REGULAR CONDITIONAL PROBABILITIES OF FUZZY PROBABILITY SPACES FORMULATED BY MEANS OF THE FUZZY RELATION ‘LESS THAN’

Abstract

Chapter 4 deals with the study of regular conditional probabilities of fuzzy probability spaces. In this chapter as a convenient departure from classical probability space, probability space structured by means of the fuzzy relation ‘less than’ is considered. With the aid of this new theoretical formulation, the properties of regular conditional probabilities of fuzzy probability spaces are obtained. Baye’s formula is established in this newly structured probability space.
4.1 Introduction

Klement et al., [29] have defined a fuzzy space as a pair 

\((\Omega, \sigma)\) where \(\Omega\) denote a fixed crisp set and \(\sigma\) is a fuzzy \(\sigma\)-algebra. 

Krzysztof Piasecki [32] has formulated fuzzy probability spaces defined by means of the fuzzy relation ‘less than’. In [32] fuzzy probability spaces are defined on the real line and fuzzy \(\sigma\)-algebras of events are introduced in an analogous way to the classical theory of probability. In [62] Zadeh has defined probability measures for fuzzy events by using the multivalued set function in the integral. In this chapter as a convenient departure from classical probability spaces Krzysztof Piasecki’s [32] probability space is considered for obtaining the properties of regular conditional probabilities of fuzzy probability spaces. 

The concept of regular conditional probabilities are introduced in the Krzysztof Piasecki’s probability space defined in terms of the fuzzy relation ‘less than’. This kind of probability space
is specifically chosen to establish Baye’s theorem. Conditional probabilities reveal the dependencies between the new and existing evidences. Since Bayesian approach originates from the ratio form of conditional probability measure, a special class of fuzzy probability measure is identified for the establishment of Baye’s theorem and similar results. The notion regular conditional probabilities to the realm of fuzzy random variables and fuzzy probability measures expose conditions under which the countably additive requirements can be met.

In Section 4.2, definitions of fuzzy algebra, fuzzy probability measures and fuzzy relation ‘less than’ are introduced. In Section 4.3 the definitions found in [32] such as fuzzy Borel intervals, fuzzy Borel family, fuzzy random variable etc are introduced. In Section 4.4 the notion of regular conditional probabilities over a fuzzy $\sigma$-algebra and regular conditional distribution function for a fuzzy random variable and its properties are de-
In section 4.5 reduced Baye's formula and properties of conditional probabilities in the probability space formulated by the fuzzy relation 'less than' are derived.

4.2 Preliminaries and Fuzzy Relation 'less than'

**Definition 4.2.1 ([32]).** Let $D$ denotes the family of all constant functions transforming the set $\Omega$ in to $[0,1]$. Let $0_\Omega : \Omega \to \{0\}$ and $1_\Omega : \Omega \to \{1\}$. A family of functions $\sigma : \{\mu : \Omega \to [0,1]\}$ is known as a fuzzy $\sigma$-algebra if it satisfies the following properties:

\[ \forall \mu \in D, \mu \in \sigma \]  
\[ \forall \mu \in \sigma, 1 - \mu \in \sigma \]  
\[ \forall \{\mu_n\} \in \sigma, \sup_n \mu_n \in \sigma \]

unless otherwise specifically stated, a fuzzy $\sigma$-algebra such that $D = \{0_\Omega, 1_\Omega\}$ will be considered.
Definition 4.2.2. A family of functions \( \hat{\sigma} = \{ \mu : \Omega \rightarrow [0,1] \} \) is known as a fuzzy algebra if it satisfies the following conditions.

\[
0_{\Omega} \in \hat{\sigma} \quad (4.2.4)
\]
\[
\forall \mu \in \hat{\sigma}, \; 1 - \mu \in \hat{\sigma} \quad (4.2.5)
\]
\[
\forall (\mu, v) \in \hat{\sigma} \times \hat{\sigma}, \; \mu \lor v \in \hat{\sigma} \quad (4.2.6)
\]

Definition 4.2.3. A fuzzy probability measure is a mapping 

\( m : \sigma \rightarrow [0,1] \) such that

\[
m(0_{\Omega}) = 0 \quad (4.2.7)
\]
\[
m(1_{\Omega}) = 1 \quad (4.2.8)
\]
\[
\forall (\mu, v) \in \sigma \times \sigma, m(\mu \lor v) + m(\mu \land v) = m(\mu) + m(v) \quad (4.2.9)
\]
\[
\forall \{\mu_n\} \in \sigma^N, \; \{\mu_n\} \uparrow \mu \Rightarrow m(\mu_n) \uparrow m(\mu) \quad (4.2.10)
\]

