CHAPTER VI
GENERALIZED PRODUCT OF FUZZY GROUPS AND P-LEVEL SUBGROUPS

Introduction: Rosenfeld [1971] used the min operating to define his fuzzy groups and showed how some basic notions of group theory should be extended in an elementary manner to develop the theory of fuzzy groups. It was extended by Antony and Sherwood [1982]. They used the t-norm operating instead of the min to define the t-fuzzy groups. Roventa and Spircu [2001] introduced the fuzzy group operating on fuzzy sets. In this chapter, we first generalized the results of the product of fuzzy groups which were done by Ray [1999]. We also define P-level subset and P-level subgroups, and then we study some of their properties.

6.2 section II Preliminaries

6.2.1 Definition: Let $\mu_{\tilde{A}} : U \rightarrow [0, 1]$ be any function and $A$ be a crisp set in the universe $U$. Then the ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in U\}$ is called a fuzzy set and $\mu_{\tilde{A}}$ is called a membership function.

6.2.2 Definition: Let $\tilde{G}_i$ be a fuzzy group under a minimum operation in a group $X_i$ ($i = 1,2,\ldots,n$). Then the membership function of the product $G = \tilde{G}_1 \times \tilde{G}_2 \times \ldots \times \tilde{G}_n$ in $X = X_1 \times X_2 \times \ldots \times X_n$ is defined by

$$(\tilde{G}_1 \times \tilde{G}_2 \times \tilde{G}_3 \times \ldots \times \tilde{G}_n) (x_1, x_2, \ldots, x_n) = \min \{\tilde{G}_1(x_1), \tilde{G}_2(x_2), \ldots, \tilde{G}_n(x_n)\}.$$  

6.2.3 Definition: A fuzzy group $\hat{G}$ of a group $X$ is said to be conjugate to a fuzzy subgroup $H$ of $X$ if there exists $x$ in $G$ such that for all $g \in X$, $\hat{G}(g) = H(x^{-1}g x)$.

6.2.4 Definition: A fuzzy subgroup $H$ of a group $X$ is called fuzzy normal if for all $x, y \in X$, it fulfills the following condition $H(xy) = H(yx)$.  

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The following are established on Product of fuzzy groups and t- fuzzy group

6.2.1 Proposition: If Ĝ₁, Ĝ₂, ..., Ĝₙ be a fuzzy groups of the groups X₁, X₂,...,Xₙ respectively, then Ĝ₁ × Ĝ₂ × ... × Ĝₙ is fuzzy groups of X₁ × X₂ × ... × Xₙ.

Proof: For all elements (x₁, x₂, ..., xₙ) and (y₁, y₂, ..., yₙ) ∈ X₁ × X₂ × ... × Xₙ.

(FG1) (Ĝ₁ × Ĝ₂ × ... × Ĝₙ) ((x₁, x₂, ..., xₙ) (y₁, y₂, ..., yₙ))

= (Ĝ₁ × Ĝ₂ × ... × Ĝₙ) (x₁y₁, x₂y₂, ..., xₙyₙ)

≥ min {Ĝ₁(x₁), Ĝ₂(x₂), ..., Ĝₙ(xₙ)}

≥ min {min {Ĝ₁(x₁), Ĝ₂(x₂), ..., Ĝₙ(xₙ)}, min {Ĝ₁(y₁), Ĝ₂(y₂), ..., Ĝₙ(yₙ)}}

≥ min {(Ĝ₁ × Ĝ₂ × ... × Ĝₙ)(x₁, x₂, ..., xₙ), (Ĝ₁ × Ĝ₂ × ... × Ĝₙ)(y₁, y₂, ..., yₙ)}

FG1 is satisfied

(FG2) (Ĝ₁ × Ĝ₂ × ... × Ĝₙ) ((x₁, x₂, ..., xₙ)^⁻¹)

= (Ĝ₁ × Ĝ₂ × ... × Ĝₙ) (x₁⁻¹, x₂⁻¹, ..., xₙ⁻¹)

= min {Ĝ₁(x₁⁻¹), Ĝ₂(x₂⁻¹), ..., Ĝₙ(xₙ⁻¹)}

= min {Ĝ₁(x₁), Ĝ₂(x₂), ..., Ĝₙ(xₙ)}

= (Ĝ₁ × Ĝ₂ × ... × Ĝₙ)(x₁, x₂, ..., xₙ)

FG2 is satisfied.

So Ĝ₁ × Ĝ₂ × ... × Ĝₙ forms a fuzzy group of X₁ × X₂ × ... × Xₙ.

6.2.2 Proposition: Let Ĝ₁ × Ĝ₂ × ... × Ĝₙ be fuzzy groups of the groups X₁ × X₂ × ... × Xₙ, respectively. Then Ĝ₁ × Ĝ₂ × ... × Ĝₙ is a fuzzy normal subgroup of X₁ × X₂ × ... × Xₙ.

