CHAPTER - V
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ON FRACTIONAL INTEGRAL OPERATORS OF THREE VARIABLES AND INTEGRAL TRANSFORMS

ABSTRACT: The studies of this chapter are in continuation to that of chapter IV where three variable analogues of certain fractional integral operators of M. Saigo were investigated. The present chapter deals with the effect of operating three variable analogues of Mellin and Laplace transforms on these three variable analogues of fractional integral operators chapter IV.

1. INTRODUCTION: Chapter IV of this thesis a study was made of certain three variables analogues of fractional integral operators of one variable due to M. Saigo where certain results including integration by parts were established in the form of certain theorems.

The aim of the present chapter is to study the effects of integral transforms say the three variable analogues of fractional integral operators of chapter IV.

We shall also need here the definition of triple hypergeometric series $\text{F}^{(3)}[x, y, z]$ (cf. Srivastava [172], p. 428) defined as:
\[
F^{(3)}[x, y, z] = F^{(3)}[(a) : (b); (b'); (b''): (d); (d'); (d'') \times, y, z]
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \Lambda(m, n, p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!},
\]

where, for convenience
\[
\Lambda(m, n, p) = \prod_{j=1}^{A} (a_j)_{m+n+p} \prod_{j=1}^{B} (b_j)_{m+n} \prod_{j=1}^{B'} (b'_j)_{n+p} \prod_{j=1}^{B''} (b''_j)_{p+m} \prod_{j=1}^{D} (d_j)_m \prod_{j=1}^{D'} (d'_j)_n \prod_{j=1}^{D''} (d''_j)_p
\]

where (a) abbreviates, the array of a parameters \(a_1, a_2, \ldots a_A\), with similar interpretations for (b), (b'), (b''), et cetera. The triple hypergeometric series in (1.3) converges absolutely when
\[
1 + E + G + G' + H - A - B - B' - C \geq 0
\]
\[
1 + E + G + G' + G'' + H' - A - B - B' - C' \geq 0
\]
\[
1 + E + G' + G'' + H' - A - B' - B'' - C'' \geq 0
\]

where the equalities hold true for suitable constrained values of \(|x|, |y|\) and \(|z|\).

A three variable analogue of Mellin transform of a function \(f(x, y, z)\) of three variables \(x, y\) and \(z\) is defined as follows:

\[
M\{f(u, v, w): r, s, t\} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} u^{r-1} v^{s-1} w^{t-1} f(u, v, w) dw \, dv \, du
\]

Similarly the triple Laplace transform of a function of three variables \(f(x, y, z)\) defined in the positive octant of the three dimensional \(xyz\)-space is defined by the equation
2. MELLIN TRANSFORMATION: In this section we shall study the effect of operating three variable analogue of Mellin transform on the operators defined in chapter IV.

The effects of operating (1.4) on the operators 4(2.1), 4(2.6), 4(2.11), 4(2.16), 4(2.21), 4(2.22), 4(2.23) and 4(2.28) are given in the form of the following theorems:

THEOREM 2.1: For $c > 0$, $c' > 0$, $c'' > 0$, $R/(r) > 0$, $R/(s) > 0$, $R/(t) > 0$, we have

\[
M \left\{ \int_{x,\infty}^{a} \int_{y,\infty}^{b} \int_{z,\infty}^{c} f(x, y, z) : r, s, t \right\} = \frac{\Gamma(r) \Gamma(s) \Gamma(t)}{\Gamma(r+c) \Gamma(s+c') \Gamma(t+c')} F^{(3)} \left[ \begin{array}{c} a; \; -; \; -; \; b; \; b'; \; b''; \; 1, 1, 1 \end{array} \right] \]

provided that term by term integration is valid and $F^{(3)}[x, y, z]$ is given by (1.1).

THEOREM 2.2: For $c > 0$, $R/(r) > 0$, $R/(s) > 0$, $R/(t) > 0$, we have

\[
M \left\{ \int_{x,\infty}^{a} \int_{y,\infty}^{b} \int_{z,\infty}^{c} f(x, y, z) : r, s, t \right\} = \frac{\Gamma(r) \Gamma(s) \Gamma(t)}{\Gamma(r+c) \Gamma(s+c) \Gamma(t+c)} F^{(3)} \left[ \begin{array}{c} \cdot \; -; \; -; \; r; \; c + s; \; c' + s; \; c'' + t; \; 1, 1, 1 \end{array} \right] \]

\[
M \left\{ x^{c-a} y^{c-a} z^{c-a} f(x, y, z) : r, s, t \right\} \]

(2.2)
provided that term by term integration is valid and \( F^{(3)} [x, y, z] \) is given by (1.1).

