Order in Probability of Fuzzy Random Variables

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Abstract:
Stochastic ordering is highly applicable in Reliability, Decision making and Life Testing Problems. In this paper we introduce a new notion of order in probability such as at most of order and derive their properties.

Key words:

1. Introduction

Stochastic ordering of fuzzy random variables a counterpart of stochastic ordering of random variables has wide and very important applicability in reliability, decision making, life testing problems and epidemic models. The notion of stochastic ordering of fuzzy random variables could be administered where parameters are not numerical but linguistic. Literatures addressing stochastic comparisons are abound in number. In [3] Gordon Pledger et al., have studied stochastic comparisons of random processes with applications in reliability in which the usual definition of stochastic comparison of random vectors is extended to stochastic comparison of Random processes and applications are made to reliability problems. In [5] Kamae et al., have studied stochastic inequalities on partially ordered spaces.

They have discussed the concept of partial ordering and its applications in the set of probabilities on a partially ordered polish space. Extensions of the stochastic ordering property of likelihood ratios was studied by Gordon Simons [4], in which he has established the fact that sequences of likelihood ratios possess the same stochastic ordering property in a multivariate sense. George Shanthikumar in [2] has provided sufficient conditions with which two random vectors could be stochastically compared. Earnest Lazarus et al., in [1] have studied stochastic orderings, Hazard rate ordering and stochastically more variable ordering to the realm of fuzzy random variables.

In this paper we introduce the notion of order in probability to the domain of fuzzy random variables. We propose the concept of order in probability such as ‘at most of
order' and derive their properties. These notions could effectively be applied to large sample theory of statistics where the parameters are not numerical but linguistic. The concept of a fuzzy random variable was introduced by Kwakernaak [6,7] and Puri and Ralescu [8]. Fuzzy random variables are random variables whose values are not real but fuzzy numbers. A fuzzy number may assume different real values with a degree of acceptability is associated to each real value under contemplation. In this paper we adopt the theoretical framework of fuzzy random variables introduced by Kwakernaak [6,7]. For notions and details of fuzzy random variables we refer to Kwakernaak [6,7]. The organization of the paper is as follows. Section 2 is employed to briefly mention the concept of Kwakernaak’s fuzzy random variables and fuzzy numbers. In Section 3 we introduce the notion of various orders in probability to the realm of Kwakernaak’s fuzzy random variables and derive its properties.

2. Preliminaries

In this section we provide the underlying concepts related to Kwakernaak’s [6,7] fuzzy random variables. A fuzzy random variable according to Kwakernaak is a fuzzy set, consisting of a membership function and a basic set whose entries are real random variables. We accommodate more real random variables in to the fold of defining a fuzzy random variables as these random variables are partially obscured by the dimness of perception. If $U_\omega$ is the underlying random variable and $\omega$ is the outcome of the random experiment, the exact value of $U_\omega(\omega)$ is imperceptible due to blurred boundaries. Instead it is assumed that a fuzzy number $\bar{e}_\omega = (R, X_\omega)$ is known which characterizes $U_\omega(\omega)$. We consider the mapping $X: \Omega \rightarrow F$ where $F$ is the set of all fuzzy numbers. the mapping $X$ supplies a membership function $X_\omega$ for each random outcome as known as the fuzzy perception function. Thus we associate with each $\omega$ a real number $U_\omega(\omega)$ but a membership function $X_\omega$. To the observer who must perceive random outcomes via $X$ to whom the identity of $U_\omega$ is blurred, naturally there may arise many reconstructions of $U_\omega$ which are governed by the fuzzy perception. Fuzzy logic generates valuation function for $X$, for which the random variables serve as entities. We consider the probability space $((\Omega, \mathcal{F}, P))$. If $U$ is a $\mathcal{F}$ measurable random variable then

$$\mu_\alpha(U) = \inf_{\omega \in \Omega} X_\omega(U(\omega))$$

(2.1)

is the valuation of its suitability as a reconstruction of $U_\omega$.

We assume that on the triple $(\Omega, \mathcal{F}, P)$ for each $\alpha \in (0, 1]$ and $\omega \in \Omega$, the functions.

$$U^+_{X, \omega}(\omega) = \inf \{x \in R; X_\omega(x) \geq \alpha\} \text{ and}$$

$$U^-_{X, \omega}(\omega) = \sup \{x \in R; X_\omega(x) \geq \alpha\}$$

(2.2)
are measurable with respect to \((\Omega, \tilde{\mathcal{F}})\).

In what follows we provide the definitions related to fuzzy numbers which are useful in the sequel.

**Definition 2.1.** A fuzzy number \(f\) in the real line \(\mathbb{R}\) is a fuzzy set \(f : \mathbb{R} \rightarrow [0, 1]\) that satisfies the following properties.

(i) \(f\) is piecewise continuous

(ii) There exists an \(x \in \mathbb{R}\) such that \(f(x) = 1\)

(iii) \(f\) is convex i.e. if \(x_1, x_2 \in \mathbb{R}\) and \(\alpha \in [0, 1]\) then

\[
f(\lambda x_1 + (1-\lambda) x_2) \geq f(x_1) \wedge f(x_2)
\]

A useful tool for the treatment of fuzzy numbers is the \(\alpha\)-level sets. The \(\alpha\)-level set \(N_j(\alpha)\) of a fuzzy number \(f\) is the non fuzzy set defined by

\[
N_j(\alpha) = \{x \in \mathbb{R}; f(x) \geq \alpha\}; \ 0 \leq \alpha \leq 1,
\]

\(\tilde{a}\) is called a fuzzy number if \(\tilde{a}\) is a normal convex fuzzy set and the \(\alpha\)-level set \(\tilde{a}_\alpha\) is bounded \(\forall \alpha \in (0, 1]\). \(\tilde{a}\) is called a closed fuzzy number if \(\tilde{a}\) is a fuzzy number and its membership function \(\tilde{a}(\cdot)\) is upper semi continuous. The fuzzy number \(\tilde{a}\) is called a bounded fuzzy number if its membership function \(\tilde{a}(\cdot)\) has compact support. If \(\tilde{a}\) is a closed fuzzy number then the \(\alpha\)-level set of \(\tilde{a}\) is a closed interval which we denote by \(\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]\).

**Definition 2.2.** \(\tilde{a}\) is called a canonical fuzzy number if it is a closed and bounded fuzzy number and its membership function is strictly increasing on the interval \([a_0^L, a_1^L]\) and strictly decreasing on the interval \([a_0^U, a_1^U]\).

**Definition 2.3.** Let \(\tilde{a}\) and \(\tilde{b}\) be two closed fuzzy numbers. Then \(\tilde{a} + \tilde{b}\), \(\tilde{a} \times \tilde{b}\) and \(\tilde{a} \cdot \tilde{b}\), are closed fuzzy numbers. Moreover we have

\[
(\tilde{a} + \tilde{b})_\alpha = [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U]
\]

\[
(\tilde{a} \times \tilde{b})_\alpha = [\min(\tilde{a}_\alpha^L \tilde{b}_\alpha^L, \tilde{a}_\alpha^L \tilde{b}_\alpha^U, \tilde{a}_\alpha^U \tilde{b}_\alpha^L, \tilde{a}_\alpha^U \tilde{b}_\alpha^U), \max(\tilde{a}_\alpha^L \tilde{b}_\alpha^L, \tilde{a}_\alpha^L \tilde{b}_\alpha^U, \tilde{a}_\alpha^U \tilde{b}_\alpha^L, \tilde{a}_\alpha^U \tilde{b}_\alpha^U)]
\]
\[
\begin{bmatrix}
\begin{bmatrix}
\min & \frac{\tilde{a}_n}{\tilde{b}_n} & \frac{\tilde{a}_n}{\tilde{b}_n} & \frac{\tilde{a}_n}{\tilde{b}_n} & \frac{\tilde{a}_n}{\tilde{b}_n} \\
\max & \frac{\tilde{a}_n}{\tilde{b}_n} & \frac{\tilde{a}_n}{\tilde{b}_n} & \frac{\tilde{a}_n}{\tilde{b}_n} & \frac{\tilde{a}_n}{\tilde{b}_n}
\end{bmatrix}
\end{bmatrix}
\]

3. **Order in probability**

Let \( \{\tilde{a}_n\} \) be a sequence of canonical fuzzy numbers and \( \{\tilde{g}_n\} \) be a sequence of canonical positive fuzzy numbers.

