CHAPTER - VI
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ON A GENERALIZATION OF Z–TRANSFORM

ABSTRACT: In the present Chapter a generalization of z-transform has been given and its properties discussed. The generalization is analogous to Taylor’s generalization of Maclaurin’s expansion.

1. INTRODUCTION: In the theory of Electrical Engineering or Telecommunication Engineering, z-transform technique is widely used for the analysis and synthesis of the sampled data system. The discrete time function which represents the sampled signal or the sampled data system is denoted by

\[ f^* (t) = \sum_{n=0}^{\infty} f(nT) \delta(t - nT) \quad (n = 0, 1, 2, \ldots) \quad (1.1) \]

The z-transform \( F(z) \) of the series \( f(nT) \) is defined as the infinite sum of

\[ F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad (n = 0, 1, 2, \ldots) \quad (1.2) \]

where \( z \) is the complex variable such as \( z = e^{sT} \) for the Laplace variables. Hence we have

\[ F(z) = \sum_{n=0}^{\infty} f(nT) e^{-sTN} = Lf^* (t), \quad (n = 0, 1, 2, \ldots) \quad (1.3) \]

and we know that the z-transform for \( f(nT) \) is suited to the Laplace transform for \( f^* (t) \). We denote (1.2) as \( F(z) = z \{ f(nT) \} \).
Thus understanding of the z-transformation is easily gained with the use of fundamental knowledge of the Laplace transformation and the function theory of complex variables.

To obtain the formulae of z-transformation or the inverse z-transformation, several methods such as power series, partial function or residues theorems are explained in [135]. Here we have generalized z-transform analogous to Taylor's generalization of Maclaurin's expansion by introducing an extra parameter. The resulting transform is expected to be more useful as compared to original z-transform.

THE $Z_a$-TRANSFORM AND ITS PROPERTIES: Let $f(t)$ be a function defined for discrete value of $t$ usually at $nT$, $n = 0, 1, 2, \ldots$, where $T$ is a fixed positive number usually referred to as the sampling period. The $Z_a$-transform which is a generalization of Z-transform of $f(t)$ is defined as

$$ Z_a f(t) = \sum_{n=0}^{\infty} f(nT) (z-a)^{-n} $$

(2.1)

The following theorems can readily be proved:

**THEOREM 1:** The $Z_a$-transform of the sum of two or more functions of $t$ is the sum of the $Z_a$-transform of the separate functions.

**THEOREM 2:** If $c$ is a constant

$$ Z_a [cf(t)] = c Z_a f(t). $$

By a joint use of these two theorems, we have

$$ Z_a [c_1 f_1(t) + c_2 f_2(t)] = c_1 Z_a f_1(t) + c_2 Z_a f_2(t). $$
Thus the $Z_a$-transform is a linear operator.

**SOME STANDARD $Z_a$-TRANSFORMS**

(i) **$Z_a$-transform of 1**: By definition it follows that

$$Z_a(1) = \frac{z-a}{z-a-1} \quad (3.1)$$

(ii) **$Z_a$-transform of powers of $t$**: Let $k$ be a positive integer. Then by definition,

$$Z_a(t^k) = \sum_{n=0}^{\infty} (nT) (z-a)^{-n} \quad (3.2)$$

$$= T(z-a) \sum_{n=0}^{\infty} (nT)^{k-1} (z-a)^{-(n+1)} \quad (3.3)$$

Changing $k$ into $k-1$ in (3.2), we get

$$Z_a(t^{k-1}) = \sum_{n=0}^{\infty} (nT)^{k-1} (z-a)^{-n} \quad (3.4)$$

Differentiating (3.4) w.r.t. $z$, we obtain

$$\frac{d}{dz} Z_a(t^{k-1}) = -\sum_{n=0}^{\infty} (nT)^{k-1} n (z-a)^{-(n+1)} \quad (3.5)$$

Substituting (3.5) into (3.3), we have

$$Z_a(t^k) = -T(z-a) \frac{d}{dz} Z_a(t^{k-1}) \quad (3.6)$$

This can be taken as a recurrence formula.

