CHAPTER-V

SEXTIC SPLINE SOLUTION OF THE FOURTH-ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

5.1 Introduction

In this chapter we consider the problem of undamped transverse vibrations of a flexible straight beam in such a way that its supports do not contribute to the strain energy of the system and is represented by the fourth order parabolic partial differential equation,

\[ \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial^4 u}{\partial x^4} = f(x,t), \quad \mu > 0, \ 0 \leq x \leq 1, \ t > 0, \]  

subject to the initial conditions

\[ u(x,0) = g_0(x), \]

and \[ u_t(x,0) = g_1(x), \quad \text{for} \ 0 \leq x \leq 1, \]  

and with boundary conditions at \( x = 0 \) and \( 1 \) of the form

\[ u(0,t) = f_0(t), \quad u(1,t) = f_1(t) \]

and \[ u_{xx}(0,t) = p_0(t), \quad u_{xx}(1,t) = p_1(t), \quad t \geq 0 \]  

where \( \mu > 0 \) is the ratio of flexural rigidity [1] of the beam to its mass per unit length, \( u \) is the transverse displacement of the beam, \( t \) and \( x \) are time and distance variables respectively, \( f(x,t) \) is dynamic driving force per unit mass and functions \( g_0(x), g_1(x), f_0(t), f_1(t), p_0(t) \) and \( p_1(t) \) are continuous functions.
Numerical solution of (5.1) based on finite difference and reduction of (5.1) into a system of second order equations have been successfully proposed by Collatz [24], Crandall [28], Conte and Royster [26], Conte [25], Albrecht [4], Evans [37], Jain et al [59] and Richtmyer [97]. While Fairweather and Gourlay [39] derived explicit and implicit finite difference methods based on the semi-explicit method of Lees [73] and high accuracy method of Douglas [36] respectively. Evans and Yousif [38] follow the Conte scheme [25] where a stable implicit finite difference approximation is presented and this scheme was unconditionally stable, and has local truncation error of $O(h^3)$. The approach, introduced in [38], has used the alternating group explicit method (AGE), achieving a better accuracy level. Wazwaz [128] approaches the problem by utilizing the Adomian decomposition method [1]. The solution by this method is derived in the form of a power series but does not include numerical results. The non-homogeneous problem has also been studied by Khan [68] based on parametric quintic spline.

We need to construct a direct numerical method for solution of equation (5.1). Direct explicit and implicit difference methods have been given by Albrecht [4], Collatz [24], Crandall [28], Jain [54], Jain et al [59] and Todd [118]. The three level explicit direct method with order of accuracy $O(k^2 + h^2)$ given by Collatz [24] is stable when the mesh ratio $(k/h^2) \leq 1/2$. The three level unconditionally stable formulas of accuracy $O(k^2 + h^2)$ and $O(k^2 + h^2 + (k/h)^2)$ are given by Todd [118], Crandall [28] and Conte [25] respectively. Five levels, unconditionally stable, explicit method with truncation error of
O(k^2+h^2+(\frac{k}{h})^2) has been given by Albrecht [4]. Direct and splitting approach finite difference methods have been proposed by Jain et al [59].

We have derived new three level methods based on sextic spline for the solution of fourth order, non-homogeneous, parabolic partial differential equation governing transverse vibrations of a flexible beam. In section 5.2, we present the formulation of our method. In section 5.3, stability analysis has been carried out. Finally in section 5.4, numerical evidence is included to demonstrate the practical usefulness and superiority of our method and confirm their theoretical behaviour.

5.2 The Method

Let the region R = [0,1] × [0,∞) be discretized by a set of points R_{h,k} which are the vertices of a grid of points (x_j, t_m), where x_j = jh, j = 0(1)N, Nh = 1 and t_m = mk, m = 0,1,2,3... The quantities h and k are mesh sizes in the space and time directions respectively.