**Definition 3.1.** \( \{\tilde{a}_n\} \) is of smaller order than \( \{\tilde{g}_n\} \) if

\[
\lim_{n \to \infty} \left( \frac{\tilde{a}_n}{\tilde{g}_n} \right) = \tilde{o}
\]

we denote it as \( \tilde{a}_n = o(\tilde{g}_n) \)

**Definition 3.2.** \( \{\tilde{a}_n\} \) is almost of order \( \{\tilde{g}_n\} \) if there exists a real \( M > 0 \) such that \( |a_n|/g_n \leq M \) for all \( n \geq n_0 \). We denote it as \( \tilde{a}_n = O(\tilde{g}_n) \)

**Lemma 3.1.** Let \( \{\tilde{a}_n\} \) and \( \{\tilde{b}_n\} \) be a sequence of canonical fuzzy numbers. Let \( \{\tilde{f}_n\} \) and \( \{\tilde{g}_n\} \) be a sequence of canonical positive fuzzy numbers.

Then (i) if \( \tilde{a}_n = o(\tilde{f}_n) \) and \( \tilde{b}_n = o(\tilde{g}_n) \) then

\[
\tilde{a}_n \cdot \tilde{b}_n = o\left(\tilde{f}_n \cdot \tilde{g}_n\right), \quad |\tilde{a}_n'| = o\left(\tilde{f}_n'\right) \quad \text{for} \quad r > 0, \quad \tilde{a}_n + \tilde{b}_n = o\left(\max\left(\tilde{f}_n, \tilde{g}_n\right)\right)
\]

(ii) if \( \tilde{a}_n = O(\tilde{f}_n) \) and \( \tilde{b}_n = O(\tilde{g}_n) \) then

\[
\tilde{a}_n \cdot \tilde{b}_n = O\left(\tilde{f}_n \cdot \tilde{g}_n\right), \quad |\tilde{a}_n'| = O\left(\tilde{f}_n'\right) \quad \text{for} \quad r \geq 0
\]

and

\[
\tilde{a}_n + \tilde{b}_n = O\left(\max\left(\tilde{f}_n, \tilde{g}_n\right)\right)
\]

(iii) if \( \tilde{a}_n = o(\tilde{f}_n) \) and \( \tilde{b}_n = O(\tilde{g}_n) \) then

\[
\tilde{a}_n \cdot \tilde{b}_n = o\left(\tilde{f}_n \tilde{g}_n\right)
\]
Order in Probability of Fuzzy Random Variables

Proof. Proofs are obvious from Definition 3.1 and 3.2. For typographical reasons in what follows we denote \( U_{X_n, \alpha}(x) \) by \( U_{X_n, \alpha} \).

Definition 3.3. Let \( \{X_n\} \) be sequence of fuzzy random variables and \( \{\tilde{g}_n\} \) be a sequence of canonical positive fuzzy number. \( \{\tilde{x}_n\} \) is of smaller order in probability than \( \{\tilde{g}_n\} \) if \( P\left\{ \left| U_{X_n, \alpha} / N_{\tilde{g}_n}(\alpha) \right| \leq \varepsilon / N_{\tilde{g}_n}(\alpha) \geq M \right\} \to 0 \) as \( n \to \infty \). We denote this as \( \tilde{x}_n = o_p(\tilde{g}_n) \).

Definition 3.4. We say the sequence of fuzzy random variables \( \{X_n\} \) is almost of order \( \{\tilde{g}_n\} \) in probability if for all \( \varepsilon > 0 \), there exists \( M > 0 \) such that

\[
P\left\{ \left| U_{X_n, \alpha} / N_{\tilde{g}_n}(\alpha) \right| \geq M \right\} \leq \varepsilon \quad \text{for all} \quad n \geq n_0.
\]

We denote this as \( X_n = Op(\tilde{g}_n) \).

Definition 3.5. If \( \{x_n\} \) is a sequence of \( K \)-dimensional fuzzy random variables, \( \{x_n\} \) is almost of order \( \{\tilde{g}_n\} \) in probability if for every \( \varepsilon > 0 \), there exists \( M_\varepsilon > 0 \) such that

\[
P\left\{ \left| U_{X_n, \alpha} / N_{\tilde{g}_n}(\alpha) \right| \geq M_\varepsilon \right\} \leq \varepsilon, \quad j = 1, 2, \ldots, K.
\]

For all \( n \) and \( \alpha \in (0, 1] \).

Definition 3.6. The sequence \( \{x_n\} \) of \( K \)-dimensional fuzzy random variables is of smaller order in probability than \( \{\tilde{g}_n\} \) if for all \( \varepsilon > 0 \) and \( \delta > 0 \) there exists an \( N \) such that for all \( n \geq N \)

\[
P\left\{ \left| U_{X_n, \alpha} / N_{\tilde{g}_n}(\alpha) \right| \leq \delta ; \quad j = 1, 2, \ldots, N \right\}
\]

Theorem 3.1. Let \( \{\tilde{f}_n\} \) and \( \{\tilde{g}_n\} \) be sequence of canonical positive numbers. Let \( \{X_n\} \) and \( \{Y_n\} \) be sequence of fuzzy random variables. If \( X_n = o_p(\tilde{f}_n) \) and \( Y_n = o_p(\tilde{g}_n) \) then

(i) \( X_n Y_n = o_p(\tilde{f}_n, \tilde{g}_n) \)
(ii) \( |X_n|^* = \sigma_p\left(\hat{f}_n^*\right) \) for \( s > 0 \) and

(iii) \( X_n + Y_n = \sigma_p\left(\max\left(\hat{f}_n, \hat{g}_n\right)\right) \)

Proof. (i) By stipulation for \( \epsilon > 0, \exists \delta > 0 \) such that

\[
P\left\{\left(\vert U_{x_{n}}^{*}\vert \vee \vert U_{y_{n}}^{*}\vert\right) > \epsilon \right\} < \delta/2 \quad \text{and}
\]

\[
P\left\{\left(\vert V_{y_{n}}^{*}\vert \vee \vert V_{y_{n}}^{*}\vert\right) > \epsilon \right\} < \delta/2 \quad \text{for } n \geq N_0
\]

where \( V \) is the underlying random variable corresponding to the fuzzy random variable \( y \).

\[
\therefore P\left\{\left(\vert U_{x_{n}}^{*}\vert \vee \vert U_{y_{n}}^{*}\vert\right) > \epsilon \right\} < \epsilon^2 N_{P_0}(\alpha) N_{P_n}(\alpha)
\]

\[
\leq P\left\{\left(\frac{U_{x_{n}}}{N_{P_0}(\alpha)} \vee \frac{U_{y_{n}}}{N_{P_n}(\alpha)}\right) > \epsilon \right\} + P\left\{\left(\frac{V_{y_{n}}}{N_{P_n}(\alpha)} \vee \frac{V_{y_{n}}}{N_{P_n}(\alpha)}\right) > \epsilon \right\} \leq \delta
\]

for all \( n \geq N_0 \) and \( \alpha \in (0, 1] \)

\[
\therefore X_n Y_n = \sigma_p\left(\hat{f}_n, \hat{g}_n\right)
\]