**THEOREM 2.3**: For \( c > 0, c' > 0, c'' > 0, R/(r) > 0, R/(s) > 0, R/(t) > 0 \), we have

\[
M \left\{ \int_{x,\infty}^{a,b,c,c'} f(x,y,z) : r,s,t \right\} = \frac{\Gamma(r) \Gamma(s) \Gamma(t)}{\Gamma(r+c) \Gamma(s+c') \Gamma(t+c'')} \Gamma^{(3)} \left[ a,b,c,b',c; b',c; c' ; 1, 1, 1 \right]
\]

\[
M \left\{ x^{c-a} y^{c'-a} z^{c''-a} f(x,y,z) : r,s,t \right\}
\]

provided that term by term integration is valid and \( F^{(3)} [x, y, z] \) is given by (1.1).

**THEOREM 2.4**: For \( c > 0, R/(r) > 0, R/(s) > 0, R/(t) > 0 \), we have

\[
M \left\{ \int_{x,\infty}^{a,b'} f(x,y,z) : r,s,t \right\} = \frac{\Gamma(r) \Gamma(s) \Gamma(t)}{\Gamma(r+c) \Gamma(s+c') \Gamma(t+c'')} \Gamma^{(3)} \left[ a,b',c,b',c; c' ; 1, 1, 1 \right]
\]

\[
M \left\{ (xyz)^{-a} f(x,y,z) : r,s,t \right\}
\]

provided that term by term integration is valid and \( F^{(3)} [x, y, z] \) is given by (1.1).

**THEOREM 2.5**: For a function of three variables \( f(x, y, z) \) defined in the positive octant of the three dimensional \( xyz \) space and \( c > 0, c' > 0, c'' > 0 \), we have

\[
\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} f(x,y,z) \Gamma^{a,b,b',c,c',c'} \left\{ x^{r-1} y^{s-1} z^{t-1} \right\} dz \, dy \, dx
\]
\[
\begin{align*}
\mathcal{M} \left\{ 1 \int_{\mathbb{R}^3} f(x, y, z) : r, s, t \right\} = M \left\{ 1 \int_{\mathbb{R}^3} f(x, y, z) : r, s, t \right\} 
\end{align*}
\]  
(2.6)

provided that the triple integrals involved exist.

**THEOREM 2.6**: Under the conditions stated in theorem 2.5, we have

\[
\begin{align*}
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \left( 1 \int_{\mathbb{R}^3} x^{r-1} y^{s-1} z^{t-1} \right) \, dz \, dy \, dx \\
= M \left\{ 1 \int_{\mathbb{R}^3} f(x, y, z) : r, s, t \right\}
\end{align*}
\]  
(2.7)

provided that the triple integrals involved exist.

**THEOREM 2.7**: For a function of three variables \( f(x, y, z) \) defined in the positive octant of the three dimensional \( xyz \)-space and \( c > 0 \), we have

\[
\begin{align*}
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \left( 1 \int_{\mathbb{R}^3} x^{r-1} y^{s-1} z^{t-1} \right) \, dz \, dy \, dx \\
= M \left\{ 2 \int_{\mathbb{R}^3} f(x, y, z) : r, s, t \right\}
\end{align*}
\]  
(2.8)

provided that the triple integrals involved exist.

**THEOREM 2.8**: Under the conditions stated in theorem 2.7, we have

\[
\begin{align*}
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \left( 2 \int_{\mathbb{R}^3} x^{r-1} y^{s-1} z^{t-1} \right) \, dz \, dy \, dx \\
= M \left\{ 2 \int_{\mathbb{R}^3} f(x, y, z) : r, s, t \right\}
\end{align*}
\]  
(2.9)

provided that the triple integrals involved exist.

**THEOREM 2.9**: Under the conditions stated in theorem 2.5, we have

\[
\begin{align*}
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \left( 3 \int_{\mathbb{R}^3} x^{r-1} y^{s-1} z^{t-1} \right) \, dz \, dy \, dx \\
\end{align*}
\]
provided that the triple integrals involved exist.

**THEOREM 2.10** : Under the conditions stated in theorem 2.5, we have

\[
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \ 3r^{a-b+c+c'} x^{r-1} y^{s-1} z^{t-1} \ dz \ dy \ dx
\]

\[
= M \left\{ 3r^{a-b+c+c'} r, s, t \right\}
\]  \hspace{1cm} (2.10)

provided that the triple integrals involved exist.

**THEOREM 2.11** : Under the conditions stated in theorem 2.7, we have

\[
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \ 4r^{a-b'+b''+c} x^{r-1} y^{s-1} z^{t-1} \ dz \ dy \ dx
\]

\[
= M \left\{ 4r^{a-b'+b''+c} r, s, t \right\}
\]  \hspace{1cm} (2.11)

provided that the triple integrals involved exist.