We next develop an approximation for (5.1) in which the time derivative is replaced by a finite difference approximation and the space derivative by the sextic spline function approximation (1.111). The equation (5.1) is then replaced by

\[ k^{-2}(1+\sigma \delta_x^2)^{-1} \delta_t^2 u_j^{m} + \mu F_j^{m} = f_j^{m} \]  

(5.4)

where \( \sigma \) is a parameter such that the finite difference approximation to the time derivative is \( O(h^2) \) for arbitrary \( \sigma \) and of \( O(h^{4}) \) for \( \sigma = 1/12 \) and for \( \sigma = 1/4, 1/6 \) the finite difference approximations reduce to parametric cubic and cubic spline approximations respectively. Also \( F_j = S^{(4)}(x_j) \) and \( S(x) \) is the sextic spline approximation given in section (1.12). Now the operator \( \Lambda \) is defined by
\[ \Lambda w_j = (w_{j+2} + w_{j-2}) + 56(w_{j+1} + w_{j-1}) + 246w_j \]  

(5.5)

Therefore from (1.111), we have

\[ (F_{j+2} + F_{j-2}) + 56(F_{j+1} + F_{j-1}) + 246F_j = \frac{360}{h^4} \left[ (u_{j+2} + u_{j-2}) - 4(u_{j+1} + u_{j-1}) + 6u_j \right] \]  

(5.6)

Using (5.5) and (5.6), we can write \[ \Lambda F_j = \frac{360}{h^4} \delta^4 u_j^m \]  

(5.7)

Operating both sides of (5.4) by \( \Lambda x \) and using (5.7) we obtain

\[ \delta^2 \left\{ (u_{j+2}^m + u_{j-2}^m) + 56(u_{j+1}^m + u_{j-1}^m) + 246u_j^m \right\} + 360r^2 (1 + \sigma \delta^2) \mu \delta^4 u_j^m = \]

\[ k^2 (1 + \sigma \delta^2) \left\{ (f_{j+2}^m + f_{j-2}^m) + 56(f_{j+1}^m + f_{j-1}^m) + 246f_j^m \right\} \]  

(5.8)

where \( r = k/h^2, u_j^m = u(x_j, t_m), \delta^2 u_j^m = u_j^{m+1} - 2u_j^m + u_j^{m-1} \).

After simplification (5.8), we get

\[ \{ (360) + (60)\delta^2 + (1 + 360r^2 \sigma \mu) \delta^4 \} \delta^2 u_j^m + 360r^2 \mu \delta^4 u_j^m = \]

\[ k^2 (1 + \sigma \delta^2) \left\{ (f_{j+2}^m + f_{j-2}^m) + 56(f_{j+1}^m + f_{j-1}^m) + 246f_j^m \right\} \]  

(5.9)

Equation (5.9) may be written in schematic form as

\[
\begin{array}{cccccc}
P2 & Q2 & S2 & Q2 & P2 \\
-2P2+r^2 & -2Q2-4r^2 & -2S2+6r^2 & -2Q2-4r^2 & -2P2+r^2 & u_j^m = \\
\end{array}
\]

\[
\begin{array}{cccccc}
P2 & Q2 & S2 & Q2 & P2 \\
K_1P & K_1Q & K_1S & K_1Q & K_1P \\
K_2P & K_2Q & K_2S & K_2Q & K_2P \\
K_1P & K_1Q & K_1S & K_1Q & K_1P \\
\end{array}
\]

\[ f_j^m \]  

82
where \( P_2 = \frac{1}{360} + \sigma \mu r^2 \), \( Q_2 = \frac{56}{360} - 4\sigma \mu r^2 \), \( S_2 = \frac{246}{360} + 6\sigma \mu r^2 \),

\( K_1 = \sigma k^2 \), \( K_2 = (1 - 2\sigma) k^2 \)

### 5.3 Truncation Error and Stability Analysis

Expanding (5.9) in Taylor series in terms of \( u(x_j, t_m) \) and its derivatives, we obtain the following relations

\[
\begin{align*}
\delta_i^4 u(x_j, t_m) &= h^4 D_i^4 + \frac{h^6 D_i^6}{6} + \frac{h^8 D_i^8}{80} + \frac{17h^{10}}{3024} D_i^{10} + \frac{62h^{12}}{10!} D_i^{12} + ... \\
\delta_i^2 u(x_j, t_m) &= -r^2 h^4 D_i^4 + \frac{1}{12} r^4 h^8 D_i^8 - \frac{1}{360} r^6 h^{12} D_i^{12} + \frac{1}{20160} r^8 h^{16} D_i^{16} + ...
\end{align*}
\]

(5.10)

\[
f_j^m = D_i^2 u_j^m + D_i^4 u_j^m
\]