Proof: For (x₁, x₂, ..., xₙ), (y₁, y₂, ..., yₙ) ∈ X₁ × X₂ × ... × Xₙ

(Ĝ₁ × Ĝ₂ × ... × Ĝₙ) ((x₁, x₂, ..., xₙ) (y₁, y₂, ..., yₙ))

= (Ĝ₁ × Ĝ₂ × ... × Ĝₙ) (x₁y₁, x₂y₂, ..., xₙyₙ)

= min {Ĝ₁(x₁y₁), Ĝ₂(x₂y₂), ..., Ĝₙ(yₙxₙ)} = min {Ĝ₁(y₁x₁), Ĝ₂(y₂x₂), ..., Ĝₙ(yₙxₙ)}

= (Ĝ₁ × Ĝ₂ × ... × Ĝₙ) ((y₁, y₂, ..., yₙ) (x₁, x₂, ..., xₙ)).

Thus Ĝ₁ × Ĝ₂ × ... × Ĝₙ is fuzzy normal subgroup of X₁ × X₂ × ... × Xₙ
6.2.3 Proposition: Let the fuzzy groups $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_n$ of $X_1, X_2, \ldots, X_n$ conjugate to fuzzy subgroups $H_1, H_2, \ldots, H_n$. Then $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ of the group $X_1 \times X_2 \times \ldots \times X_n$ is conjugate to the $H_1 \times H_2 \times \ldots \times H_n$ of $X_1 \times X_2 \times \ldots \times X_n$.

Proof: By definition (6.2.4) and a fuzzy group $\hat{G}_i$ conjugates to a fuzzy group $H_i$ of $X_i$, then there exists $x_i \in X_i$ such that $g_i$ in $X_i$, $\hat{G}_i (g_i) = H_i (x_i^{-1} g_i x_i)$, $i = 1, 2, \ldots, n$.

Thus $$(\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (g_1, g_2, \ldots, g_n)$$

$$= \min \{ \hat{G}_1 (g_1), \hat{G}_2 (g_2), \ldots, \hat{G}_n (g_n) \}$$

$$= \min \{ H_1 (x_1^{-1} g_1 x_1), H_2 (x_2^{-1} g_2 x_2^{-1}), \ldots, H_n (x_n^{-1} g_n x_n) \}$$

$$= (H_1 \times H_2 \times \ldots \times H_n) (x_1^{-1} g_1 x_1, x_2^{-1} g_2 x_2, \ldots, x_n^{-1} g_n x_n)$$

$\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ is fuzzy normal subgroup of $H_1 \times H_2 \times \ldots \times H_n$.

6.2.4 Proposition: Let $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_n$ be fuzzy subsets of the groups $X_1, X_2, \ldots, X_n$ respectively. Suppose that $e_1, e_2, \ldots, e_n$ are identity elements of $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_n$ respectively. If $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ is a fuzzy group $X_1 \times X_2 \times \ldots \times X_n$. Then for at least one $i = 0, 1, 2, \ldots, n$, $\hat{G}_i (x_i) \leq (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_{i-1}, e_n, \ldots, e_n)$ --- (1)

Proof: Let $\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n$ be a fuzzy group of $X_1 \times X_2 \times \ldots \times X_n$ and $x_i y_i$ in $X$.

Then $(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n), (e_1, e_2, \ldots, e_{i-1}, y_i, e_{i+1}, \ldots, e_n) \in X_1 \times X_2 \times \ldots \times X_n$.

Now using (1), it follows that

(FG1) $G_i (x_i, y_i) = \min \{ G_i (x_i), (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_{i-1} \times \hat{G}_{i+1} \times \ldots \times \hat{G}_n) \}$

$$= (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_2, \ldots, e_n)$$

$$\geq \min \{ (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_2, x_i, \ldots, e_n), (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_2, y_i, \ldots, e_n) \}$$

$$= \min \{ \min \{ \hat{G}_1 (x_i) (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_2, \ldots, y_i, \ldots, e_n) \}, \min \{ \hat{G}_1 (x_i) (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_2, \ldots, x_i, \ldots, e_n) \} \}$$

$$\geq \min \{ \hat{G}_1 (x_i), \hat{G}_1 (y_i) \}$$

(FG2) $\hat{G}_i (x_i^{-1}) = \min \{ \hat{G}_i (x_i^{-1}), (\hat{G}_1 \times \hat{G}_2 \times \hat{G}_3 \times \ldots \times \hat{G}_n) (e_1^{-1}, e_2^{-1}, \ldots, e_1^{-1}, \ldots, e_n^{-1}) \}$

$$= (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1^{-1}, e_2^{-1}, \ldots, e_n^{-1})$$

$$\geq (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_n) (e_1, e_2, \ldots, x_i, \ldots, e_n)^{-1}$$

$$= \min \{ G_i (x_i), (\hat{G}_1 \times \hat{G}_2 \times \ldots \times \hat{G}_{i-1} \times \hat{G}_i \times \ldots \times \hat{G}_n) (e_1, e_2, e_{i-1}, \ldots, e_n) \}$$

$$= \hat{G}_i (x_i)$. Hence $\hat{G}_i$ is fuzzy group of $X$. 126
6.2.5 **Definition:** Let X be a groupoid and T a t-norm. A function A : X → [0, 1] is called a t-fuzzy group of ‘X’ if and only if (i) A(xy) ≥ T{A(x), A(y)}