In [29] it is established that in general case a fuzzy probability measure fails to satisfy the equality

\[
m(1 - \mu) = 1 - m(\mu) \quad \text{for all } \mu \in \sigma. \quad (4.2.11)
\]

Since (4.2.11) is a necessary condition for the Baye’s formula
fuzzy probability measure does not satisfy Baye’s formula. In this chapter a class of fuzzy probability measures that satisfy (4.2.11) is considered.

Let $X$ be a given set. A fuzzy relation on $X$ is defined as a mapping $\Psi : X \times X \rightarrow [0,1]$. The inverse and complement of the fuzzy relation are defined respectively by the membership functions.

$$\Psi^{-1}(x, y) = \Psi(y, x) \quad \text{and} \quad \Psi(x, y) = 1 - \Psi(x, y).$$

A fuzzy relation $\Psi$ is called antireflexive if for each

$$x \in X, \quad \Psi(x, x) = 0 \quad (4.2.12)$$

A fuzzy relation $\Psi$ is called antisymmetric if for each pair

$$(x, y) \in X \times X$$

$$\Psi(x, y) \wedge \Psi(y, x) = 0 \quad (4.2.13)$$
A fuzzy relation $\Psi$ is called transitive if for each pair

$$(x, y) \in X \times X.$$  

$$\Psi(x, y) \geq \sup_{z \in X} \{\Psi(x, z) \land \Psi(z, y)\} \quad (4.2.14)$$

The definition of the fuzzy relation ‘less than’ $\rho$ below leaves out the properties antireflexivity, antisymmetry and transitivity. It lays emphasis on connections between inverse relations and complement relations.

**Definition 4.2.4 ([32])**. A fuzzy relation ‘less than’ is a mapping $\rho : \bar{R} \times \bar{R} \to [0,1]$ satisfying the following two conditions.

$$\forall (x, y) \in \bar{R} \times \bar{R}, \rho(-x, -y) = \rho(y, x) = \rho^{-1}(x, y)$$

$$\leq \bar{\rho}(x, y) = 1 - \rho(x, y) \quad (4.2.15)$$

$$\forall \{y_n\} \in \bar{R}^N, \quad \{y_n\} \uparrow y \Rightarrow \{\rho(., y_n)\} \uparrow \rho(., y), \quad (4.2.16)$$

$$\lim_{y \uparrow \infty} \rho(., y) = 1_{\bar{R}} \quad (4.2.17)$$
4.3 Fuzzy Borel Intervals

Definition 4.3.1. A mapping $\phi[a, b] : \bar{R} \rightarrow [0, 1]$ defined for any pair $(a, b) \in \bar{R} \times \bar{R}$ by

$$\phi[a, b](x) = \rho(x, b) \land (1 - \rho(x, a))$$ \hspace{1cm} (4.3.1)

is known as a fuzzy Borel interval.

The smallest fuzzy $\sigma$-algebra $\beta_\rho$ containing all fuzzy Borel intervals exists for each fuzzy relation ‘less than’ $\rho$.

For each $n \in N$, let $N_n = \{k; k \in N; k \leq n\}$. For all sequences $\{(a_k, b_k)\}_{k \in N_n}$ such that $(a_k, b_k) \in \bar{R}^2$ the inequality

$$a \geq c \quad \text{and} \quad b \leq d \Rightarrow \phi[a, b] \leq \phi[c, d]$$ \hspace{1cm} (4.3.2)

implies

$$\max_{k \in N_n}\{\phi[a_k, b_k]\} = \max_{k \in \bar{N}_n}\{\phi[a_k, b_k]\}$$

where $\bar{N}_n = \{k; k \in N_n; \exists l \in N_n; a_k \geq a_l \text{ and } b_k \leq b_l\}$.
Definition 4.3.2 ([32]). If $\bar{\beta}\rho = \{\mu : \bar{R} \to [0, 1]\}$ is the family of all fuzzy subsets $\mu$ such that

$$
\mu = \max_{k \in N_n}\{\phi(a_k, b_k)\}
$$

where $n$ is any positive integer and the sequences $\{a_k\}_{k \in N_n}$ and $\{b_k\}_{k \in N_n}$ are increasing then $\bar{\beta}\rho$ is known as a fuzzy Borel family.