**THEOREM 2.12** : Under the conditions stated in theorem 2.7, we have

\[
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \ 4r^{a-b'+b''+c} x^{r-1} y^{s-1} z^{t-1} \ dz \ dy \ dx
\]

\[
= M \left\{ 4r^{a-b'+b''+c} r, s, t \right\}
\]  \hspace{1cm} (2.12)

provided that the triple integrals involved exist.

**THEOREM 2.13** : For functions of three variables \( f(x, y, z) \) and \( g(x, y, z) \) defined in the positive octant of the three dimensional \( xyz \) – space and \( c > 0, c' > 0, c'' > 0 \), we have
THEOREM 2.14: For functions of three variables $f(x, y, z)$ and $g(x, y, z)$ defined in the positive octant of the three dimensional $xyz$-space and $c > 0$, we have

$$M \left[ f(x, y, z) \right] \Gamma_{0, x; 0, y; 0, z}^{a, b, b', c, c', c'} \left\{ x^{r-1} y^{s-1} z^{t-1} g(x, y, z) \right\} : r, s, t$$

$$= M \left[ g(x, y, z) \right] \Gamma_{x, \infty; y, \infty; z, \infty}^{a, b, b', c, c', c'} \left\{ x^{r-1} y^{s-1} z^{t-1} f(x, y, z) \right\} : r, s, t$$

(2.14)

THEOREM 2.15: Under the conditions stated in theorem 2.13, we have

$$M \left[ f(x, y, z) \right] \Gamma_{0, x; 0, y; 0, z}^{a, b, b', c, c', c'} \left\{ x^{r-1} y^{s-1} z^{t-1} g(x, y, z) \right\} : r, s, t$$

$$= M \left[ g(x, y, z) \right] \Gamma_{x, \infty; y, \infty; z, \infty}^{a, b, b', c, c', c'} \left\{ x^{r-1} y^{s-1} z^{t-1} f(x, y, z) \right\} : r, s, t$$

(2.15)

THEOREM 2.16: Under the conditions stated in theorem 2.14, we have

$$M \left[ f(x, y, z) \right] \Gamma_{0, x; 0, y; 0, z}^{a, b, b', c, c', c'} \left\{ x^{r-1} y^{s-1} z^{t-1} g(x, y, z) \right\} : r, s, t$$

$$= M \left[ g(x, y, z) \right] \Gamma_{x, \infty; y, \infty; z, \infty}^{a, b, b', c, c', c'} \left\{ x^{r-1} y^{s-1} z^{t-1} f(x, y, z) \right\} : r, s, t$$

(2.16)

(2.17)

It is interesting to note that in terms of triple Mellin transforms the results 4(3.1), 4(3.2), 4(3.3), 4(3.4), 4(3.5), 4(3.6), 4(3.7) and 4(3.8) respectively be written as
\[
M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\]

\[
= M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\] (2.18)

\[
M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\]

\[
= M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\] (2.19)

\[
M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\]

\[
= M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\] (2.20)

\[
M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\]

\[
= M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\] (2.21)

\[
M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\]

\[
= M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\] (2.22)

\[
M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\]

\[
= M \left[ \mathcal{J} \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] f(x, y, z) : a - c, a - c', a - c''
\] (2.23)
M \left[ f \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] = \frac{1}{\Gamma(1, b, b', b''; c) g(x, y, z); a - c, a - c, a - c}

= M \left[ g \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] = \frac{1}{\Gamma(1, b, b', b''; c) g(x, y, z); a - c, a - c, a - c}

= M \left[ g \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right] = \frac{1}{\Gamma(1, b, b', b''; c) g(x, y, z); a - c, a - c, a - c}

(2.24)

(2.25)

3. LAPLACE TRANSFORMATION

Making use of results of theorems 4.1, 4.2, 4.3 and 4.4 of chapter IV, the relationships of (1.5) with the operators 4(2.1), 4(2.6), 4(2.11), 4(2.16), 4(2.21) 4(2.22), 4(2.23) and 4(2.28) are given in the form of the following theorems:

THEOREM 3.1: For a function of three variables \( f(x, y, z) \) defined in the positive octant of the three dimensional \( xyz \) - space and \( c > 0, c' > 0, c'' > 0 \), we have

\[
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) \, \Gamma(1, b, b', b''; c, c', c''; z) \, e^{-sx - sy - sz} \, dz \, dy \, dx
\]

\[
= L \left[ \int_0^\infty \Gamma(1, b, b', b''; c, c', c''; z) \, f(x, y, z) \, r, s, t \right]
\]

(3.2)

provided that the triple integrals involved exist.