(ii) A(x⁻¹) ≥ A(x) for every x, y in ‘X’

6.2.6 **Definition:** Let Ĝ_i be a t-fuzzy group of X_i for each i=1, 2, 3, …, n and T be t-norm. The T product of Ĝ_i (i = 1, 2, 3, …, n) is the function (Ĝ_1× Ĝ_2× …× Ĝ_n) : (X_1× X_2×……×X_n)→ [0, 1] defined by (Ĝ_1× Ĝ_2× …×Ĝ_n) (x_1, x_2, …, x_n) = T(Ĝ_1(x_1), Ĝ_1(x_2), …, Ĝ_n(x_n)).

6.3 section III: P-level subgroups

In this section , the idea of a P-level subset of a fuzzy subset is introduced and discussed. Then some of its important algebraic results are given.

6.3.1 **Definition:** Given a fuzzy set A: X → [0,1] and let P ∈ [0,1]. The set A_p = {x ∈ X / A(x) ≥ p} is called a P-cut of a fuzzy set A.

6.3.2 **Definition:** Let A be a fuzzy subset of a set X , T a t-norm and c ∈ [0,1], Then a P-level subset of a fuzzy set A is defined as A_cT = {x ∈ X : T(A(x), c) ≥ c}.

The following are the propositions of p-level subgroups

6.3.1 **Proposition:** Let X a group and A be a t-fuzzy group of X. Then the P-level subset A_cT , for c ∈ [0,1] , c < T(A(e), r) , is a subgroup of X, where e is the identity of X.

**Proof:** A_cT = {x ∈ X : T(A(x), c) ≥ c} is clearly non-empty.

Let x, y ∈ A_cT. Then T(A(x),c) ≥ c and T(A(y),c) ≥ c, since A is a t-fuzzy group of X.

A(xy) ≥ T{A(x), A(y)} is satisfied. This means

T(A(xy), c) ≥ T(T(A(x), A(y)), c)

= T(A(x) , (T(A(y), c))

≥ T(A(x), c) ≥ c  Hence x, y ∈ A_cT

Again x, y ∈ A_cT implies T(A(x),c) ≥ c. Since A is a t-fuzzy group, A(x⁻¹) = A(x) and hence T(A(x),c) = T(A(x),c). This means that x⁻¹ ∈ A_cT is a subgroup of X.
6.3.2 Proposition: Let A and B be P-level subsets of the sets X and Y respectively and let \( c \in [0, 1] \). Then \( A \times B \) is also a P-level subset of \( X \times Y \).

**Proof:** Any t-norm \( T \) is associative. Using definition (6.2.2) and (6.3.2), the following statements are written.

\[
T((A \times B)(a,b), c) = T(T(A(a), B(b), c)
\]

\[
= T(A(a), T(B(b), c))
\]

\[
T(A(a), c) \geq c
\]

This completes the proof.

6.3.3 Proposition: Let \( X \) and \( Y \) be two groups, \( A \) and \( B \) a t-fuzzy group of \( X \) and \( Y \) respectively. Then the P-level subset \((A \times B)_c\), for \( c \in [0,1] \) is a fuzzy group of \( X \times Y \), where \( e_x \) and \( e_y \) are identities of \( X \) and \( Y \) respectively.

**Proof:** \((A \times B)_c \) T is non empty. Let \((x_1, y_1), (x_2, y_2) \in (A \times B)_c T\)

Then \( T((A \times B)(x_1, y_1), c) > c \) and \( T((A \times B)(x_2, y_2)) > c \).

Since \( A \times B \) is a t-fuzzy group of \( X \times Y \), we get \((A \times B)((x_1, y_1), (x_2, y_2)) = (A \times B)(x_1 x_2, y_1 y_2)\)

\[
= T(A(x_1 x_2), B(y_1 y_2))
\]

Using A and B are t-fuzzy group, it implies that

\[
(A \times B)(x_1 x_2, y_1 y_2) > T((T(A(x_1 x_2), B(y_1 y_2)), c)
\]

\[
= T(A(x_1 x_2), T(B(y_1 y_2), c))
\]

\[
= T(A(x_1 x_2), c)) \geq c.
\]

Hence \((x_1 y_1), (y_1 y_2) \in (A \times B)_c T\).