Using (5.9) and (5.10) we obtain the truncation error

\[
T_j^m = \{ (360) + (60)\delta_i^2 + (1 + 360r^2\sigma\mu)\delta_i^4 \} \delta_i^2 u_j^m + 360r^2\mu\delta_i^4 u_j^m
\]

\[
- k^2 (1 + \sigma\delta_j^2) \{ (f_{j-2}^m + f_{j-1}^m) + 56(f_{j+1}^m + f_{j+2}^m) + 246f_j^m \}
\]
Using Von Neumann’s method the characteristic equation of the scheme (5.9)
is obtained as:
\[
\xi^2 + 2\gamma\xi + 1 = 0
\]  
(5.12)

where

\[
\gamma = -1 \left( \frac{8r^2 \sin^4 \phi}{16\left( \frac{1}{360} + r^2\sigma \right) \sin^4 \phi - \frac{2}{3} \sin^2 \phi + 1} \right)
\]

\[
\phi = \frac{1}{2} \theta h, \text{ where } \theta \text{ is the variable in the Fourier expansion.}
\]

Applying the Routh-Hurwitz criterion to (5.12) we get the necessary and sufficient conditions for (5.9) to be stable as:

\[
-1 \leq \gamma \leq 1
\]

Simplifying, we obtain from the left inequality

\[
[2 + 360(2\sigma - 1)r^2] \sin^4 \phi - 30 \sin^2 \phi + 45 \geq 0
\]  
(5.13)

We deduce that the scheme (5.9) is unconditionally stable if

\[
\sigma < \frac{1}{12}, \quad r^2 \geq \frac{\cos \phi(2 - 3\csc^2 \phi)}{24(2\sigma - 1)}
\]

and conditionally stable if

\[
\sigma < \frac{1}{12}, \quad r^2 \geq \frac{\cos \phi(2 - 3\csc^2 \phi)}{24(2\sigma - 1)}
\]

5.4 Numerical Illustrations and Discussions

In this section we consider the numerical results obtained by the method discussed above by applying them to the following fourth order initial boundary value problem.
Example 5.1

We consider the non-homogeneous fourth order parabolic partial differential equation introduced by [38].

\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = (\pi^4 - 1) \sin \pi x \cos t, \quad 0 \leq x \leq 1, \quad t > 0 \]  
(5.14)

with the initial conditions

\[ u(x,0) = \sin \pi x, \quad u_t(x,0) = 0, \quad 0 \leq x \leq 1 \]  
(5.15)

and the boundary conditions

\[ u(0,t) = u(1,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0, \quad t \geq 0 \]  
(5.16)

The exact solution of the above problem is

\[ u(x,t) = \sin \pi x \cos t, \]  
(5.17)

For solving (5.14) we use scheme (5.9). The first two boundary conditions in (5.16) are replaced by

\[ u_0^n = u_N^n = 0, \quad t \geq 0 \]  
(5.18)

We discretize the last two boundary conditions in (5.16) by the following equations

(i) \[ 2u_0^n - 5u_1^n + 4u_2^n - u_3^n = h^2 u''_0, \]  
(ii) \[ -u_{N-3}^n + 4u_{N-2}^n - 5u_{N-1}^n + 2u_N^n = h^2 u''_N \]  
(5.19)

For high accuracy formulas, we use the following equations for approximating the boundary conditions:

(i) \[ 45u_0^n - 154u_1^n + 214u_2^n - 156u_3^n + 61u_4^n - 10u_5^n = 12h^2 u''_0, \]  
(ii) \[ -10u_{N-5}^n + 61u_{N-4}^n - 156u_{N-3}^n + 214u_{N-2}^n - 154u_{N-1}^n + 45u_N^n = 12h^2 u''_N \]  
(5.20)