Let $(\bar{R}, \beta\rho, m)$ be a fuzzy probability space with a fixed relation ‘less than’.

Definition 4.3.3 ([32]). A fuzzy probability measure

$$
m : \beta\rho \to [0, 1]
$$

satisfying the following properties is known as a fuzzy $P$-measure

$$
\forall \alpha \in \bar{R}, \ m(\phi[\alpha, \alpha]) = 0 \quad (4.3.4)
$$

$$
\forall \alpha \in \bar{R}, \ m(\phi[-\infty, a) \lor \phi[a, \infty)) = 1 \quad (4.3.5)
$$

Definition 4.3.4 ([32]). Let $(\Omega, \sigma, m)$ be a fuzzy probability space. A fuzzy random variable is a function $\chi : \Omega \to \bar{R}$ satisfying the condition

$$
\forall x \in \bar{R}, \ \rho(\chi(.), x) \in \sigma \quad (4.3.6)
$$
Definition 4.3.5 ([32]). The cumulative distribution function of the fuzzy random variable $\chi$ is a mapping $F : \bar{R} \to [0, 1]$ defined as

$$F(x) = m\left(\rho(\chi(.), x)\right)$$

(4.3.7)

Theorem 4.3.1 ([32]). Each cumulative distribution function $F : \bar{R} \to [0, 1]$ is a non-decreasing function and continuous from below. Then

$$\lim_{x \to \infty} F(x) = 1$$

(4.3.8)

$$\exists \alpha \in [0, 1] \lim_{x \to -\infty} F(x) = \alpha \geq 0 = F(-\infty)$$

(4.3.9)

Definition 4.3.6 ([32]). The mapping $\rho(\cdot \mid v) : \sigma \to [0, 1]$ defined for a fuzzy subset $v \in \sigma$ such that $m(v) \neq 0$ by the identity

$$\rho(\mu \mid v) = \frac{m(\mu \wedge v)}{m(v)}$$

(4.3.10)

is known as the conditional probability given $v$. 
4.4 Regular Conditional Probabilities

Let $G$ be an arbitrary fuzzy $\sigma$-algebra. The conditional probability $p(\mu \mid v)$ is a more intuitive object than the conditional probability $p(\mu \mid G)$. Hence for formal arguments in which $G$ is an arbitrary fuzzy $\sigma$-algebra, we make use of $p(\mu \mid G)$.

If $\mu_1, \mu_2, \ldots$ are disjoint sets in $\sigma$, then

$$p\left(\bigcup_{n=1}^{\infty} \mu_n \mid G\right) = \sum_{n=1}^{\infty} p(\mu_n \mid G) \text{ a.e.}$$

This does not imply that we will be able to choose $p(\mu \mid G)(\omega)$, $\mu \in \sigma$, $\omega \in \Omega$, so that it is a measure in $\mu$ for all $\omega \in \Omega$. The difficulty that could emerge under this circumstance is that for a fixed $\omega$, $p(\cdot \mid G)(\omega)$ need not be countably additive on $\sigma$. Thus there is no guarantee that we can specify the $p(\mu \mid G)$ to be countably additive in $\mu$. In the following definition suitable conditions are imposed under which the countable additivity requirement is realized.
**Definition 4.4.1.** Let $Y$ be a fuzzy random variable on $(\Omega, \sigma, m)$ and $G$ is a fuzzy sub $\sigma$-algebra of $\sigma$. The function

$$F = F(\omega, y) = m\{\rho(\chi(\cdot), y)\}(\omega)$$

where $\omega \in \Omega$, $y \in \mathbb{R}$ is called a regular conditional distribution function for $Y$ given $G$ iff the following two conditions are satisfied.

1. For each $\omega$, $F(\omega, \bullet)$ is a proper cumulative distribution function of the fuzzy random variable $Y$, that is increasing and right continuous, with

   $$F(\omega, \infty) = m\{\rho(\chi(\bullet), \infty)\}(\omega) = 1$$

   and  $$F(\omega, -\infty) = m\{\rho(\chi(\bullet), -\infty)\}(\omega) = 0$$

2. For each $y$, $F(\omega, y) = m\{\rho(\chi(\bullet), y) \mid G\}(\omega)$ for almost every $\omega$.

**Theorem 4.4.1.** If $Y$ is a fuzzy random variable on $(\Omega, \sigma, m)$, $G$ a fuzzy sub $\sigma$-algebra of $\sigma$, there is always a regular conditional distribution function for $Y$ given $G$. 
Proof. Choose a version of $F_r(\omega) = m(\rho(\chi(\bullet), r) \mid G)(\omega)$ for each fixed rational $r$. If $r_1, r_2, \ldots$ is a list of the rationals.