(ii) For each \( \alpha \in (0, 1] \) and \( \epsilon > 0 \) we have

\[
P\left\{\left(\vert U_{x_{n}}^{*}\vert \vee \vert U_{y_{n}}^{*}\vert\right) > \epsilon \right\} = P\left\{\left(\vert U_{x_{n}}^{*}\vert \vee \vert U_{y_{n}}^{*}\vert\right) > \epsilon^* (N_{P_0}(\alpha))^*\right\}
\]

\[
\therefore |X_n|^* = \sigma_p\left(\hat{f}_n^*\right)
\]

(iii) we take \( \hat{g}_n = \max\left(\hat{f}_n, \hat{g}_n\right) \). Then given \( \epsilon > 0 \) and \( \delta > 0 \) there exists integer \( n \) such that

\[
P\left\{\left(\vert U_{x_{n}}^{*}\vert \vee \vert U_{y_{n}}^{*}\vert\right) > \epsilon/2 q_n\right\} < \delta/2
\]

and

\[
P\left\{\left(\vert V_{y_{n}}^{*}\vert \vee \vert V_{y_{n}}^{*}\vert\right) > \epsilon/2 q_n\right\} < \delta/2
\]

for all \( n \geq N_0 \) and \( \alpha \in (0, 1] \).

\[
\therefore P\left\{\left(\vert U_{x_{n}}^{*}\vert + \vert V_{y_{n}}^{*}\vert \vee \vert U_{y_{n}}^{*}\vert + \vert V_{y_{n}}^{*}\vert\right) > \epsilon\right\} \leq q_n
\]
\[ \leq P \left( \left( U_{X_n,\alpha} \lor U_{X_n,\alpha}^{\prime} \right) \right) > \varepsilon / 2 q_n + P \left( \left( V_{Y_n,\alpha} \lor V_{Y_n,\alpha}^{\prime} \right) \right) > \varepsilon / 2 q_n \]

\[ < \delta / 2 + \delta / 2 = \delta , \forall n > N_0 . \]

This proves the result.

**Theorem 3.2.** Let \( \{ \tilde{f}_n \} \) and \( \{ \tilde{g}_n \} \) be sequences of canonical positive numbers and let \( \{ X_n \} \) and \( \{ Y_n \} \) be sequences of fuzzy random variables. If \( X_n = O_p \left( \tilde{f}_n \right) \) and \( Y_n = O_p \left( \tilde{g}_n \right) \). Then

\[ X_n Y_n = O_p \left( \tilde{f}_n \tilde{g}_n \right) , \quad |X_n|^s = O_p \left( \tilde{f}_n^s \right) \text{ for } s \geq 0 \text{ and, } X_n + Y_n = O_p \left( \max \left( \tilde{f}_n, \tilde{g}_n \right) \right) . \]

**Proof.** Proof is similar to Theorem 3.1.

**Theorem 3.3.** Let \( \{ \tilde{f}_n \} \) and \( \{ \tilde{g}_n \} \) be sequences of canonical positive fuzzy numbers and let \( \{ X_n \} \) and \( \{ Y_n \} \) be sequences of fuzzy random variables. If \( X_n = o_p \left( \tilde{f}_n \right) \) and \( Y_n = O_p \left( \tilde{g}_n \right) \) then \( X_n Y_n = o_p \left( \tilde{f}_n \tilde{g}_n \right) . \)

**Proof.** By stipulation \( X_n = o_p \left( \tilde{f}_n \right) \). Then for \( \varepsilon > 0 , \exists \delta > 0 \) such that

\[ P \left( \left( U_{X_n,\alpha} \lor U_{X_n,\alpha}^{\prime} \right) \right) > \varepsilon N_p \left( \alpha \right) < \delta . \]

Since \( Y_n = O_p \left( \tilde{g}_n \right) \)

\[ P \left( \left( V_{Y_n,\alpha} \lor V_{Y_n,\alpha}^{\prime} \right) \right) \geq MN_p \left( \alpha \right) \leq \forall n \geq N_0 . \]

Then

\[ P \left( \left( U_{X_n,\alpha} V_{Y_n,\alpha} \lor U_{X_n,\alpha}^{\prime} V_{Y_n,\alpha}^{\prime} \right) \right) \geq \varepsilon MN_p \left( \alpha \right) N_p \left( \alpha \right) \leq \delta \]

i.e.

\[ P \left( \left( U_{X_n,\alpha} V_{Y_n,\alpha} \lor U_{X_n,\alpha}^{\prime} V_{Y_n,\alpha}^{\prime} \right) \right) \geq \varepsilon \rightarrow 0 \]

as \( n \rightarrow \infty \), which completes the proof.

**Theorem 3.4.** Let \( \{ X_n \} \) be a sequence of fuzzy random variables and \( \{ \tilde{a}_n \} \) be a sequence canonical positive fuzzy numbers such that \( E \tilde{a}_n^2 = O \left( \tilde{a}_n^2 \right) . \) Then \( X_n = O_p \left( \tilde{a}_n \right) . \)
Proof. By stipulation \( \text{E}x_n = O(\tilde{a}_n) \) i.e. there exists \( M_2 > 0 \) such that \( E\tilde{x}_n^2 \leq M_2^2 \tilde{a}_n^2 \) for all \( n \geq N_0 \). By Chebychev’s inequality

\[
P\left(\left|U_{X,\alpha}^{*} - EU_{X,\alpha}^{*}\right| \geq \frac{E\tilde{x}_n^2}{M_2^2 \tilde{a}_n^2}\right) \leq M_2 \tilde{a}_n
\]

Taking \( M_2 \geq M_1/e^{\xi^2} \). We have

\[
P\left(\left|U_{X,\alpha}^{*} - EU_{X,\alpha}^{*}\right| \geq \frac{E\tilde{x}_n^2}{M_1^2 \tilde{a}_n^2}\right) \leq E\tilde{x}_n^2 / M_1^2 \tilde{a}_n^2 \leq \forall n \geq N_0
\]

\[\therefore X_n = O(p(\tilde{a}_n)).\]

References

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HELLY'S THEOREM ON FUZZY VALUED FUNCTIONS

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Helly’s theorem on fuzzy valued functions

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ABSTRACT. Some recently developed notions of fuzzy valued functions, fuzzy distribution functions proposed by H. C.Wu [9] are employed to establish Helly’s theorem on fuzzy valued functions. To establish Helly’s theorem, and for a convenient discussion of fuzzy random variables, a more strong sense of measurability for fuzzy valued functions is introduced.

1. INTRODUCTION

The notion of fuzzy random variables, with necessary theoretical framework was introduced by Kwaikenaak [5] and Puri and Ralescu [8]. The notion of normality of fuzzy random variables was also discussed by M.L.Puri et.al [7]. To make fuzzy random variables amenable to statistical analysis for imprecise data where dimness of perception is prevalent, H.C.Wu[9-12] has contributed a variety of research papers on fuzzy random variables which expose various rudiments of fuzzy random variables such as, weak and strong law of large numbers, weak and strong convergence with probability, fuzzy distribution functions, fuzzy probability density functions, fuzzy expectation, fuzzy variance and fuzzy valued functions governed by strong measurability conditions. In [9] H.C.Wu has introduced the notion of fuzzy distribution functions for fuzzy random variables.

Since the α-level set of a closed fuzzy number is a compact interval, in order to make the end points of the α-level set of a fuzzy random variables to be the usual random variables H.C.Wu [9-12] has introduced the concept of strong measurability for fuzzy random variables. In this paper, Helly’s theorem, and Helly Bray theorem for fuzzy valued functions and fuzzy probability distribution function are introduced. These results are based on the concept of strong measurability for fuzzy random variables.