Putting $k = 1, 2, 3$, etc. in (3.6), we get

$$Z_a(t) = \frac{T(z-a)}{(z-a-1)^2} \quad (3.7)$$
\[ Z_a(t^2) = \frac{T^2 (z-a)(z-a+1)}{(z-a-1)^2} \] (3.8)

\[ Z_a(t^3) = \frac{T^3 (z-a)\{(z-a)^2 + 4(z-a)+1\}}{(z-a-1)^4} \] (3.9)

etc.

**SHFITING THEOREM AND ITS APPLICATIONS**: The \( Z_a \)-transform of a function multiplied by \( e^{-\lambda t} \) is called shifting theorem and is stated below:

**THEOREM 3**: If the \( Z_a \)-transform of \( f(t) \) is \( F\{(z-a)\} \), then the \( Z_a \)-transform of \( e^{-\lambda t} f(t) \) is \( F\{ e^{-\lambda t}(z-a)\} \).

The proof of this theorem is straightforward.

Applying the above theorem to (3.1), (3.7) and (3.8) one can easily obtain the following results:

\[ Z_a(e^{-\lambda t}) = \frac{z-a}{z-a-e^{-\lambda T}} \] (4.1)

\[ Z_a(te^{-\lambda t}) = \frac{T(z-a)e^{-\lambda T}}{(z-a-e^{-\lambda T})^2} \] (4.2)

\[ Z_a(t^2e^{-\lambda t}) = \frac{T^2(z-a)e^{-\lambda T}\{(z-a)+e^{-\lambda T}\}}{(z-a-e^{-\lambda T})^3} \] (4.3)

Replacing \( \lambda \) by \( i\lambda \) in (4.1) and separating the real and imaginary parts the following results are obtained

\[ Z_a(\cos \lambda t) = \frac{(z-a)(z-a-\cos \lambda T)}{(z-a)^2 - 2(z-a)\cos \lambda T + 1} \] (4.4)
\[ Z_\lambda (\sin \lambda t) = \frac{(z-a) \sin \lambda T}{(z-a)^2 - 2(z-a) \cos \lambda T + 1} \] (4.5)

Replacing \( \lambda \) and \( \beta \) in (4.4) and (4.5) and applying shifting theorem one gets the following results:

\[ Z_\lambda \left( e^{-\alpha T} \cos \beta t \right) = \frac{(z-a) e^{\alpha T} \left( (z-a) e^{\alpha T} - \cos \beta T \right)}{(z-a)^2 e^{2\alpha T} - 2(z-a) e^{\alpha T} \cos \beta T + 1} \] (4.6)

\[ Z_\lambda \left( e^{-\alpha T} \sin \beta t \right) = \frac{(z-a) e^{\alpha T} \sin \beta T}{(z-a)^2 e^{2\alpha T} - 2(z-a) e^{\alpha T} \cos \beta T + 1} \] (4.7)

**ANOTHER SHIFTING THEOREM:** If \( Z_\lambda f(t) = F((z-a)) \), then

\[ Z_\lambda \{f(t + T)\} = (z - a)[F((z-a) - f(0))]. \] (5.1)

**PROOF:** By definition,

\[ Z_\lambda \{f(t + T)\} = \sum_{n=0}^{\infty} f(nT + T)(z-a)^{-n} \]

\[ = (z-a) \sum_{n=0}^{\infty} f[(n+1)T](z-a)^{-(n+1)} \]

\[ = (z-a) \sum_{k=1}^{\infty} f[kT](z-a)^{-k} \text{ by putting } n + 1 = k. \]

\[ = (z-a) \left[ \sum_{k=0}^{\infty} f(kT)(z-a)^{-k} - f(0) \right] \]

\[ = (z-a) [F\{(z-a)\} - F(0)]. \]