Let $A_{ij} = \{\omega; m(\rho(\chi(\bullet), r_j))(\omega) < m(\rho(\chi(\bullet), r_i))(\omega)\}$

$$A = \bigcup i j \{A_{ij}; \rho(r_i, r_j)\}$$

Since $\rho(r_i, r_j) > 0$,

$$m(\rho(\chi(\bullet), r_i) \mid G) \leq m(\rho(\chi(\bullet), r_j) \mid G) \text{ a.e.}$$

and so $m(A) = 0$.

Let $B_i = \{\omega; \lim_{n \to \infty} m(\rho(\chi(\bullet), r_i + 1/n))(\omega) \neq m(\rho(\chi(\bullet), r_i))(\omega)\}$

$$B = \bigcup_{i=1}^{\infty} B_i.$$ 

Since $\rho(\chi(\bullet), r_i + 1/n) \downarrow \rho(\chi(\bullet), r_i)$ as $n \to \infty$.

∴ $m(\rho(\chi(\bullet), r_i + 1/n))(\omega) \to m(\rho(\chi(\bullet), r_i))(\omega)$ a.e.

Hence $m(B) = 0$.

If $C = \{\omega; \lim_{n \to \infty} m(\rho(\chi(\bullet), n) \mid G)(\omega) \neq 1\}$,

then $m(C) = 0$.

Similarly $D = \{\omega; \lim_{n \to -\infty} m(\rho(\chi(\bullet), n) \neq 0)\}$ has fuzzy probability measure zero.
Define $F(\omega, y) = \begin{cases} 
\lim_{r \to y^+} F_r(\omega); & \text{if } \omega \notin A \cup B \cup C \cup D \\
G(y); & \text{if } \omega \in A \cup B \cup C \cup D. \end{cases}$

where $G(y)$ is any proper cumulative distribution function of a fuzzy random variable.

Then $F$ is well defined for if $\omega \notin A$, Then $F_r(\omega)$ is monotone in $r$, so that $\lim_{r \to y^+} F_r(\omega)$ exists. Also if $\omega \notin A \cup B$ then

$$\lim_{r \to y^+} F_r(\omega) = F_y(\omega) \text{ if } y \text{ is rational.}$$

Similarly if $\omega \notin A \cup C \cup D$ then

$$\lim_{r \to \infty} F_r(\omega) = 1 \lim_{r \to -\infty} F_r(\omega) = 0.$$  

Fix $\omega \in A \cup B \cup C \cup D$. Then $F(\omega, \cdot)$ is clearly increasing.

If $y < y' \leq r$ then $F(\omega, y) \leq F(\omega, y') \leq F(\omega, r) = F_r(\omega) \to F(\omega, y)$ as $r \to y$.

Thus $F(\omega, \cdot)$ is right continuous. If $r \leq y$ then

$$F(\omega, y) \geq F(\omega, r) = F_r(\omega) \to 1 \text{ as } r \to \infty.$$  

Hence $F(\omega, y) \to 1$ as $y \to \infty$. Similarly $F(\omega, y) \to 0$ as $y \to -\infty$. Hence the first requirement is satisfied.
By construction of $F$ we note

$$m\{\rho(\chi(\bullet), r) \mid G\}(\omega) = F_r(\omega) = F(\omega, r)$$

As $r \downarrow y$, $F(\omega, r) \to F(\omega, y)$ for all $\omega$ by right continuity and

$$m\{\rho(\chi(\bullet), r) \mid G\} \leq m\{\rho(\chi(\bullet), y) \mid G\} \text{ a.e.}$$

Thus $m\{\rho(\chi(\bullet), y) \mid G\}(\omega) = F(\omega, y)$ for almost every $\omega$. This establishes the second requirement and the proof is complete. ■

**Definition 4.4.2.** Let $Y : (\Omega, \sigma) \to (\Omega', \sigma')$ be a random object and $G$ a fuzzy sub $\sigma$-field of $\sigma$. The function $Q : \Omega \times \sigma' \to [0, 1]$ is called a regular conditional probability for $Y$ given $G$ iff

1. $Q(\omega, B)$ is a fuzzy probability measure in $B$ for each fixed $\omega \in \Omega$ and

2. For each fixed $B \in \sigma'$,

$$Q(\omega, B) = \{b \in B; m\{\rho(\chi(.), b) \mid G\}(\omega) \text{ a.e.}$$
Theorem 4.4.2. Let $Y$ be a fuzzy random variable on $(\Omega, \sigma, m)$, $G$ a fuzzy sub $\sigma$-algebra of $\sigma$. There exists a regular conditional probability for $Y$ given $G$.