In section 2, we introduce some preliminaries related to fuzzy numbers, such as strong and weak convergence of sequence of fuzzy numbers to a fuzzy numbers. Section 3 is devoted to fuzzy random variables and its fuzzy distribution functions. The theoretical settings of fuzzy random variables are derived from H.C. Wu [9-12].

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In section 3 we introduce the definition of a fuzzy valued function and other notions related to the measurability of a fuzzy valued function. The notion of fuzzy distribution function is also introduced in this section. This section conclude with the definitions of strong and weak convergence in distribution of fuzzy random variables. In section 4 we present the Helly's theorem and Helly-Bray theorem for fuzzy valued functions and fuzzy distribution functions. Throughout this paper we denote the indicator function of the set $A$ by $1_A$.

2. FUZZY NUMBERS

In this section we provide some limit properties of fuzzy numbers by applying the Hausdorff metric. We also introduce the notions of fuzzy real numbers and fuzzy random variables.

Definition: 2.1.
(i) Let $f$ be a real valued function on a topological space. If $\{x; f(x) \geq \alpha\}$ is closed for each $\alpha$, then $f$ is said to be upper semi continuous.
(ii) A real valued function $f$ is said to be upper semi continuous at $y$ if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that $|x - y| < \delta$ implies $f(x) < f(y) + \varepsilon$. 
(iii) $f(x)$ is said to be lower semicontinuous if $-f(x)$ is upper semicontinuous.

Definition: 2.2. Let $F : X \rightarrow \mathbb{R}^n$ be a set valued mapping. $F$ is said to be continuous at $x_0 \in X$ if $F$ is both upper semi continuous and lower semi continuous at $x_0$.

Theorem: 2.1. (Bazarr and Shetty [2]). Let $S$ be a compact set in $\mathbb{R}^n$. If $f$ is upper semicontinuous on $S$ then $f$ assumes maximum over $S$ and if $f$ is lower semi-continuous on $S$ then $f$ assumes minimum over $S$.

Definition: 2.3. Let $X$ be a universal set. Then a fuzzy subset $A$ of $X$ is defined by its membership function $\mu_A : X \rightarrow [0, 1]$. We denote $A_\alpha = \{x; \mu_A(x) \geq \alpha\}$ as the $\alpha$-level set of $A$ where $A_0$ is the closure of the set $\{x; \mu_A(x) \neq 0\}$.

Definition: 2.4.
(i) $A$ is called a normal fuzzy set if their exist $x$ such that $\mu_A(x) = 1$.
(ii) $A$ is called a convex fuzzy set if $\mu_A(\lambda x + (1 - \lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for $\lambda \in [0, 1]$.

Theorem: 2.2. [13]. $A$ is a convex fuzzy set if and only if $\{x; \mu_A(x) \geq \alpha\}$ is a convex set for all $\alpha$.

Definition: 2.5. Let $X = \mathbb{R}$
(i) $\tilde{m}$ is called a fuzzy number if $\tilde{m}$ is a normal convex fuzzy set and the $\alpha$-level set $\tilde{m}_\alpha$ is bounded $\forall \alpha \neq 0$. 
(ii) \( \tilde{m} \) is called a closed fuzzy number if \( \tilde{m} \) is a fuzzy number and its membership function \( \mu_{\tilde{m}} \) is upper semicontinuous.

(iii) \( \tilde{m} \) is called a bounded fuzzy number if \( \tilde{m} \) is a fuzzy number and its membership function \( \mu_{\tilde{m}} \) has compact support.

**Theorem 2.3.** [10]. If \( \tilde{m} \) is a closed fuzzy number then the \( \alpha \)-level set of \( \tilde{m} \) is a closed interval, which is denoted by

\[
\tilde{m} = [\tilde{m}_0^L, \tilde{m}_0^U]
\]

**Definition 2.6.** \( \tilde{m} \) is called a canonical fuzzy number, if it is a closed and bounded fuzzy number and its membership function is strictly increasing on the interval \([\tilde{m}_0^L, \tilde{m}_1^L]\) and strictly decreasing on the interval \([\tilde{m}_0^U, \tilde{m}_0^U]\).

**Theorem 2.4.** [10]. Suppose that \( \tilde{a} \) is a canonical fuzzy number. Let \( g(\alpha) = \tilde{a}_0^L \) and \( h(\alpha) = \tilde{a}_0^U \). Then \( g(\alpha) \) and \( h(\alpha) \) are continuous functions of \( \alpha \).

**Theorem 2.5.** (Zadeh [14] Resolution Identity).

(i) Let \( A \) be a fuzzy set with membership function \( \mu_A \) and \( A_\alpha = \{ x ; \mu_A(x) \geq \alpha \} \). Then

\[
\mu_A(x) = \sup_{\alpha \in [0,1]} \alpha 1_{A_\alpha}(x)
\]

(ii) (Negoita and Relescu [6]). Let \( A \) be a set and \( \{ A_\alpha : 0 \leq \alpha \leq 1 \} \) be a family of subsets of \( A \) such that the following conditions are satisfied.

\begin{enumerate}
\item \( A_0 = A \)
\item \( A_\alpha \subseteq A_\beta \) for \( \alpha > \beta \)
\item \( A_\alpha = \bigcap_{n=1}^\infty A_{\alpha_n} \) for \( \alpha_n \uparrow \alpha \)
\end{enumerate}

Then the function \( \mu : A \to [0,1] \) defined by \( \mu(x) = \sup_{\alpha \in [0,1]} \alpha 1_{A_\alpha}(x) \) has the property that

\[
A_\alpha = \{ x ; \mu(x) \geq \alpha \} \ \forall \alpha \in [0,1]
\]

With the help of \( \alpha \)-level sets of a fuzzy set \( A \) we can construct closed fuzzy number. Let \( g \) and \( h \) be two functions from \([0,1]\) into \( R \).

Let \( \{ A_\alpha = [g(\alpha), h(\alpha)] : 0 \leq \alpha \leq 1 \} \) be a family of closed intervals. Then we can induce a fuzzy set \( A \) with the membership function

\[
\mu_A(x) = \sup_{0 \leq \alpha \leq 1} \alpha 1_{A_\alpha}(r)
\]

via the form of resolution identity.
Theorem: 2.6. [11]. Let \( \{ A_\alpha : 0 \leq \alpha \leq 1 \} \) be a set of decreasing closed intervals, i.e., \( A_\alpha \subseteq A_\beta \) for \( \alpha > \beta \). Then \( f(\alpha) = \alpha 1_{A_\alpha} (x) \) is upper semicontinuous for any fixed \( x \).