THEOREM 3.2: Under the conditions stated in theorem 3.1, we have
\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} f(x, y, z) \left[ e^{-rx - sy - tz} \right] dz \, dy \, dx \]

\[ = L \left[ J_{a,a',a',b',b';c} f(x, y, z) : r, s, t \right] \quad (3.3) \]

provided that the triple integrals involved exist.

**THEOREM 3.3** : For a function of three variables \( f(x, y, z) \) defined in the positive octant of the three dimensional \( xyz \)-space and \( c > 0 \), we have

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} f(x, y, z) \left[ e^{-rx - sy - tz} \right] dz \, dy \, dx \]

\[ = L \left[ J_{a,a',a',b',b';c} f(x, y, z) : r, s, t \right] \quad (3.4) \]

provided that the triple integrals involved exist.

**THEOREM 3.4** : Under the conditions stated in theorem 3.3, we have

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} f(x, y, z) \left[ e^{-rx - sy - tz} \right] dz \, dy \, dx \]

\[ = L \left[ J_{a,a',a',b',b';c} f(x, y, z) : r, s, t \right] \quad (3.5) \]

provided that the triple integrals involved exist.

**THEOREM 3.5** : Under the conditions stated in theorem 3.1, we have

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} f(x, y, z) \left[ e^{-rx - sy - tz} \right] dz \, dy \, dx \]

\[ = L \left[ J_{a,b,c,c',c'} f(x, y, z) : r, s, t \right] \quad (3.6) \]

provided that the triple integrals involved exist.
THEOREM 3.6: Under the conditions stated in theorem 3.1, we have

\[ \int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) J_{x, y, z}^{a, b, c} e^{-rx-sy-tz} \, dz \, dy \, dx \]

\[ = L \left[ J_{0, x, 0, y, 0, z}^{a, b, c} e^{r,s,t} f(x, y, z) : r, s, t \right] \quad (3.7) \]

provided that the triple integrals involved exist.

THEOREM 3.7: Under the conditions stated in theorem 3.3, we have

\[ \int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) J_{x, y, z}^{a, b, c} e^{-rx-sy-tz} \, dz \, dy \, dx \]

\[ = L \left[ J_{0, x, 0, y, z, 0}^{a, b, c} e^{r,s,t} f(x, y, z) : r, s, t \right] \quad (3.8) \]

provided that the triple integrals involved exist.

THEOREM 3.8: Under the conditions stated in theorem 3.3, we have

\[ \int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) J_{x, y, z}^{a, b, c} e^{-rx-sy-tz} \, dz \, dy \, dx \]

\[ = L \left[ J_{0, x, 0, y, 0, z}^{a, b, c} e^{r,s,t} f(x, y, z) : r, s, t \right] \quad (3.9) \]

provided that the triple integrals involved exist.

We further give relationships among triple Laplace transform, triple Mellin transform and the operators 4(2.1), 4(2.6), 4(2.11), 4(2.16), 4(2.21), 4(2.22) 4(2.23) and 4(2.28) in the form of the following theorems:
**THEOREM 3.9**: For functions of three variables \( f(x, y, z) \) and \( g(x, y, z) \)
defined in the positive octant of the three dimensional xyz-space and \( c > 0, \)
\( c' > 0, c'' > 0, \) we have

\[
M \left[ f(x, y, z) \right] = L \left[ g(x, y, z) \right] \\
= L \left[ g(x, y, z) \right]
\]

**THEOREM 3.10**: For functions of three variables \( f(x, y, z) \) and \( g(x, y, z) \)
defined in the positive octant of the three dimensional xyz-space and \( c > 0, \)
we have

\[
M \left[ f(x, y, z) \right] = L \left[ g(x, y, z) \right]
\]

**THEOREM 3.11**: Under the conditions stated in theorem 3.9, we have

\[
M \left[ f(x, y, z) \right] = L \left[ g(x, y, z) \right]
\]

**THEOREM 3.12**: Under the conditions stated in theorem 3.10, we have

\[
M \left[ f(x, y, z) \right] = L \left[ g(x, y, z) \right]
\]

**THEOREM 3.13**: Under the conditions stated in theorem 3.9, we have
THEOREM 3.14: Under the condition stated in theorem 3.10, we have

\[
L\left[ f(x, y, z) \right] = L\left[ g(x, y, z) \right]
\]

THEOREM 3.15: Under the conditions stated in theorem 3.9, we have

\[
L\left[ f(x, y, z) \right] = L\left[ g(x, y, z) \right]
\]

THEOREM 3.16: Under the conditions stated in theorem 3.10, we have

\[
L\left[ f(x, y, z) \right] = L\left[ g(x, y, z) \right]
\]