CHAPTER-IV

SPLINE FINITE DIFFERENCE METHOD FOR A CLASS OF BOUNDARY VALUE PROBLEMS

4.1 Introduction

In this chapter, we consider the class of two-point boundary-value problem

\[ e^{-ax} \left( e^{ax} u' \right)' = f(x, u), \quad a \leq x \leq b, \]
\[ u(a) = A, \quad u(b) = B. \tag{4.1} \]

Here A, B are finite constants. We assume that for \((x, u) \in \{[a,b] \times \mathbb{R}\}; \) \(f(x, u)\) is continuous, \(\frac{\partial f}{\partial u}\) exists, continuous and \(\frac{\partial f}{\partial u} \geq 0\). Such problems arise in the study of generalized axially symmetric potentials after separation of variables has been employed [60]. The discrete variable numerical solution of two-point boundary value problems by finite differences and spline methods has been considered by many authors, see for examples [16-17,53,70,72,89,94] and references given in these papers.

The use of cubic splines for the solution of (regular) linear two-point boundary-value problems was suggested by Bickley [15]. Later Fyfe [45] discussed the application of deferred corrections to the method suggested by Bickley by considering again the case of (regular) linear boundary-value problems. In comparison with the finite difference methods spline solution has its own advantages. For examples, once the solution has been computed, the information required for spline interpolation between mesh points is available. This is particularly significant when the solution of the boundary-value problem is required at various locations in the interval \([a,b]\). An important instance also
is the use of an automatic plotter that frequently requires interpolation at great many intermediate points. However, it is well known since then that the cubic spline method Bickley gives only second order convergent approximations. But cubic spline itself is a fourth order process.

In the present chapter, in section 4.2, we construct splines and the three-point finite difference method for the solution of (4.1). If we take $\alpha=0$ then our method reduces to the well-known Bickley problem. In section 4.3, we show that the scheme is of $O(h^2)$ convergent under appropriate conditions. The advantage of the spline approximation is that (4.1) may be solved with a particular step length $h$ and the intermediate values if required can be computed using splines. In section 4.4, the three-point finite difference method is applied on two examples. The numerical results confirm the theoretical analysis of our method and support the conclusion that splines give better approximation at the intermediate points.

4.2 Spline Finite Difference Method

We consider a uniform mesh $\Delta$: $a = x_0 < x_1 < x_2 < \ldots \ldots \ldots < x_n = b$,

where $x_j = a + jh$, $h = (b-a)/N$, $j = 0(1)N$.

Let $u_j$ and $f_j$ denote approximations for $u(x_j)$ and $f(x_j, u_j)$ respectively.

Using a piecewise linear interpolating polynomial for $e^{-ax} (e^{ax} u')'$, we write

$$e^{-ax} (e^{ax} u')' = \frac{M_j}{h} (x_{j+1} - x) + \frac{M_{j+1}}{h} (x - x_j)$$

where $h = x_{j+1} - x_j$ and

$$[ e^{-ax} (e^{ax} u')' ]_{x_{j+1}} = M_{j+1}$$

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Integrating (4.2) twice and setting the interpolatory conditions \( u_j = u(x_j) \) and 
\( u_{j+1} = u(x_{j+1}) \), we obtain a spline approximation for \( u(x) \) in the form

\[
u(x) = -S_j \left[ \left( e^{-\alpha x} - e^{-\alpha x_j} \right) u_{j+1} + \left( e^{-\alpha x_{j+1}} - e^{-\alpha x} \right) u_j \right] \\
+ M_j \left[ S_j^* \left\{ 2 x - \alpha (x_{j+1} - x_j)^2 \right\} + a_j + a_j^* \right] \\
+ M_{j+1} \left[ S_j^* \left\{ \alpha (x - x_j)^2 - 2x \right\} + b_j + b_j^* \right]
\]

where

\[
S_j = \frac{e^{\alpha x_j}}{1 - e^{-\alpha h}}, \quad S_j^* = \frac{1}{2\alpha^2 h}
\]