(FG2) Let \((x,y) \in (A \times B)_c T\) implies

\[
T(A \times B)(xy)^{-1}, c) = T((A \times B)(x^{-1} y^{-1}), c)
\]

\[
= T((A(x^{-1}), B(y^{-1})), c)
\]

\[
= T(A(x^{-1}), T(B(y^{-1}), c))
\]

\[
\geq T(A(x^{-1}), c) \geq c.
\]

This means that \((xy)^{-1} \in (A \times B)_c T\). It forms a fuzzy group of \( X \times Y \)
**6.3.4 Proposition:** Let $X$ be a group and $A_{c}T$ be a P-level subgroup of $X$. If $A$ is a normal t-fuzzy subgroup of $X$, then $A_{c}T$ is a normal subgroup of $X$.

**Proof:** By proposition (6.3.1), $A_{c}T$ is a P-level subgroup of $X$.

Now let show that $A_{c}T$ is normal. for all $a \in X$ and $x \in A_{c}T$.

$$T(A(axa^{-1}), c) = T(A(a^{-1}xa), c) = T(A(x), c) \geq c.$$ 

Thus $axa^{-1} \in A_{c}T$. hence $A_{c}T$ is normal subgroup.

**6.3.5 Proposition:** Let $A$ and $B$ be fuzzy subsets of $X$ and $Y$ respectively, $T$ be a t-norm and $c \in [0,1]$. Then $A_{c}T \times B_{c}T = (AB)_{c}T$.

**Proof:** Let $(a, b) \in A_{c}T \times B_{c}T$ then $a \in A_{c}T$. By definition (6.3.2) we can write

$T(A(a), c) \geq c$ and $T(A(b), c) \geq c$. using the definition (6.2.2) and (6.3.2) we get

$$T((A \times B)(a, b), c) = T(T(A(a),B(b), c) \geq T(A(a), c) \geq c$$

Thus $(a, b) \in (A \times B)_{c}T$ Now let $(a, b) \in (A \times B)_{c}T$, This is required following statements:

$T((A \times B)(a, b), c) = T(T(A(a),B(b), c) \geq T(A(a), c) \geq c = T(1, c).$

Thus the inequalities $T(B(b),c) \geq c$ and $T(A(a), c) \geq c$ is satisfied. Hence $(a, b) \in A_{c}T \times B_{c}T$.

This completes the proof

**CONCLUSION:** We extend fuzzy sets to define the concept of product of fuzzy groups.

We give the sufficient condition for a fuzzy set to be a fuzzy group. By using this fuzzy group, then define a generalized product of fuzzy groups and investigate its structure properties with applications. One can obtain similar result from fuzzy normal subgroup by using the definition of t-fuzzy groups.
6.4 Section IV Fuzzy left h-ideals over hemi rings and interval valued anti fuzzy characteristics function

Introduction: Fuzzy set theory has been developed in many directions by many researchers and has evoked great interest among mathematicians working in different fields of mathematics, such as topological spaces, functional analysis, loop, group, ring, near ring, vector spaces, automation. There have been wide-ranging applications if the theory of fuzzy sets, from the design of robots and computer simulations to engineering and water resource planning. Since then many researchers have been involved in extending the concepts and results of abstract algebra to the broader frame work of the fuzzy setting.

The notion of fuzzy left h-ideals in hemi ring was introduced by Jun [2004]. The notion of (i-v) fuzzy set, a kind of well-known generalization of ordinary fuzzy set, was introduced by zadeh [1965]. Biswas [1994] investigate (i-v) fuzzy subgroup, wang and Li [2000] investigated TH-interval valued fuzzy subgroup and SH – interval valued fuzzy subgroup zeng [2006] proposed concepts of cut set of (i-v) fuzzy set and investigated decompositions of theorems and representation theorems of (i-v) fuzzy set and so on. These works show the importance of (i-v) fuzzy set.

Jun. Y.B. [2004] showed that the fuzzy setting of a left h-ideal in a hemi ring is constructed and basic properties are investigated. Using a collection of left h-ideals of a hemi ring S are established. He also explained the notion of a finite valued fuzzy left h-ideals and its characterization. fuzzy relations on a hemi ring S are also discussed. X.P.Li, [2000] explained the idempotent interval co norm SH induced by a T-co norm on the space on the interval valued
fuzzy sets on fuzzy groups and SH interval valued fuzzy groups. In the mean time, some of its basic properties and structural characterizations are discussed. Also he showed that the theorems of the homomorphic image and the inverse image are given. D.M.Olson [1978] showed that the fundamental homomorphisms theorems for rings is not generally applicable in hemi ring theory. He explained also the class of N- homomorphisms of hemi rings the fundamental theorem is valid. In addition, the concept of N- homomorphism is used to prove that every hereditarily semi subtractive hemi ring is of type (k). W.J.Liu [1987] proved that some basic concepts of fuzzy algebra as a fuzzy invariant subgroups, fuzzy ideals and some fundamental properties. He also showed that characteristic of a field by fuzzy ideals.