Theorem: 2.7. [11]. Let \( \tilde{a} \) and \( \tilde{b} \) be two canonical fuzzy numbers. Then \( \tilde{a} \oplus \tilde{b} \), \( \tilde{a} \odot \tilde{b} \) and \( \tilde{a} \otimes \tilde{b} \) are also canonical fuzzy numbers. Furthermore we have for \( \alpha \in [0, 1] \):

\[
\left( \tilde{a} \oplus \tilde{b} \right) = \left[ a^L_\alpha + b^L_\alpha, a^U_\alpha + b^U_\alpha \right]
\]

\[
\left( \tilde{a} \odot \tilde{b} \right) = \left[ a^L_\alpha - b^L_\alpha, a^U_\alpha - b^U_\alpha \right]
\]

\[
\left( \tilde{a} \otimes \tilde{b} \right) = \left[ \min \left\{ a^L_\alpha b^L_\alpha, a^L_\alpha b^U_\alpha, a^U_\alpha b^L_\alpha, a^U_\alpha b^U_\alpha \right\}, \max \left\{ a^L_\alpha b^L_\alpha, a^L_\alpha b^U_\alpha, a^U_\alpha b^L_\alpha, a^U_\alpha b^U_\alpha \right\} \right]
\]

Let \( A \subseteq R^n \) and \( B \subseteq R^n \). The Hausdorff metric is defined by

\[
d_H (A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\}
\]

Let \( \mathcal{F} \) be the set of all fuzzy numbers and \( \mathcal{F}_b \) be the set of all bounded fuzzy numbers. Puri and Ralescu [8] have defined the metric \( d_3 \) in \( \mathcal{F} \) as

\[
d_3 (\tilde{a}, \tilde{b}) = \sup_{0 < \alpha \leq 1} d_H (\tilde{a}_\alpha, \tilde{b}_\alpha) \quad \text{and} \quad \quad d_3 (\tilde{a}, \tilde{b}) \quad \text{in} \quad \mathcal{F}_b \quad \text{as} \quad \quad d_3 (\tilde{a}, \tilde{b}) = \sup_{0 < \alpha \leq 1} d_H (\tilde{a}_\alpha, \tilde{b}_\alpha),
\]

Lemma. 2.1. [9]. Let \( \tilde{a} \) and \( \tilde{b} \) be two closed fuzzy numbers. Then we have

\[
d_H (\tilde{a}_\alpha, \tilde{b}_\alpha) = \max \left\{ |a^L_\alpha - b^L_\alpha|, |a^U_\alpha - b^U_\alpha| \right\}
\]

Definition: 2.7. [9]

(i) Let \( \{ \tilde{a}_n \} \) be a sequence of closed (canonical) fuzzy numbers. \( \{ \tilde{a}_n \} \) is said to converge strongly if there is a closed (canonical) fuzzy number \( \tilde{a} \) with the following property \( \forall \varepsilon > 0, \exists N > 0 \) such that for \( n > N \), we have

\[
d_3 (\tilde{a}_n, \tilde{a}) < \varepsilon \quad (d_3 (\tilde{a}_n, \tilde{a}) < \varepsilon).
\]

We say that the sequence \( \{ \tilde{a}_n \} \) converges to \( \tilde{a} \) strongly and it is denoted as \( \lim_{n \to \infty} \tilde{a}_n = \tilde{a} \).

(ii) Let \( \{ \tilde{a}_n \} \) be a sequence of closed (canonical) fuzzy numbers.
\( \{ \tilde{a}_n \} \) is said to converge weakly if there is a closed (canonical) fuzzy number \( \tilde{a} \) with the following property: \( \lim_{n \to \infty} (\tilde{a}_n)_{\alpha}^L = \tilde{a}_{\alpha}^L \) and \( \lim_{n \to \infty} (\tilde{a}_n)_{\alpha}^U = \tilde{a}_{\alpha}^U \) for all \( \alpha \in [0,1] \). We say that the sequence \( \{ \tilde{a}_n \} \) converges to \( \tilde{a} \) weakly and it is denoted as \( \lim_{n \to \infty} \tilde{a}_n = \tilde{a} \). We note that \( \lim_{n \to \infty} \tilde{a}_n = \tilde{a} \) is equivalent to \( \lim_{n \to \infty} d_{\tilde{a}_n, \tilde{a}} = 0 \).

3. FUZZY RANDOM VARIABLE AND ITS DISTRIBUTION FUNCTION

In this section we provide the theoretical framework of fuzzy random variables and its distribution functions proposed by H.C. Wu [9]. Given a real number \( x \) one can induce a fuzzy number \( \tilde{x} \) with membership function \( \mu_\tilde{x}(x) \) such that \( \mu_\tilde{x}(x) = 1 \) and \( \mu_\tilde{x}(x) < 1 \) for \( x \neq \tilde{x} \). We call \( \tilde{x} \) as a fuzzy real number induced by the real number \( x \). Let \( \mathfrak{S}_R \) be a set of all fuzzy real numbers induced by the real number system \( R \). We define the relation \( \sim \) on \( \mathfrak{S}_R \) as \( \tilde{x}_1 \sim \tilde{x}_2 \) if and only if \( \tilde{x}_1 = \tilde{x}_2 \) are induced by the same real number \( x \). Then \( \sim \) is an equivalence relation. This equivalence relation induces the equivalence classes \( [\tilde{x}] = \{ \tilde{x} \mid \tilde{x}_1 \sim \tilde{x}_2 \} \). The quotient set \( \mathfrak{S}_{R/N} \) is the set of all equivalence classes. Then the cardinality of \( \mathfrak{S}_{R/N} \) is equal to the cardinality of the real number system \( R \), since the map \( R \to \mathfrak{S}_{R/N} \) specified by \( x \to [\tilde{x}] \) is a bijection. We call \( \mathfrak{S}_{R/N} \) as the fuzzy real number system. For practical purposes we take only one element \( \tilde{x} \) from each equivalence class \( [\tilde{x}] \) to form the fuzzy real number system \( (\mathfrak{S}_{R/N})_R \),

i.e. \( (\mathfrak{S}_{R/N})_R = \{ \tilde{x} \mid \tilde{x} \in [\tilde{x}] \} \), \( \tilde{x} \) is the only element from \( [\tilde{x}] \).

If the fuzzy real number system \( (\mathfrak{S}_{R/N})_R \) consists of canonical fuzzy real numbers, then we call \( (\mathfrak{S}_{R/N})_R \) as the canonical fuzzy real number system. Let \( (X, \mathcal{F}) \) be a measurable space and \( (R, \mathcal{B}) \) be a Borel measurable space. Let \( f : X \to \mathcal{P}(R) \) (power set of \( R \)) be a set valued function. According to Aumann [1] the set valued function \( f \) is called measurable if and only if \( \{(x, y) \mid y \in f(x)\} \) is \( \mathcal{F} \times \mathcal{B} \) measurable. The function \( \tilde{f}(x) \) is called a fuzzy valued function if \( f : x \to \mathfrak{S} \) (the set of all fuzzy numbers).

**Definition:** 3.1. [9]. Let \( (\mathfrak{S}_{R/N})_R \) be a canonical fuzzy real number system and: \( \tilde{X} : \Omega \to (\mathfrak{S}_{R/N})_R \) be a closed-fuzzy valued function. \( \tilde{X} \) is called a fuzzy random variable if \( \tilde{X} \) is measurable (or equivalently strongly measurable).

**Theorem 3.1.** [9]. Let \( (\mathfrak{S}_{R/N})_R \) be a canonical fuzzy real number system and: \( \tilde{X} : \Omega \to (\mathfrak{S}_{R/N})_R \) be a closed-fuzzy valued function. \( \tilde{X} \) is a fuzzy random variable if and only if \( \tilde{X}_{\alpha}^L \) and \( \tilde{X}_{\alpha}^U \) are random variables for all \( \alpha \in [0,1] \).
If $\hat{x}$ is a canonical fuzzy real number, then $\hat{x}_\alpha = \hat{x}^L_\alpha$. Let $\tilde{X}$ be a fuzzy random variable. By theorem 3.1, $\tilde{X}^U_\alpha$ and $\tilde{X}^L_\alpha$ are random variables for all $\alpha$. Let $\tilde{X}^L_\alpha$ and $\tilde{X}^U_\alpha$ have the same distribution function $F(x)$ for all $\alpha \in [0, 1]$. If $\tilde{x}$ is any fuzzy observation of a fuzzy random variable $\tilde{X}$, then the $\alpha$-level set $\tilde{x}_\alpha$ is $\tilde{x}_\alpha = [\tilde{x}^L_\alpha, \tilde{x}^U_\alpha]$. By a fuzzy observation we mean an imprecise data. We can see that $\tilde{x}^L_\alpha$ and $\tilde{x}^U_\alpha$ are the observations of $\tilde{X}^L_\alpha$ and $\tilde{X}^U_\alpha$ respectively. From Theorem 2.4, $\tilde{x}^L_\alpha(w) = \tilde{x}^L_\alpha$ and $\tilde{x}^U_\alpha(w) = \tilde{x}^U_\alpha$ are continuous with respect to $\alpha$ for fixed $w$. Thus $[\tilde{x}^L_\alpha, \tilde{x}^U_\alpha]$ is continuously shrinking with respect to $\alpha$. The disjoint union of $[\tilde{x}^L_\alpha, \tilde{x}^U_\alpha]$ and $[\tilde{x}^L_\alpha, \tilde{x}^U_\alpha]$ exists. Therefore for any real number $x \in [\tilde{x}^L_\alpha, \tilde{x}^U_\alpha]$, we have $x = \tilde{x}^L_\beta$ or $x = \tilde{x}^U_\beta$ for some $\beta \geq \alpha$. This confirms the existence of a suitable $\alpha$-level set for which $x$ coincides with the lower end of that $\alpha$-level set or with the upper end of that $\alpha$-level set. Hence for any $x \in [\tilde{x}^L_\alpha, \tilde{x}^U_\alpha]$, we can associate an $F(\tilde{x}^L_{\beta})$ or $F(\tilde{x}^U_{\beta})$ with $x$.