\[
a_j = 2S_j S_j^* \left[ x_j e^{-\alpha x_{j+1}} - x_{j+1} e^{-\alpha x_j} \right] + 2S_j S_j^* h e^{-\alpha x}
\]

\[
a_j^* = S_j S_j^* \alpha h^2 e^{-\alpha x} - S_j \alpha h^2 e^{-\alpha x_{j+1}}
\]

\[
b_j = 2S_j S_j^* \left[ x_{j+1} e^{-\alpha x_j} - x_j e^{-\alpha x_{j+1}} \right] - 2S_j S_j^* h e^{-\alpha x}
\]

\[
b_j^* = S_j S_j^* \alpha h^2 e^{-\alpha x} - S_j S_j^* \alpha h^2 e^{-\alpha x_j}
\]

Taking limits as \( \alpha \to 0 \), it is easily verified that \( u(x) \) reduces to a cubic spline.

Replacing \( j \) by \( j-1 \) in (4.3) we get the spline approximation in the interval \((x_{j-1}, x_j)\) where expressions for \( S_j, S_j^*, a_j, a_j^*, b_j, b_j^* \) are modified accordingly. From the continuity conditions of the first and second derivatives at the node \( x_j \) we get

\[-S_j u_{j-1} + (S_j + S_{j+1}) u_j - S_{j+1} u_{j+1} = A_j M_{j+1} + B_j M_j + C_j M_{j-1} \]

\[j = 1, 2, \ldots, N - 1 \]

where

\[
A_j = S_{j+1}^* \left[ \frac{-2}{\alpha} e^{\alpha x_j} + 2h S_{j+1} - \alpha h^2 S_{j+1} \right]
\]
\[ B_j = \left[ S_{j+1}^* \left\{ \frac{2}{\alpha} e^{\alpha x_j} + 2h e^{\alpha x_j} - 2h S_{j+1} - \alpha h^2 S_{j+1} \right\} \right. \]
\[ \left. - S_j^* \left\{ \frac{2}{\alpha} e^{\alpha x_j} + 2h S_j - \alpha h^2 S_j \right\} \right]. \]

Setting \( M_{j-1} = f_{j-1} = f(x_{j-1}, u(x_{j-1})) \) etc, we get the three-point finite difference approximation

\[ -S_j u_{j-1} + (S_j + S_{j+1}) u_j - S_{j+1} u_{j+1} = A_j f_{j-1} + B_j f_j + C_j f_{j+1} \]
\[ j = 1, 2, \ldots, N - 1. \] \tag{4.5}

Taking limits as \( \alpha \to 0 \), the method (4.5) reduces to

\[ u_{j-1} - 2u_j + u_{j+1} = \frac{h^2}{6} (f_{j-1} + 4f_j + f_{j+1}) \] \tag{4.6}

which is a well-known Bickley scheme [15].

Truncation error of the method (4.5) is

\[ t_j(h) = \frac{h^3}{12} e^{\alpha x_j} f_j + \ldots. \] \tag{4.7}

**4.3 Convergence of the method**

The spline difference scheme (4.5) can be written in the form

\[ S U + M f(U) + T = R \] \tag{4.8}

where \( S, U, M, f, T \) and \( R \) are defined by
For the scheme (4.6) we have

\[
R = \left[(S^2 + C,0,0),(S,B + A,f,)\right]
\]

\[
S = \begin{bmatrix}
S_1 + S_2 & -S_2 & 0 \\
-S_2 & S_3 + S_3 & -S_3 \\
-S_3 & S_4 + S_4 & -S_4 \\
0 & -S_{N-1} & S_{N-1} + S_N
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
-B_1 & -A_1 & 0 \\
-C_2 & -B_2 & -A_2 \\
-C_3 & -B_3 & -A_3 \\
0 & -C_{N-1} & -B_{N-1}
\end{bmatrix}
\]

We note that \( S_j > 0, A_j < 0, B_j < 0, C_j < 0 \)

Hence we have, \( M > 0. \) Dropping the truncation error in (4.8) we get

\[
S \bar{U} + M f(\bar{U}) = R,
\]

where \( \bar{U} \) denotes the approximate solution vector.