In this section, we apply the notion of (i-v) fuzzy sets to anti fuzzy left h- ideals of hemi ring. We introduce the notion of (i-v) anti fuzzy left h- ideals of R with respect to max norm and investigate some of their properties. Using lower level set, we give a characterization of max i – anti fuzzy left h- ideal. Finally we establish the theorems of the homomorphic image and the inverse image. In this chapter, the concept of max i - interval valued anti fuzzy left h- ideals in a hemi rings and extension principle of interval valued fuzzy set are introduced. Some of their properties and structural characteristics, some theorems for homomorphic image are investigated and its inverse image on max i – interval valued anti fuzzy left h- ideals of a hemi rings is verified. Relationship between anti fuzzy left h- ideals in a hemi ring and fuzzy left h- ideals is also given. Using lower level set, a characterization of interval value anti fuzzy left h ideals is given.
6.4.1 Definition: A mapping $A: X \rightarrow [0, 1]$ is called interval valued fuzzy set (i-v fuzzy set) of $X$ and $A(x) = [A^-(x), A^+(x)]$, for all $x \in X$ where $A^-$ and $A^+$ are fuzzy sets in a set $X$.

6.4.2 Definition: A mapping $\max_i: [0,1] \times [0,1] \rightarrow [0,1]$ given by $\max_i(a, b) = [\max(a^-, b^-), \max(a^+, b^+)]$ for all $a, b$ in $[0,1]$ is called interval max-norm.

6.4.3 Definition: Let $\max_i$ be a norm. For arbitrary $a \in [0,1]$ and it satisfies $\max_i(a, a) = a$, then $\max_i$-norm is called idempotent norm.

6.4.4 Definition: An interval number $a$ on $[0,1]$ say $a = [a^-, a^+]$ where $0 < a^- < a^+ < 1$. For any interval numbers $a = [a^-, a^+]$ and $b = [b^-, b^+]$ on $[0,1]$, we define $a < b$ if and only if $a^- < b^-$ and $a^+ < b^+$ and $a^+ = b^+ \Rightarrow a + b = [a^- + b^-, a^+ + b^+]$, whenever $a^- + b^- < 1$ and $a^+ + b^+ < 1$.

6.4.5 Definition: Let $X$ be a fuzzy set. A mapping $A: X \rightarrow [0,1]$ is called a fuzzy set in $X$. For $\alpha \in [0,1]$, $L(A; \alpha) = \{x \in X / A(x) \leq \alpha\}$ is called lower level set of $A$.

6.4.6 Definition: A semi ring $(R, +, \cdot)$ is called a hemi ring if $+$ is commutative and there exists an element $0 \in R$ such that $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$.

6.4.7 Definition: A fuzzy set $A$ of a semi ring is said to be fuzzy left ideal of $R$ if $A(x + y) \geq \min_i \{A(x), A(y)\}$ and $A(xy) \geq A(y)$ for all $x, y$ in $R$. Note that if $A$ is a fuzzy left h-ideal of a hemi ring ‘R’ then $A(0) \geq A(x)$.

Example: Let $R = \{0,1,2,3,4\}$ be a hemi ring with zero multiplication and addition by the following table

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We define a fuzzy set $\mu : R \rightarrow [0,1]$ by letting $\mu(0) = t_1$ and $\mu(x) = t_2$ for all $x \neq 0$, $t_1 < t_2$. By routine computations, we can also easily check that $\mu$ is fuzzy left h-ideals of hemi ring $R$.

6.4.8 Definition: A fuzzy subset $A$ of a hemi ring ‘R’ is said to be max $i$- anti fuzzy left h-ideal of $R$ (i) $A(x + y) \leq \max_i \{A(x), A(y)\}$ (ii) $A(xy) \leq A(y)$ (iii) $x + a + z = b + z \rightarrow A(x) \leq \max_i \{A(a), A(b)\}$ for all $x, y, a, b$ in $R$. 

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The following are the properties of interval valued anti fuzzy left h-ideals

6.4.1 Proposition: Let R be a hemi ring and A be a fuzzy set in R. Then A is max\(^i\) - anti fuzzy h-ideal in R if and only if \(A^c\) is a fuzzy left h-ideal in R.

**Proof:** Let A be a max\(^i\) - anti fuzzy left h-ideal in R. For any \(x, y \in R\),

1. \(A^c(x + y) = 1 - A(x + y)\)
   \[\geq 1 - \max^i \{ A(x), A(y) \}\]
   \[\geq \min^i \{ 1 - A(x), 1 - A(y) \}\]
   \[\geq \min^i \{ A^c(x), A^c(y) \}\]

2. \(A^c(xy) = 1 - A(xy)\)
   \[\geq 1 - A(y)\]
   \[\geq A^c(y).\]

Let \(x, z, a, b \in R\) be such that \(x + a + z = b + z\) Then

\(A^c(x) = 1 - A(x)\)
\[\geq 1 - \max^i \{ A(a), A(b) \}\]
\[\geq \min^i \{ 1 - A(a), 1 - A(b) \}\]
\[\geq \min^i \{ A^c(a), A^c(b) \}. \text{ Thus } A^c \text{ is a fuzzy left h-ideal in } R.\]

Conversely, let \(A^c\) be a fuzzy left h-ideals in R.