Proof. Let $F$ be a regular conditional distribution function for $Y$ given $G$.

Define $Q(\omega, B) = \int_{y \in B} dF(\omega, y)$.

Thus for each $\omega$, $Q(\omega, \cdot)$ is the Lebesgue stieltjes measure corresponding to $F(\omega, \cdot)$, hence $Q$ is a fuzzy probability measure in $B$ if $\omega$ is fixed.

Now let $\mathcal{C} = \{ B \in \bar{\beta}_p; Q(\omega, B) = \rho\{ y \in B \mid G \}(\omega) \text{ a.e.} \}$

Then $\mathcal{C}$ contains all fuzzy borel intervals $(-\infty, y]$.

If $A, B \in \mathcal{C}, A \subset B$ then $B - A \in \mathcal{C}$. This shows that $\mathcal{C}$ contains all fuzzy borel intervals $(a, b]$, hence all finite disjoint unions are right semi closed intervals. Thus $Q$ is a regular conditional probability for $Y$ given $G$. $\blacksquare$
Theorem 4.4.3. Let $Y: (\Omega, \sigma) \rightarrow (\Omega', \sigma')$ be a fuzzy random object and $G$ a fuzzy sub $\sigma$-algebra of $\sigma$. Suppose there is a map $\psi: (\Omega', \sigma') \rightarrow (R, \tilde{\beta}_\rho)$ such that $\psi$ is one-to-one, $E = \psi(\Omega')$ is a fuzzy Borel subset of $R$ and $\psi^{-1}$ is measurable.

Then there is a regular conditional probability for $Y$ given $G$.

Proof. Let $Q_0 = Q_0(\omega, B), B \in \tilde{\beta}_\rho, \omega \in \Omega$ be a regular conditional probability for the fuzzy random variable $\psi(Y)$ given $G$.

Define $Q(\omega, A) = Q_0(\omega, \psi(A)); A \in \sigma'$.

Since $\psi^{-1}$ is measurable $\psi(A) \in \tilde{\beta}_\rho(E) \subset \tilde{\beta}_\rho$ and $Q$ is well defined.

Now $Q$ is a probability measure in $A$ for $\omega$ fixed. If $A$ is fixed then

$$Q(\omega, A) = m\{a \in \psi(A); \rho(\psi(\chi(\bullet), a)) \mid G\}(\omega)$$

$$= m\{a \in A; \rho(\chi(\bullet), a) \mid G\}(\omega) \quad \text{a.e.}$$

This completes the proof.
4.5 Reduced Baye’s Formula

Theorem 4.5.1.

(i) \( 0 \leq p(\mu \mid v) \leq 1 \)

(ii) \( p(\mu v(1 - \mu) \mid v) = 1 \)

(iii) \( p((\mu \mid v) \lor (1 - \mu \mid v)) = p(\mu \mid v) + p(1 - \mu \mid v). \)

Proof. (i) We claim \( 0 \leq \frac{m(\mu \land v)}{m(v)} \leq 1 \).

The left side is obvious. For the right side, \((\mu \land v)(x) \leq v(x)\).

This shows that \(m(\mu \land v) \leq m(v)\)

\[ \therefore 0 \leq \frac{m(\mu \land v)}{m(v)} \leq 1 \]

\[ \Rightarrow 0 \leq p(\mu \mid v) \leq 1 \]

(ii) \( p((\mu v(1 - \mu)) \lor v) = \frac{m(\mu v(1 - \mu) \land v)}{m(v)} = \frac{m(v)}{m(v)} = 1 \)

(iii) \( p((\mu \mid v) \lor (1 - \mu) \mid v) = \frac{m((\mu \lor (1 - \mu)) \land v)}{m(v)} \)

\[ = \frac{m(\mu \land v) + m((1 - \mu) \land v)}{m(v)} \]

\[ = p(\mu \mid v) + p(1 - \mu \mid v) \]

\[ \blacksquare \]
Definition 4.5.1. Let \((\Omega, \sigma, m)\) be any fuzzy probability space. The mapping \(p(. \mid v) : \sigma \rightarrow [0, 1]\) defined for a fuzzy subset \(v \in \sigma\) such that \(m(v) \neq 0\) by the identity

\[ p(\mu \mid v) = \frac{m(\mu \land v)}{m(v)} \]

is called the conditional probability given \(v\).