**FURTHER THEOREM:** The value \( f(0) \) and \( f(\infty) \), when the latter exists, of \( f(t) \) are related to the properties of \( Z_\lambda \)-transform of \( f(t) \).
INITIAL VALUE THEOREM: If $F \{(z - a)\}$ is the $Z_a$-transform of $f(t)$, then

$$f(0) = \lim_{(z-a) \to \infty} F\{(z-a)\}.$$  

PROOF: $F\{(z-a)\} = Z_a f(t)$

$$= \sum_{n=0}^{\infty} f(nT) (z-a)^n$$

$$= f(0) + \frac{f(T)}{z-a} + \frac{f(2T)}{(z-a)^2} + \ldots$$

Let $(z-a) \to \infty$ on both sides, then

$$\lim_{(z-a) \to \infty} F\{(z-a)\} = f(0).$$

FINAL VALUE THEOREM: If $F \{(z - a)\}$ is the $Z_a$-transform of $f(t)$, then

$$\lim_{t \to \infty} f(t) = \lim_{z \to a+1} (z-a-1) F\{(z-a)\}.$$  

PROOF: By definition.

$$Z_a [f (t + T) - f (t)] = \sum_{n=0}^{\infty} [f(nT + T) - f(nT)] (z-a)^{-n}$$

or

$$Z_a f(t + T) - Z_a f(t) = \sum_{n=0}^{\infty} [f(nT + T) - f(nT)] (z-a)^{-n}.$$  

In view of the result (5.1) of the second shifting Theorem, we have

$$(z-a) [F \{(z-a)\} - f(0)] - F \{(z-a)\}$$

$$= \sum_{n=0}^{\infty} [f(nT + T) - f(nT)] (z-a)^{-n}$$
Taking limit of both sides as $z \rightarrow a + 1$, we get

$$
\lim_{z \to a+1} (z-a-1) F \{(z-a)\} - \lim_{z \to a+1} (z-a) f(0) = \sum_{n=0}^{\infty} [f(nT + T) - f(nT)] \times 1
$$

$$
= \lim_{z \to a+1} \{f(T) - f(0)\} + [f(2T - f(T)) + f(3T) - f(2T)]
$$

$$
+ \ldots + [f(n + 1) T - f(nT)]\}
$$

$$
= f(\infty) - f(0)
$$

or

$$
\lim_{z \to a+1} (z-a-1) F \{(z-a)\} - f(0) = f(\infty) - f(0).
$$

$$
f(\infty) = \lim_{z \to a+1} (z-a-1) F \{(z-a)\}.
$$

**CONVOLUTION THEOREM**: If $F_1 \{(z-a)\}$ and $F_2 \{(z-a)\}$ are respectively the $Z_a$-transform of $f_1(t)$ and $f_2(t)$, then $F_1 \{(z-a)\} F_2 \{(z-a)\}$ will be the $Z_a$-transform of

$$
\sum_{k=0}^{n} f_1 \{(n-k)T\} f_2(kT).
$$

**PROOF**: By definition,

$$
F_1 \{(z-a)\} = \sum_{n=0}^{\infty} f_1(nT) (z-a)^{-n}
$$

$$
F_2 \{(z-a)\} = \sum_{k=0}^{\infty} f_2(kT) (z-a)^{-k}
$$

Hence, $F_1 \{(z-a)\} F_2 \{(z-a)\} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f_1(nT) f_2(kT) (z-a)^{-(n+k)}$
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_1 \{(n - k) T\} f_2(kT) (z - a)^{-n} \]

\[ = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} f_1 \{(n - k) T\} f_2(kT) \right] (z - a)^{-n} \]

\[ = \sum_{n=0}^{\infty} A_n (z - a)^{-n}, \text{ where } A_n = \sum_{k=0}^{n} f_1 \{(n - k) T\} f_2(kT) \]

\[ = Z_a (A_n) \]

i.e., \( F \{(z - a)\} F_2 \{(z - a)\} \) is the \( Z_a \)-transform of \( \sum_{k=0}^{n} f_1 \{(n - k) T\} f_2(kT) \).

This completes the proof.