For any \(x, y \in R\), it gives that

\(A(x + y) = 1 - A^c(x + y)\)
\[\leq 1 - \min^i \{ A^c(x), A^c(y) \}\]
\[\leq \max^i \{ 1 - A^c(x), 1 - A^c(y) \}\]
\[\leq \max^i \{ A(x), A(y) \}\]
(ii) \( A(xy) = 1 - A^c(xy) \)
\[ \leq 1 - A(xy) \]
\[ \leq A(y) . \]

(iii) Let \( x, z, a, b \in R \) be such that \( x + a + z = b + z \). Then
\[ A(x) = 1 - A^c(x) \]
\[ \leq 1 - \min_i \{ A^c(a), A^c(b) \} \]
\[ \leq \max_i \{ A(x), A(y) \} . \] Thus \( A \) is a \( max^i \) - anti fuzzy left h- ideal of \( R \).

6.4.9 Definition: \( (A \cup B)(x) = \{ \max_i \{ A^c(x), B^c(x) \}, \max_i \{ A^+(x), B^+(x) \} \} \) and
\( (A \cap B)(x) = \{ \min_i \{ A(x), B^c(x) \}, \min_i \{ A^+(x), B^+(x) \} \} \).

6.4.2 Proposition: Let \( A \) and \( B \) are \( max^i \) - anti fuzzy left h- ideal of \( R \). Then \( A \cup B \) also \( max^i \)-anti fuzzy left h- ideal in \( R \).

Proof: For any \( x, y \in R \),

(i) \( (A \cup B)(x + y) = \{ \max_i \{ A^c(x + y), B^c(x + y) \}, \max_i \{ A^+(x + y), B^+(x + y) \} \} \)
\[ \leq \max_i \{ \max_i \{ A^c(x), A^c(y) \}, B^c(x) \}, \max_i \{ A^+(x), A^+(y) \}, \max_i \{ B^+(x), B^+(y) \} \} \]
\[ \leq \max_i \{ \max_i \{ \{ A^c(x), B^c(x) \}, \max_i \{ A^+(x), B^+(x) \} \}, \max_i \{ \{ A^c(y), B^c(y) \}, \max_i \{ B^+(x), B^+(y) \} \} \}
\[ \leq \max_i \{ (A \cup B)(x), (A \cup B)(y) \} \]

\( (A \cup B)(xy) = \{ \max_i \{ A^c(xy), B^c(xy) \}, \max_i \{ A^+(xy), B^+(xy) \} \}
\[ = \{ \max_i \{ A^c(xy), A^+(xy) \}, \max_i \{ B^c(xy), B^+(xy) \} \}
\[ \leq \{ \max_i \{ A^c(y), B^c(y) \}, \max_i \{ A^+(y), B^+(y) \} \}
\[ \leq (A \cup B)(y) . \)
For any $x, z, a, b \in \mathbb{R}$, $x + a + z = b + z \rightarrow$

$$(A \cup B) (x) = \{ \max^i \{ A^- (x), B^- (x) \}, \max^i \{ A^+ (x), B^+ (x) \}$$

$$\leq \max^i \{ \max^i \{ A^- (a), A^- (b) \}, \max^i \{ B^- (a), B^- (b) \} \}$$

$$\leq \max^i \{ \max^i \{ A^- (a), B^- (a) \}, \max^i \{ A^+ (a), B^+ (a) \}, \max^i \{ A^- (b), B^- (b) \}, \max^i \{ A^+ (b), B^+ (b) \} \}$$

$$\leq \max^i \{ (A \cup B) (a), (A \cup B) (b) \}.$$  

### 6.4.10 Definition: 
Let $R_1$ and $R_2$ be two hemi rings and $f$ be a function of $R_1$ into $R_2$. If $A$ is a fuzzy subset in $R_2$, then pre image of $A$ under $f$ is the fuzzy set in $R_1$ defined by $f^{-1} (A) (x) = A f(x)$, for all $x$ in $\mathbb{R}$.

### 6.4.3 Proposition: 
Let $f : R_1 \rightarrow R_2$ be an onto homomorphism of hemi ring. If $A$ is max $^i$ - anti fuzzy left h-ideal of $R_2$, then $f^{-1} (A)$ is a max $^i$ - anti fuzzy left h-ideal of $R_1$.