Theorem 4.5.2. For fuzzy subsets \(\mu_1, \mu_2, \ldots, \mu_n; \mu_i \in \sigma\),

\[ i = 1, 2, \ldots, n, \]

\[ p(\mu_1 \mu_2 \cdots \mu_n) = p(\mu_1)p(\mu_2 \mid \mu_1)p(\mu_3 \mid \mu_1 \mu_2) \cdots p(\mu_n \mid \mu_1 \mu_2 \cdots \mu_{n-1}) \]

provided \(p(\mu_1 \mu_2 \cdots \mu_{n-1}) > 0\).

Proof. Since \(p(\mu_1) \geq p(\mu_1 \mu_2) \cdots \geq p(\mu_1 \mu_2 \cdots \mu_{n-1}) > 0\) call the conditional probabilities stated above are well defined.

\[ \therefore \]

\[ p(\mu_1)p(\mu_2 \mid \mu_1)p(\mu_3 \mid \mu_1 \mu_2) \cdots p(\mu_n \mid \mu_1 \mu_2 \cdots \mu_{n-1}) \]

\[ = \frac{m(\mu_1)}{m(\Omega)} \frac{m(\mu_1 \land \mu_2)}{m(\mu_1)} \frac{m(\mu_1 \land \mu_2 \land \mu_3)}{m(\mu_1 \land \mu_2)} \cdots \frac{m(\mu_1 \land \mu_2 \land \cdots \land \mu_n)}{m(\mu_1 \land \mu_2 \cdots \mu_{n-1})} \]

\[ = \frac{m(\mu_1 \land \mu_2 \land \cdots \land \mu_n)}{m(\Omega)} \]

\[ = p(\mu_1 \mu_2 \cdots \mu_n) \]

\(\blacksquare\)
Theorem 4.5.3. If a fuzzy probability measure $m$ satisfies

$$m(\mu \lor (1 - \mu)) = 1 \quad (4.5.1)$$

$$m(\mu \land (1 - \mu)) = 0 \quad (4.5.2)$$

Then for any $\nu \in \sigma$,

$$m(\nu) = m(\mu)p(\nu | \mu) + m(1 - \mu)p(\nu | 1 - \mu)$$

Proof.

$$\nu = (\mu \lor (1 - \mu)) \land \nu$$

$$= (\mu \land \nu) \lor ((1 - \mu) \land \nu)$$

$$m(\nu) = m(\mu \land \nu) + m((1 - \mu) \land \nu)$$

$$m(\nu) = m(\mu)\frac{m(\mu \land \nu)}{m(\mu)} + m(1 - \mu)\frac{m((1 - \mu) \land \nu)}{m(1 - \mu)}$$

$$m(\nu) = m(\mu)p(\nu | \mu) + m(1 - \mu)p(\nu | 1 - \mu)$$

$$\Rightarrow \quad m(\nu) = m(\mu)p(\nu | \mu) + m(1 - \mu)p(\nu | 1 - \mu)$$
Theorem 4.5.4 (Reduced Baye’s Formula). If a fuzzy probability measure \( m \) satisfies (4.5.1) and (4.5.2) then

\[
p(\mu \mid v) = \frac{m(\mu).p(v \mid \mu)}{m(\mu)p(v \mid \mu) + m(1 - \mu)p(v \mid 1 - \mu)}
\]

(4.5.3)

for any pair \((\mu, v) \in \sigma \times \sigma\) such that \(m(\mu) \neq 0\), \(m(\mu) \neq 1\) and \(m(v) \neq 0\).

**Proof.** \(m(\mu)p(v \mid \mu) + m(1 - \mu)p(v \mid 1 - \mu) = m(v)\)

Then equation (4.5.3) becomes

\[
m(v)p(\mu \mid v) = m(\mu)p(v \mid \mu).
\]

This is true because both sides are equal to \(m(\mu \land v)\). ■