x;0,y;0,z} f(x,y,z) = \frac{\mathcal{E}^{\alpha,\eta}_{0,x;0,y;0,z}}{\Gamma(\alpha)^3} f(x,y,z) \]

\[ = \frac{x^{-\eta} y^{-\alpha} z^{-\alpha}}{\Gamma(\alpha)^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} \ u^n f(u,v,w) \ dw \ dv \ du \]

(9.32)

(iii) For $a = c = \alpha$, $b = b'' = 0$, $b' = -\eta$, (9.30) gives

\[ 4 \mathcal{I}^\alpha_{0,x;0,y;0,z} f(x,y,z) = \frac{\mathcal{E}^{\alpha,\eta}_{0,x;0,y;0,z}}{\Gamma(\alpha)^3} f(x,y,z) \]

\[ = \frac{x^{-\alpha} y^{-\alpha} z^{-\alpha}}{\Gamma(\alpha)^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} \ v^n f(u,v,w) \ dw \ dv \ du \]

(9.33)

(iv) For $a = c = \alpha$, $b = b' = 0$, $b'' = -\eta$, (9.30) yields

\[ 4 \mathcal{I}^\alpha_{0,x;0,y;0,z} f(x,y,z) = \frac{\mathcal{E}^{\alpha,\eta}_{0,x;0,y;0,z}}{\Gamma(\alpha)^3} f(x,y,z) \]

\[ = \frac{x^{-\alpha} y^{-\alpha} z^{-\alpha}}{\Gamma(\alpha)^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1} (y-v)^{\alpha-1} (z-w)^{\alpha-1} \ w^n f(u,v,w) \ dw \ dv \ du \]

(9.34)
Here (9.32), (9.33) and (9.34) may be considered as third three-variable analogues of Erdelyi–Kober fractional integral operator \( E_{0,x}^{\alpha,\eta} \).

It may be remarked here that (9.32), (9.33) and (9.34) can also be obtained from (9.10), (9.11) and (9.12) respectively by taking \( \alpha = \beta = \gamma \).

Under the same condition of (9.30), a fourth three-variable analogue of another fractional integral operator \( J_{x,\infty}^{\alpha,\beta,\gamma} \) due to M. Saigo is defined as follows:

\[
J_{x,\infty}^{a,b,b',b'';c} f(x,y,z) = \frac{1}{\Gamma(c)^3} \int_0^\infty \int_0^\infty \int_0^\infty (u-x)^{c-1} (v-y)^{c-1} (w-z)^{c-1} f(u,v,w) \, dw \, dv \, du
\]

\[
F_{(3)}^{a,b,b',b'',1-x,1-y,1-z} f(u,v,w) \, dw \, dv \, du
\]

\[
(9.35)
\]

**SPECIAL CASES:**

(i) For \( a = b = b' = b'' = 0, c = \alpha \), (9.35) reduces to

\[
J_{x,\infty}^{0,0,0;\alpha} f(x,y,z) = \frac{1}{\Gamma(\alpha)^3} \int_0^\infty \int_0^\infty \int_0^\infty (u-x)^{\alpha-1} (v-y)^{\alpha-1} (w-z)^{\alpha-1} f(u,v,w) \, dw \, dv \, du
\]

\[
(9.36)
\]

which is (9.24) i.e. a second three-variable analogue of Weyl fractional integral operator \( L_x^{\alpha,\eta} \). It can also be obtained from (9.14) by taking \( \alpha = \beta = \gamma \).
(ii) For \( a = c = \alpha, b = -\eta, b' = b'' = 0 \), (9.35) becomes

\[
4J_{x,\infty; y,\infty; z,\infty}^{\alpha,-\eta,0,0; \alpha} f(x, y, z) = \frac{x^\eta}{\Gamma(\alpha)} \int \int \int (u-x)^{\alpha-1} (v-y)^{\alpha-1}(w-z)^{\alpha-1} u^{-\alpha-\eta} v^{-\alpha} w^{-\alpha} f(u, v, w) \, dw \, dv \, du
\]

(9.37)