**Proof:** (i) Let $x, y \in R_1$, Then

$$f^{-1} (A) \ (x + y) = A f (x + y) = A (f(x) + f(y))$$

$$\leq \max^i \{Af(x), Af(y)\}$$

$$\leq \max^i \{f^{-1} (A) (x), f^{-1} (A) (y)\}.$$ 

$$f^{-1} (A) \ (xy) = A f (x + y) = A (f(x)+f(y))$$

$$\leq Af(y)$$

$$\leq f^{-1} (A) \ (y).$$

Let $x, y, a, b \in R_1$ be such that $x + a + z = b + z$. Then

$$f^{-1} (A) \ (x) = A f(x) \leq \max^i \{Af(a), A(b)\}$$

$$\leq \max^i \{f^{-1} (A)(a), f^{-1} (A)(b)\}.$$ 

Hence $f^{-1} (A)$ is max $^i$- anti fuzzy left h-ideal.
6.4.11 Definition: Let $R_1$ and $R_2$ be any two sets and let $f: R_1 \to R_2$ be any function. A fuzzy subset $A$ of $R_1$ is called $f$– invariant if $f(x) = f(y) \implies A(x) = A(y)$.

6.4.4 Proposition: Let $f: R_1 \to R_2$ be an epimorphisms of hemi ring, and $A$ be an $f$- invariant max $^i$ - anti fuzzy left h- ideal of $R_1$. Then max $^i$ - anti fuzzy left h- ideal of $R_2$.

Proof: Let $x, y \in R_2$. Then there exists $a, b \in R_1$ such that $f(a) = x$ and $f(b) = y$.

\[ F(a + b) = x + y \text{ and } f(ab) = xy. \]

Since ‘$A$’ is invariant, it follows that

(i) \[ f(A)(x+y) = A(x+y) \leq \max^i \{A(x), A(y)\} \leq \max^i \{f(A)(x), f(A)(y)\}. \]

(ii) \[ f(A)(xy) = A(xy) \leq A(y) \leq f(A)(y). \]

(iii) Let $x, y, a, b \in R_2$ be such that $x + a + z = b + z \implies f(A)(x) = A(x) \leq \max^i \{A(a), B(b)\} \leq \max^i \{f(A)(a), f(B)(b)\}.

f(A) is max $^i$ - anti fuzzy left h- ideal of $R_2$.

6.4.5 Proposition: Let $A$ be max $^i$ – anti fuzzy left h – ideal in a hemi ring $R$ such that $L(A; \alpha)$ is a left h–ideal of $R$, for each $\alpha \in \text{Im}(A), \alpha \in [0, 1]$. Then $A$ is max $^i$- anti fuzzy left h – ideal of $R$. 

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Proof: Let \(x, y \in R\) such that \(A(x) = \alpha_1\) and \(A(y) = \alpha_2\). Then \(x + y \in L(A; \alpha)\). Without loss of generality, \(\alpha_1 \geq \alpha_2\). It follows \(L(A; \alpha_2) \subseteq L(A; \alpha_1)\), so that \(x \in L(A; \alpha_1)\) and \(y \in L(A; \alpha_2)\). Since \(L(A; \alpha_2)\) is a left \(\alpha\)-ideal of \(R\), then \(x + y \in L(A; \alpha_1)\).

Thus (i) \(A(x + y) \leq \alpha_1 = \max \{A(x), A(y)\}\) (ii) \(A(xy) \leq A(y)\).

(iii) Let \(x, z, a, b \in R\) be such that \(x + a + z = b + z\). Then \(A(x) \leq \alpha_1 = \max \{A(a), A(b)\}\).

This shows that \(A\) is max\(^i\)-anti fuzzy left \(\alpha\)-ideal of \(R\).

6.4.6 Proposition: Let \(A\) be max\(^i\)-anti fuzzy left \(\alpha\)-ideal in a hemi ring \(R\). Let \(A^+\) be a fuzzy lower cut set in \(R\) defined by \(A^+(x) = A(x) + 1 - A(0)\) for all \(x \in R\). Then \(A^+\) is lower cut of max\(^i\)-anti fuzzy left \(\alpha\)-ideal in ‘\(R\)’ which contains \(A\).

Proof: For any \(x, y \in R\), \(A^+(0) = A(0) + 1 - A(0) = 1\) and \(A^+(x) \leq \alpha\).

\[
A^+(x + y) = A(x + y) + 1 - A(0) \leq \max \{A(x), A(y)\} + 1 - A(0) \\
\leq \max \{A(x) + 1 - A(0), A(y) + 1 - A(0)\} \\
\leq \max \{A^+(x), A^+(y)\} \\
\leq \max \{\alpha, \alpha\} \leq \alpha.
\]

\[
A^+(xy) = A(xy) + 1 - A(0) \leq A(y) + 1 - A(0) \leq A^+(y) \leq \alpha.
\]

Let \(x, z, a, b \in R\) such that \(x + a + z = b + z\). Then

\[
A^+(x) = A(x) + 1 - A(0) \\
\leq \max \{A(a), A(b)\} + 1 - A(0) \\
\leq \max \{A(a) + 1 - A(0), A(b) + 1 - A(0)\} \\
\leq \max \{A^+(a), A^+(b)\} \leq \max \{\alpha, \alpha\} \leq \alpha.
\]

Hence \(A^+\) is lower level cut of max\(^i\) anti fuzzy left \(\alpha\)-ideal in \(R\).
6.4.12 Definition: max\(^i\)-anti fuzzy left h-ideal of A of hemi ring R is said to be max\(^i\)-anti fuzzy characteristic if \(A^f(x) = A(x)\) for \(x \in R\) and \(f \in \text{Aut}(R)\).