If $\tilde{f}$ is a fuzzy valued function then $\tilde{f}_\alpha$ is a set valued function for all $\alpha \in [0, 1]$. $\tilde{f}$ is called (fuzzy-valued) measurable if and only if $\tilde{f}_\alpha$ is (set-valued) measurable for all $\alpha \in [0, 1]$. To make fuzzy random variables more governable mathematically a more strong sense of measurability for fuzzy valued function is required. $\tilde{f}(x)$ is called a closed-fuzzy-valued function if $\tilde{f} : X \to \mathcal{S}_\alpha (\text{the set of all closed fuzzy numbers})$. Let $\tilde{f}(x)$ be a closed fuzzy-valued function defined on $X$ from H.C. Wu [11] the following two statements are equivalent.

(i) $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ are (real-valued) measurable for all $\alpha \in [0, 1]$. (ii) $\tilde{f}(x)$ is (fuzzy-valued) measurable and one of $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(x)$ is (real valued) measurable for all $\alpha \in [0, 1]$. Then $\tilde{f}(x)$ is called strongly measurable if one of the above two conditions is satisfied. It is easy to see that the strong measurability implies measurability.

Let $(X, \mathcal{F}, \mu)$ be a measure space and $(R, \mathcal{B})$ be a Borel measurable space. Let $f : X \to \mathcal{P}(R)$ be a set valued function. For $K \subseteq R$, the inverse image of $f$ is defined by

$$f^{-1}(K) = \{x \in X, f(x) \cap K \neq \emptyset\}.$$ 

Let $(X, \mathcal{F}, \mu)$ be a complete $\sigma$-finite measure space. From Hiai and Umegaki [3] the following two statements are equivalent.

a). For each Borel set $K \subseteq R$, $f^{-1}(K)$ is measurable (i.e. $f^{-1}(K) \in \mathcal{F}$)

b). $\{(x, y) : y \in f(x)\}$ is $\mathcal{F} \times \mathcal{B}$ measurable.

If we construct an interval
\[ A_\alpha = \]
\[
= \left[ \min \left\{ \inf_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^L), \inf_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^U) \right\}, \max \left\{ \sup_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^L), \sup_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^U) \right\} \right]
\]

Then this interval will contain all of the distributions associated with each of \( x \in [\tilde{x}_\alpha^L, \tilde{x}_\alpha^U] \). We denote by \( \tilde{F}(\tilde{x}) \) the fuzzy distribution function of the fuzzy random variable \( \tilde{x} \). Then we define the membership function of \( \tilde{F}(\tilde{x}) \) for any fixed \( \tilde{x} \) by

\[
\mu_{\tilde{F}(\tilde{x})}^{(r)}(r) = \sup_{\alpha \leq \beta \leq 1} \mu_{A_\alpha}(r)
\]

we also say that the fuzzy distribution function \( \tilde{F}(\tilde{x}) \) is induced by the distribution function \( F(x) \). Since \( F(x) \) is continuous from theorem 2.4 and 2.1, we can write \( A_\alpha \) as

\[ A_\alpha = \]
\[
= \left[ \min \left\{ \min_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^L), \min_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^U) \right\}, \max \left\{ \max_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^L), \max_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^U) \right\} \right]
\]

For typographical reasons we employ the following notations.

\[
F^{\min}(\tilde{x}_\beta^{(•)}) = \min \left\{ \min_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^L), \min_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^U) \right\}
\]

\[
F^{\max}(\tilde{x}_\beta^{(•)}) = \max \left\{ \max_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^L), \max_{\alpha \leq \beta \leq 1} F(\tilde{x}_\beta^U) \right\}
\]

Let \( \tilde{X} \) and \( \tilde{Y} \) be two fuzzy random variables. We say that \( \tilde{X} \) and \( \tilde{Y} \) are independent if and only if each random variable in the set

\[
\left\{ \tilde{X}_\alpha^L, \tilde{X}_\alpha^U : 0 \leq \alpha \leq 1 \right\}
\]

is independent of each random variable in the set

\[
\left\{ \tilde{Y}_\alpha^L, \tilde{Y}_\alpha^U : 0 \leq \alpha \leq 1 \right\}.
\]

We say that \( \tilde{X} \) and \( \tilde{Y} \) are identically distributed if and only if \( \tilde{X}^L_\alpha \) and \( \tilde{Y}^L_\alpha \) are identically distributed and \( \tilde{X}^U_\alpha \) and \( \tilde{Y}^U_\alpha \) are identically distributed for all \( \alpha \in [0, 1] \).

**Definition:** 3.2. Let \( \tilde{X} \) and \( \{\tilde{X}_n\} \) be fuzzy random variables defined on the same probability space \((\Omega, \mathcal{A}, P)\).
(i) We say that \( \{ \tilde{X}_n \} \) converges in distribution to \( \tilde{X} \) level-wise if \( (\tilde{X}_n)_{\alpha}^L \) and \( (\tilde{X}_n)_{\alpha}^U \) converge in distribution to \( \tilde{X}_\alpha^L \) and \( \tilde{X}_\alpha^U \) respectively for all \( \alpha \in (0,1] \).

Let \( \tilde{F}_n (\tilde{x}) \) and \( \tilde{F} (\tilde{x}) \) be the respective fuzzy distribution functions of \( \tilde{X}_n \) and \( \tilde{X} \).

(ii) We say that \( \{ \tilde{X}_n \} \) converge in distribution to \( \tilde{X} \) strongly if

\[
\lim_{n \to \infty} \tilde{F}_n (\tilde{X}) = s \tilde{F} (\tilde{x})
\]

i.e.

\[
\lim_{n \to \infty} \sup_{0 \leq \beta \leq 1} \left\{ \max \left\{ \left\| (\tilde{F}_n)_{\alpha}^L (\tilde{x}) - \tilde{F}_{\alpha}^L (\tilde{x}) \right\|, \left\| (\tilde{F}_n)_{\alpha}^U (\tilde{x}) - \tilde{F}_{\alpha}^U (\tilde{x}) \right\| \right\} = 0
\]

(iii) We say that \( \{ \tilde{X}_n \} \) converges in distribution to \( \{ \tilde{X} \} \) weakly if

\[
\lim_{n \to \infty} \tilde{F}_n (\tilde{x}) = w \tilde{F} (\tilde{x})
\]

i.e.

\[
\lim_{n \to \infty} (\tilde{F}_n)_{\alpha}^L (\tilde{X}) = \tilde{F}_{\alpha}^L (\tilde{x}) \quad \text{and} \quad \lim_{n \to \infty} (\tilde{F}_n)_{\alpha}^U (\tilde{X}) = \tilde{F}_{\alpha}^U (\tilde{x}) \quad \text{for all} \quad \alpha \in [0,1].
\]

4. HELLY'S THEOREMS

In this section based on the theoretical framework of section 3 and section 4 we have established the Helly's theorem and Helly Bray theorem for fuzzy valued functions and fuzzy distribution functions.