(iii) For \( a = c = \alpha, b = b'' = 0, b' = -\eta \), (9.35) gives

\[
4J_{x,\infty; y,\infty; z,\infty}^{\alpha,0,-\eta,0; \alpha} f(x, y, z) = \frac{y^\eta}{\Gamma(\alpha)} \int \int \int (u-x)^{\alpha-1} (v-y)^{\alpha-1}(w-z)^{\alpha-1} u^{-\alpha-\eta} v^{-\alpha} w^{-\alpha} f(u, v, w) \, dw \, dv \, du
\]

(9.38)

(iv) For \( a = c = \alpha, b = b' = 0, b'' = -\eta \), (9.35) yields

\[
4J_{x,\infty; y,\infty; z,\infty}^{\alpha,0,0,-\eta; \alpha} f(x, y, z) = \frac{z^\eta}{\Gamma(\alpha)} \int \int \int (u-x)^{\alpha-1} (v-y)^{\alpha-1}(w-z)^{\alpha-1} u^{-\alpha} v^{-\alpha-\eta} w^{-\alpha} f(u, v, w) \, dw \, dv \, du
\]

(9.39)

Here (9.37), (9.38) and (9.39) may be considered as third three-variable analogues of Erdelyi–Kober fractional integral operator \( K_{x,\infty}^{\alpha,\eta} \).

Further (9.37), (9.38) and (9.39) can also be obtained from (9.15), (9.16) and (9.17) respectively by taken \( \alpha = \beta = \gamma \).

The chapter IV establishes certain results in the form of theorems including integration by parts.

10. ON FRACTIONAL INTEGRAL OPERATORS OF THREE VARIABLES AND INTEGRAL TRANSFORMS: In chapter IV of the present thesis a study was made of certain three variable analogues of fractional
integral operators of one variable due to M. Saigo. Continuing the study of these operators, chapter V of this thesis establishes the effects of integral transforms say the three variable analogues of Mellin and Laplace transforms on the three variable analogues of fractional integral operators of the preceding chapter. A three variable analogue of Mellin transform of a function $f(x, y, z)$ of three variables $x$, $y$ and $z$ is defined as follows:

$$M \{ f(u, v, w) : r, s, t \} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} u^{-1} v^{-1} w^{-1} f(u, v, w) \, dw \, dv \, du \quad (10.1)$$

Similarly the triple Laplace transform of a function of three variables of $f(x, y, z)$ defined in the positive octant of the three dimensional space is defined by the equation

$$L \{ f(x, y, z) : r, s, t \} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-r \cdot x - s \cdot y - t \cdot z} f(x, y, z) \, dz \, dy \, dx \quad (10.2)$$

11. ON A GENERALIZATION OF Z – TRANSFORM : In the theory of Electrical Engineering of Telecommunication Engineering, $z$-transform technique is widely used for the analysis and synthesis of the sampled data system. The discrete time function which represents the sampled signal or the sampled data system is denoted by

$$f^* (t) = \sum_{n=0}^{\infty} f(nT) \delta(t - nT) \quad (n = 0, 1, 2, .....) \quad (11.1)$$

The $z$-transform $F(z)$ of the series $f(nT)$ is defined as the infinite sum of

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad (n = 0, 1, 2, .....) \quad (11.2)$$
where \( z \) is the complex variable such as \( z = e^{st} \) for the Laplace variables. Hence we have

\[
F(z) = \sum_{n=0}^{\infty} f(nT) e^{-stn} = Lf^*(t), \quad (n = 0, 1, 2, \ldots) \quad (11.3)
\]

and we know that the \( z \)-transform for \( f(nT) \) is suited to the Laplace transform for \( f^*(t) \). We denote (1.2) as \( F(z) = z \{f(nT)\} \).

Thus understanding of the \( z \)-transformation is easily gained with the use of fundamental knowledge of the Laplace transformation and the function theory of complex variables.

To obtain the formulae of \( z \)-transformation or the inverse \( z \)-transformation, several methods such as power series, partial function or residues theorems are explained in [135]. Chapter VI deals with a generalization of \( z \)-transform. The generalization is analogous to Taylor's generalization of Maclaurin's expansion. Certain properties of this generalized \( z \)-transform have been discussed in this chapter. Contents of this chapter are already published in *Acta Ciencia India* [109].