Subtracting (4.8) from (4.9), we have

\[
(S + MF)E = T
\]

where \( E = \bar{U} - U \) and \( FE = f(\bar{U}) - f(U) \)
and \( F = \text{diag} \left[ \frac{\partial f_1}{\partial u_1}, \frac{\partial f_2}{\partial u_2}, \ldots, \frac{\partial f_{N-1}}{\partial u_{N-1}} \right] \)

Since we have assumed \( \frac{\partial f}{\partial u} > 0 \), it follows that \( F \geq 0 \), \( MF \geq 0 \) and so \( S + MF \geq S \), where \( S \) is irreducible and monotone.

Therefore, \( (S + MF)^{-1} \leq S^{-1} \) \hspace{1cm} (4.11)

From (4.10) and (4.11), we find

\[
\|E\|_\infty = \left\| (S + MF)^{-1}T \right\|_\infty \leq \left\| (S + MF)^{-1} \right\|_\infty \left\| T \right\|_\infty \leq \left\| S^{-1} \right\|_\infty \left\| T \right\|_\infty \] \hspace{1cm} (4.12)

It may be shown that \( \left\| S^{-1} \right\|_\infty < K \), \( K \) being a constant. Using (4.7), we obtain

\[
\|E\|_\infty = O(h^2) .
\]

4.4 Numerical Illustrations

To illustrate the method (4.5), we consider the following two-point boundary-value problems. The numerical results confirm the second order convergence of the method.

**Example 4.1** \( e^{-\alpha x} (e^{\alpha x} u')' = 2u(1 + \alpha + 2x^2) \).

The exact solution is given by \( u = e^{2x} \) with the boundary conditions

\[
u(0) = 1, \quad u(1) = e \]

We have solved the above problem using the method (4.5) for \( \alpha = .5, 2, 10 \) and \( N = 16, 32, 64, 128 \). We found that the results given in Table 4.1 are accurate and show the second-order convergence of method (4.5).

**Example 4.2** \( u'' + \alpha u' = u \beta x^{\beta-1} \left\{ (\beta - 1) + \alpha x + \beta x^\beta \right\} \).

The exact solution is given by \( u = e^{x^\beta} \) with the boundary conditions

\[
u(0) = 1, \quad u(1) = e \]
We solved this example using the method (4.5) for three sets of values as given in Table 4.2. These results also confirm the second-order convergence of method (4.5).

**Table 4.1 Absolute errors $||E||$ in Ex. 4.1**

| $N$  | $||E||$ $\alpha = 0.5$ | $||E||$ $\alpha = 2$ | $||E||$ $\alpha = 10$ |
|------|------------------------|----------------------|---------------------|
| 16   | $1.53 \times 10^{-3}$  | $1.85 \times 10^{-3}$ | $2.41 \times 10^{-3}$ |
| 32   | $3.82 \times 10^{-4}$  | $4.62 \times 10^{-4}$ | $6.01 \times 10^{-4}$ |
| 64   | $9.58 \times 10^{-5}$  | $1.15 \times 10^{-4}$ | $1.50 \times 10^{-4}$ |
| 128  | $2.39 \times 10^{-5}$  | $2.89 \times 10^{-5}$ | $3.75 \times 10^{-5}$ |

**Table 4.2 Absolute errors $||E||$ in Ex. 4.2**

| $N$  | $||E||$ $\alpha = 0.5, \beta = 4$ | $||E||$ $\alpha = 0.75, \beta = 3.75$ | $||E||$ $\alpha = 2.75, \beta = 3.75$ |
|------|----------------------------------|---------------------------------|---------------------------------|
| 16   | $9.98 \times 10^{-3}$           | $8.63 \times 10^{-3}$          | $9.65 \times 10^{-3}$          |
| 32   | $2.45 \times 10^{-3}$           | $2.12 \times 10^{-3}$          | $2.38 \times 10^{-3}$          |
| 64   | $6.02 \times 10^{-4}$           | $5.31 \times 10^{-4}$          | $5.93 \times 10^{-4}$          |
| 128  | $1.52 \times 10^{-4}$           | $1.32 \times 10^{-4}$          | $1.48 \times 10^{-4}$          |