**Theorem 4.1.** (Helly's Theorem). If (i) non-decreasing sequence of fuzzy probability distribution function \( \{ F_n (x) \} \) converges to the fuzzy probability distribution function \( F(x) \) (ii) the fuzzy valued function \( g(x) \) is everywhere continuous and (iii) \( a, b \) are continuity points of \( F(x) \) then

\[
\lim_{n \to \infty} \int_a^b g_n^L (x) dF_n^{\text{min}} (x) \land g_n^U (x) dF_n^{\text{max}} (x) \land g_n^U (x) dF_n^{\text{max}} (x) = \\
= \int_a^b g_n^L (x) dF^{\text{min}} (x) \land g_n^U (x) dF^{\text{max}} (x) \land g_n^U (x) dF^{\text{max}} (x)
\]

(3.1).

**Proof.** By stipulation \( F_n (x) \) and \( F_n (x) \) is non-decreasing.

\( F (x) \) and \( F (x) \) is also non-decreasing.
For all \( n \geq 1 \), 
\[
F_n \left( x^L_{\beta} + h \right) - F_n \left( x^L_{\beta} \right) \geq 0; \text{ if } h > 0.
\]
\[
F_n \left( x^U_{\beta} + h \right) - F_n \left( x^U_{\beta} \right) \geq 0; \text{ if } h > 0.
\]
Letting \( n \to \infty \) we have
\[
F \left( x^L_{\beta} + h \right) - F \left( x^L_{\beta} \right) \geq 0; \text{ if } h > 0
\]
\[
F \left( x^U_{\beta} + h \right) - F \left( x^U_{\beta} \right) \geq 0; \text{ if } h > 0
\]
we take \( a = x_0 < x_1 < x_2 < \ldots < x_k = b \) where \( x_0, x_1, \ldots \) are the continuity points of the fuzzy probability distribution function \( F \), then for \( \alpha \leq \beta \leq 1 \)
\[
\int_a^b g^L \alpha \left( x \right) \, dF^{\min} \left( x \right) \land g^U \beta \left( x \right) \, dF^{\max} \left( x \right) =
\]
\[
= \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} g^L \alpha \left( x \right) \, dF^{\min} \left( x \right) \land g^U \beta \left( x \right) \, dF^{\max} \left( x \right) =
\]
\[
= \sum_{i=0}^{k-1} \left( \int_{x_i}^{x_{i+1}} \left( g^L \alpha \left( x \right) - g^L \alpha \left( x_i \right) \right) \, dF^{\min} \left( x \right) + \int_{x_i}^{x_{i+1}} g^U \beta \left( x \right) \, dF^{\max} \left( x \right) \right) \land
\]
\[
= \sum_{i=0}^{k-1} \left( \int_{x_i}^{x_{i+1}} \left( g^L \alpha \left( x \right) - g^L \alpha \left( x_i \right) \right) \, dF^{\min} \left( x \right) + \sum_{i=0}^{k-1} g^U \beta \left( x_i \right) \int_{x_i}^{x_{i+1}} dF^{\min} \left( x \right) \right) \land
\]
\[
= \sum_{i=0}^{k-1} \left( \int_{x_i}^{x_{i+1}} \left( g^L \alpha \left( x \right) - g^L \alpha \left( x_i \right) \right) \, dF^{\min} \left( x \right) + \sum_{i=0}^{k-1} g^U \beta \left( x_i \right) \int_{x_i}^{x_{i+1}} dF^{\max} \left( x \right) \right) =
\]
\[
= \sum_{i=0}^{k-1} \left( g^L \alpha \left( x_i \right) - g^L \alpha \left( x_i \right) \right) dF^{\min} \left( x \right) + \sum_{i=0}^{k-1} g^L \beta \left( x_i \right) \left( F^{\min} \left( x+1 \right) - F^{\min} \left( x \right) \right)
\]
\[ \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (g_{\alpha_i}^L(x) - g_{\alpha_i}^U(x_i)) \, dF_{\text{max}} \left( x_{\beta_i}^\bullet \right) + \sum_{i=0}^{k-1} g_{\alpha_i}^U(x_i) (F_{\text{max}}(x_{i+1}) - F_{\text{max}}(x_i)) \]

By stipulation the fuzzy valued function \( g \) is continuous everywhere, i.e., for each \( \alpha \in [0, 1] \)

\[ |g_{\alpha_i}^L(x) - g_{\alpha_i}^U(x_i)| < \frac{\varepsilon}{3} \frac{1}{F(b) - F(a)} \quad \text{for} \quad x_i \leq x \leq x_{i+1}. \]

We take \( |\theta_1| \leq 1 \). Then

\[ \int_a^b g_{\alpha_i}^L(x) \, dF_{\text{min}} \left( x_{\beta_i}^\bullet \right) \wedge g_{\alpha_i}^U(x) \, dF_{\text{max}} \left( x_{\beta_i}^\bullet \right) = \]

\[ = \frac{\theta_1 \varepsilon}{3} + \sum_{i=0}^{k-1} g_{\alpha_i}^L(x_i) (F_{\text{min}}(x_{i+1}) - F_{\text{min}}(x_i)) \wedge \]

\[ \wedge \sum_{i=0}^{k-1} g_{\alpha_i}^U(x_i) (F_{\text{max}}(x_{i+1}) - F_{\text{max}}(x_i)) \quad (3.2). \]

Similarly

\[ \int_a^b g_{\alpha_i}^L(x) \, dF_{\text{min}} \left( x_{\beta_i}^\bullet \right) \wedge g_{\alpha_i}^U(x) \, dF_{\text{max}} \left( x_{\beta_i}^\bullet \right) \leq \]

\[ \leq \frac{\theta_2 \varepsilon}{3} + \sum_{i=0}^{k-1} g_{\alpha_i}^L(x_i) (F_{\text{min}}(x_{i+1}) - F_{\text{min}}(x_i)) \wedge \]

\[ \wedge \sum_{i=0}^{k-1} g_{\alpha_i}^U(x_i) (F_{\text{max}}(x_{i+1}) - F_{\text{max}}(x_i)) \quad (3.3). \]

Since \( F_{\text{min}}(x_{\beta_i}^\bullet) \to F_{\text{min}}(x_{\beta_i}^\bullet) \) at continuity points of \( F \) and

\[ F_{\text{max}}(x_{\beta_i}^\bullet) \to F_{\text{max}}(x_{\beta_i}^\bullet) \) at continuity points of \( F \)

\[ F_{\text{min}}(x_{i+1}) - F_{\text{min}}(x_{i+1}) < \frac{\varepsilon}{6} \sum_i g_{\alpha_i}^L(x_i) \quad \text{for all} \quad i \]
and large values of $n$. Similarly

$$F_n^{\max}(x_{i+1}) - F_n^{\max}(x_i) < \frac{\varepsilon}{\sum \alpha_i g_i(x_i)}$$

for all $i$.

and large values of $n$.