6.4.7 Proposition: Let \(f : X \rightarrow Y\) be a hemi ring homomorphism. If A is a max\(^i\)-anti fuzzy left h-ideal of Y, then \(A^f\) is also max\(^i\)-anti fuzzy left h-ideal of X.

Proof: For \(x, y \in R\), (i) \(A^f(x + y) = A(x + y)\)

\[
\leq \max_i \{ A(x), A(y) \} \leq \max_i \{ A^f(x), A^f(y) \}.
\]

(ii) \(A^f(xy) = A(xy) \leq A(y) \leq A^f(y)\)

(iii) For \(x, z, a, b \in R\), it gives that

\(x + a + z = b + z \rightarrow A^f(x) = A(x) \leq \max_i \{ A(a), A(b) \} \leq \max_i \{ A^f(a), A^f(b) \}.
\]

\(A^f\) is a max\(^i\)-anti fuzzy left h-ideal of X.

6.4.8 Proposition: Let \(f : X \rightarrow Y\) be an epimorphism of hemi ring R. If \(A^f\) is max\(^i\)-anti fuzzy left h-ideal of X, then A is max\(^i\)-anti fuzzy left h-ideal of Y.

Proof: Let \(x, y \in Y\), then there exist \(a, b \in X\) such that \(f(a) = x\) and \(f(b) = y\). It follows that \(A(x) = Af(a) = A^f(a)\).

(i) \(A(x + y) = Af(a) = A^f(a) \leq \max_i \{ Af(a), Af(b) \}\)

\[
\leq \max_i \{ A^f(a), A^f(b) \} \leq \max_i \{ A(x), A(y) \}.
\]
(ii) $A(xy) = Af(b) = A^f(b) \leq A^f(y) \leq A(y)$.

(iii) For any $x, z, a, b \in R$, $x + z + a = b + z \rightarrow A(x) = Af(a) = A^f(a)$

\[ \leq \max_i \{ A^f(a), A^f(b) \} \]

\[ \leq \max_i \{ A(a), A(b) \} \]

Thus ‘$A’ is max $^i$ - anti fuzzy left h- ideal of $Y$.

6.4.13 Definition : If a fuzzy set $A$ is normal interval valued anti fuzzy left h- ideal of $R$, then $A(0) = 1$.

6.4.9 Proposition: Let $A$ be max $^i$ – anti fuzzy left $h$ – ideal in a hemi ring $R$. Let $A^+$ be a fuzzy set in $R$ defined by $A^+(x) = A(x) + 1 - A(0)$ for all $x$ in $R$. Then $A^+$ is normal max $^i$ – anti fuzzy left $h$ – ideal in ‘$R’$.

Proof: For any $x, y \in R$ we have $A^+(0) = A(0) + 1 - A(0) = 1$ and

\[ A^+(x + y) = A(x + y) + 1 - A(0) \]

\[ \leq \max^i \{ A(x), A(y) \} + 1 - A(0) \]

\[ \leq \max^i \{ A(x) + 1 - A(0), A(y) + 1 - A(0) \} \]

\[ \leq \max^i \{ A^+(x), A^+(y) \}. \]

\[ A^+(xy) = A(xy) + 1 - A(0) \]

\[ \leq A(y) + 1 - A(0) \]

\[ \leq A^+(y). \]

Let $x, z, a, b \in R$ such that $x + a + z = b + z$. 

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Then

\[ A^+(x) = A(x) + 1 - A(0) \]

\[ \leq \max_i \{A(a), A(b)\} + 1 - A(0) \]

\[ \leq \max_i \{A(a) + 1 - A(0), A(b) + 1 - A(0)\} \]

\[ \leq \max_i \{A^+(a), A + (b)\}. \]

Hence \( A^+ \) is normal \( \max^i \) – anti fuzzy left h– ideal in \( R \) and clearly \( A \subset A^+ \).

**Conclusion:** Li, X.P, Wang G.J [2000] introduced the concept of SH interval valued fuzzy subgroup and TH – interval valued fuzzy subgroups. In this chapter, we investigate the interval valued anti fuzzy left h- ideals in a hemi ring with respect to \( \max \) – norm and characterization of them.