We solved example 5.1 with \( h = 0.05 \) and \( k = 0.005 \) giving \( r = 2 \), and by choosing
\(\sigma = 1/4\), the results are presented in table 5.1. The errors in the solutions computed by our method (5.9) and the AGE method [38] have been presented in table 5.1 for 10 time steps and \(x = 0.1(0.1)0.5\) and in table 5.2 for \(x = 0.5\) and larger time steps. In a second series of experiments, calculations are carried out for \(h = 0.05\) and \(k = 0.00125\) giving \(r = 0.5\), and results are presented in table 5.1. The absolute errors in the solution are shown in table 5.1 for 16 time steps and \(x = 0.1(0.1)0.5\) and in table 5.2 for \(x = 0.5\) and larger time steps. From tables 5.1 and 5.2, it is evident that our method is superior. Moreover, we solved the same problem with different values of \(\sigma\) and mesh ratio \(r\) and carrying out the computations for different time steps. In table 5.3 we have tabulated the absolute errors at \(x = 0.5\) for different values of \(\sigma\) and mesh ratio \(r\) for \(h = 0.1\). The errors in displacement function \(u(x,t)\) at midpoint of the interval \([0,1]\) are given in table 5.2 & 5.3.

**Table 5.1 Absolute Errors, Example 5.1**

<table>
<thead>
<tr>
<th>Our Method</th>
<th>(r)</th>
<th>Time Steps</th>
<th>(x = .10)</th>
<th>.20</th>
<th>.30</th>
<th>.40</th>
<th>.50</th>
</tr>
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<tbody>
<tr>
<td>(\sigma)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>2.0</td>
<td>10</td>
<td>1.87×10^{-5}</td>
<td>2.13×10^{-5}</td>
<td>1.49×10^{-5}</td>
<td>8.60×10^{-6}</td>
<td>5.96×10^{-6}</td>
</tr>
<tr>
<td>1/4</td>
<td>0.5</td>
<td>16</td>
<td>9.07×10^{-6}</td>
<td>7.79×10^{-6}</td>
<td>2.75×10^{-6}</td>
<td>1.01×10^{-6}</td>
<td>2.59×10^{-6}</td>
</tr>
<tr>
<td>Evans [38]</td>
<td>2.0</td>
<td>10</td>
<td>2.2×10^{-4}</td>
<td>4.1×10^{-4}</td>
<td>5.4×10^{-4}</td>
<td>6.2×10^{-4}</td>
<td>6.5×10^{-4}</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>16</td>
<td>2.5×10^{-5}</td>
<td>4.7×10^{-5}</td>
<td>6.6×10^{-5}</td>
<td>7.8×10^{-5}</td>
<td>8.2×10^{-5}</td>
</tr>
</tbody>
</table>
Table 5.2 Absolute Errors at Mid Points, $x = 0.5$, Example 5.1

$h = 0.05$

<table>
<thead>
<tr>
<th>Our Method</th>
<th>$\sigma$</th>
<th>$r = 2$</th>
<th>$r = 0.5$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td>No. of Time Steps</td>
<td>No. of Time Steps</td>
</tr>
<tr>
<td></td>
<td></td>
<td>25  75  100</td>
<td>32  48  64</td>
</tr>
<tr>
<td>$1/4$</td>
<td></td>
<td>$1.02 \times 10^{-4}$</td>
<td>$1.94 \times 10^{-6}$</td>
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<td></td>
<td></td>
<td>$3.53 \times 10^{-4}$</td>
<td>$2.68 \times 10^{-5}$</td>
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<td></td>
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<td>$1.37 \times 10^{-4}$</td>
<td>$5.11 \times 10^{-5}$</td>
</tr>
<tr>
<td>Evans [38]</td>
<td></td>
<td>$3.3 \times 10^{-3}$</td>
<td>$3.1 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$4.1 \times 10^{-3}$</td>
<td>$6.9 \times 10^{-4}$</td>
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<tr>
<td></td>
<td></td>
<td>$3.9 \times 10^{-3}$</td>
<td>$1.2 \times 10^{-3}$</td>
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</tbody>
</table>

Table 5.3 Absolute Errors at Mid Points, $x = 0.5$, Example 5.1

$h = 0.1$

<table>
<thead>
<tr>
<th>Our Method</th>
<th>$\sigma$</th>
<th>$r$</th>
<th>No. of Time Steps</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>10  20  30</td>
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<tr>
<td></td>
<td></td>
<td>$\sqrt{1/6}$</td>
<td>$4.43 \times 10^{-5}$</td>
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<td>$\sqrt{1/84}$</td>
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<td>$1.51 \times 10^{-3}$</td>
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<td>$1/12$</td>
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<td>$2.93 \times 10^{-3}$</td>
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<td>$9.26 \times 10^{-5}$</td>
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<td>$9.10 \times 10^{-5}$</td>
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