Hence letting $n \to \infty$ the absolute difference of (3.2) and (3.3) is

$$\int_a^b g^L_\alpha(x) dF_n^{\min}(x_{\beta}^{(*)}) - dF_n^{\min}(x_{\beta}^{(*)}) \wedge g^U_\alpha(x) dF_n^{\max}(x_{\beta}^{(*)}) - dF_n^{\max}(x_{\beta}^{(*)})$$

$$\leq \frac{\varepsilon}{3} [\theta_2 - \theta_1] + \sum_{i=0}^{k-1} g^L_{\alpha_i}(x_i) |F_n^{\min}(x_{i+1}) - F_n^{\min}(x_i) - F_n^{\max}(x_{i+1}) + F_n^{\max}(x_i)| \wedge$$

$$\sum_{i=0}^{k-1} g^L_{\alpha_i}(x_i)|F_n^{\max}(x_{i+1}) - F_n^{\max}(x_i) - F_n^{\max}(x_{i+1}) + F_n^{\max}(x_i)|$$

Then

$$\int_a^b g^L_\alpha(x) dF_n^{\min}(x_{\beta}^{(*)}) - dF_n^{\min}(x_{\beta}^{(*)}) \wedge g^L_\alpha(x) dF_n^{\max}(x_{\beta}^{(*)}) - dF_n^{\max}(x_{\beta}^{(*)})$$

This show that

$$\lim_{n \to \infty} \int_a^b g^L_\alpha(x) dF_n^{\min}(x_{\beta}^{(*)}) \wedge g^U_\alpha(x) dF_n^{\max}(x_{\beta}^{(*)}) =$$

$$= \int_a^b g^L_\alpha(x) dF_n^{\min}(x_{\beta}^{(*)}) \wedge g^U_\alpha(x) dF_n^{\max}(x_{\beta}^{(*)})$$

which completes the proof.

**Theorem 4.2.** (Helly Bray Theorem).

If (i) the fuzzy valued function $g(x)$ is continuous.

(ii) the fuzzy probability distribution function $F_n(x) \to F(x)$ in each continuity point of $F(x)$ and

(iii) for any $\varepsilon > 0$ we can find $A$ such that
\[
\int_{-\infty}^{A} \left| g_{\alpha}^{L}(x) \right| dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge \left| g_{\alpha}^{R}(x) \right| dF_{n}^{\max} \left( x_{\beta}^{(*)} \right) + \\
+ \int_{-\infty}^{\infty} \left| g_{\alpha}^{L}(x) \right| dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge \left| g_{\alpha}^{R}(x) \right| dF_{n}^{\max} \left( x_{\beta}^{(*)} \right) < \epsilon 
\]
for all \( n = 1, 2, 3, \ldots \) then
\[
\lim_{n \to \infty} \int_{-\infty}^{x_{\beta}^{(*)}} g_{\alpha}^{L}(x) dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge g_{\alpha}^{R}(x) dF_{n}^{\max} \left( x_{\beta}^{(*)} \right) = \\
= \int_{-\infty}^{x_{\beta}^{(*)}} g_{\alpha}^{L}(x) dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge g_{\alpha}^{R}(x) dF_{n}^{\max} \left( x_{\beta}^{(*)} \right)
\]

**Proof.** The theorem is proved for \( F_{n}(x) \to F(x) \) for all \( x \).

Letting \( n \to \infty \) and from condition (3.4) we have
\[
\int_{-\infty}^{A} \left| g_{\alpha}^{L}(x) \right| dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge \left| g_{\alpha}^{R}(x) \right| dF_{n}^{\max} \left( x_{\beta}^{(*)} \right) + \\
+ \int_{A}^{\infty} \left| g_{\alpha}^{L}(x) \right| dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge \left| g_{\alpha}^{R}(x) \right| dF_{n}^{\max} \left( x_{\beta}^{(*)} \right) < \epsilon 
\]

By theorem (3.1)
\[
\lim_{n \to \infty} \int_{a}^{b} g_{\alpha}^{L}(x) dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge g_{\alpha}^{R}(x) dF_{n}^{\max} \left( x_{\beta}^{(*)} \right) = \\
= \int_{a}^{b} g_{\alpha}^{L}(x) dF_{n}^{\min} \left( x_{\beta}^{(*)} \right) \wedge g_{\alpha}^{R}(x) dF_{n}^{\max} \left( x_{\beta}^{(*)} \right)
\]

By the expression (3.4) it follows that for continuity points \( b > a \), the fuzzy valued function \( g(x) \) is integrable over \( (-\infty, \infty) \), with respect to the fuzzy probability distribution function \( F(x) \). If in (3.6) we take \( a = -A \) and \( b = A \) we have
\[
\lim_{n \to \infty} \int_{-A}^{A} g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) = \\
= \int_{-A}^{A} g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \\
- \int_{-\infty}^{\infty} g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) - \\
- \int_{-\infty}^{\infty} g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) = \\
= \left( \int_{-\infty}^{\infty} + \int_{-A}^{A} + \int_{A}^{\infty} \right) \left( g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) - \\
- \left( \int_{-\infty}^{\infty} + \int_{-A}^{A} + \int_{A}^{\infty} \right) \left( g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) 
\]

Then

\[
\lim_{n \to \infty} \left[ \int_{-\infty}^{\infty} \left( g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) \right] \\
- \left[ \int_{-\infty}^{\infty} \left( g_{\alpha}^L (x) \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge g_{\alpha}^U (x) \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) \right] \leq \\
\leq \lim_{n \to \infty} \left[ \int_{-\infty}^{\infty} \left( |g_{\alpha}^L (x)| \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge |g_{\alpha}^U (x)| \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) \right] + \\
+ \left[ \int_{-\infty}^{\infty} \left( |g_{\alpha}^L (x)| \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge |g_{\alpha}^U (x)| \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) \right] + \\
+ \left[ \int_{-\infty}^{\infty} \left( |g_{\alpha}^L (x)| \, dF_n^{\min} \left( x_{\beta}^{(\bullet)} \right) \wedge |g_{\alpha}^U (x)| \, dF_n^{\max} \left( x_{\beta}^{(\bullet)} \right) \right) \right] 
\]
Helly’s theorem on fuzzy valued functions

\[ + \lim_{n \to \infty} \left| \int_A \left( g^{I_n}_\alpha (x) \, dF^{\min}_n \left( x^{(\bullet)}_\beta \right) - g^{I_n}_\alpha (x) \, dF^{\min}_n \left( x^{(\bullet)}_\beta \right) \right) \right| = \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \]

if \( n \) and \( A \) are both large.

Thus we have proved the theorem if

\[ F^{\min}_n \left( x^{(\bullet)}_\beta \right) \to F^{\min} \left( x^{(\bullet)}_\beta \right) \text{ and} \]

\[ F^{\max}_n \left( x^{(\bullet)}_\beta \right) \to F^{\max} \left( x^{(\bullet)}_\beta \right) \text{ at all } x^{(\bullet)}_\beta \]

\[ \therefore \lim_{n \to \infty} \int_{-\infty}^{\infty} g^{I_n}_\alpha (x) \, dF^{\min}_n \left( x^{(\bullet)}_\beta \right) \wedge g^{I_n}_\alpha (x) \, dF^{\max}_n \left( x^{(\bullet)}_\beta \right) = \]

\[ = \int_{-\infty}^{\infty} g^{I_n}_\alpha (x) \, dF^{\min}_n \left( x^{(\bullet)}_\beta \right) \wedge g^{I_n}_\alpha (x) \, dF^{\max}_n \left( x^{(\bullet)}_\beta \right). \]

Now we can choose only continuity points of \( F(x) \) viz., \( x_1, x_2, \ldots \). Between two discontinuity points one can choose a continuity point and for monotonic \( F_n \), the points of discontinuities are at most countable. Hence the result is true if \( F^{\min}_n (x) \to F^{\min} (x) \) and \( F^{\max}_n (x) \to F^{\max} (x) \) at all continuity points of \( F \), and the proof is complete.

REFERENCES


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