CHAPTER I

INTRODUCTION

The economic welfare of the people of any nation or country is a key factor for the social welfare of the citizens of the country. The economic welfare primarily depends upon the natural resources available in any country. Proper utilization of the same as well as economic policies of the government helps the process of achieving economic and social welfare. It is in this context the planning of various policies and the implementation of the same play a vital role. Efficient decision making is necessary to achieve the results which are important for the achievement of economic welfare.

In the words of T.L. Saaty “Operations Research is the art of giving bad answers to problems which otherwise have worse answers”. Operations Research has developed into a new Science providing innumerable techniques and methods to solve complex and vital managerial and other decisions. The outbreak of the Second World War has given rise to multifarious branches of scientific approach to practical and real problems of life. Chief among them are Operations Research, Industrial Statistical Quality Control etc. The military management
called on Scientists from various disciplines and organized them into teams to assist in solving strategic and tactical problems. The mathematicians and statisticians faced with real life problems conceptualized them as mathematical models and in solving such problems; they used probability theory and statistics to a large extent. After obtaining a solution, the implementation of the same into real life situations showed the extent of deviation from the optimal solutions, and this called for a review and improvement of the solutions by iterative and other methods. This constituted the subject matter of Operations Research. The importance of this science and its development came to be recognized by academic institutions during the 1950’s.

Operations Research has a number of branches like Mathematical programming, Game theory, Queueing Problems, Reliability theory, Inventory Control etc. All these disciplines depended more and more, for their development and sophistication, the use of advanced probability theory for which Stochastic Process is a basic structure. Many of the real life problems which are governed by chance mechanism are deeply involved with the concept of Stochastic Processes. So in conceptualizing real life problems, as mathematical models, stochastic process plays a predominant role. An important aspect in the theory of stochastic process
in the renewal theory which is lively from the mathematical view point and at the same time is a handy tool to solve many problems of stochastic process.

Many of the available stochastic models and their solutions are used here to conceptualize some interesting new problems and solve them. The problems which are conceptualized on certain hypothetical assumptions are in Inventory Control, Reliability Theory and in Queues. Also the dependent random variables are used here for the application of renewal theory concepts in some of these areas.

The outbreak of Second World War in 1940’s has made it imperative to have vital planning and execution in the various fronts of economic welfare. Since the availability of resources both national and man made were scarce during the World War II, the people in power and governance had the necessity to go in for proper planning. It aimed at the proper and judicious planning of the utilization of the available resources. Hence the combined effort in decision making was a necessity. The combined efforts of the scientists, economists, mathematicians, and statisticians were found to be very important. This has resulted in several
new methods of resource allocation and planning by the different groups of individuals.

As a result of the same the subject ‘Operations Research’ which is a branch of science has emerged as a separate discipline.

It may be observed that many areas of science and technology have been developed taking into account the emerging pattern of demand for the commodities, products and services but resource management has been considered as a basic need.

According to Ackoff, Arnoff and Churchman “Operation Research is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem”.

It may be observed that model building is the essence of Operational Research approach. Operation Research is mainly concerned with the techniques of applying scientific knowledge, besides the development of science. It provides an understanding which gives the expert/ manager new insights and capabilities to determine better solutions in his/her decision making process with great speed, competence and confidence.
Operations Research comprises of various divisions or branches. Suitable methods and techniques are developed to solve problems and decision making. They are

1) Mathematical Programming
2) Transportation problems
3) Sequencing problems
4) Assignment problems
5) Inventory Control
6) Queueing Theory
7) Replacement problems
8) Network problems

In all these branches many real life problems are converted in the form of a mathematical or stochastic model. The purpose is to find out the solution to such problems by using the available standard mathematical tools. Finding the optimal solution involves the following steps.
Any real life problem is conceptualized as a mathematical or stochastic model. The solution is derived by using the standard mathematical tools and techniques. If the derived solution is optimal the process of search of solution ends. If the solution is not optimal then the problem is suitably reformulated and solved. This interactive process continues till the optimal solution is obtained.

It may be observed that the use of Mathematics and Statistics has been the basis of developing many mathematical techniques even during the past three centuries. The initial mathematical models of primitive character have been due to Quesnay in the 18th century and Walras in 19th century. The great mathematicians like Jordan in 1873 and Minkowski in 1896 have developed the mathematical basis for the linear models. The concept of dynamic programming in the initial stage has been developed
by Markoff from 1856 to 1922. The initial work in the field of queueing theory has been due to Erlang in 1909.

It would be interesting to note that all algorithms in the area of Operations Research (OR) are based on the concept of model building which is the essence of Operations Research approach. Constructing a mathematical model helps to put the complexities and possible uncertainties in making any decision making problem into a logical framework amenable to comprehensive analysis and finally implementing the findings updating the model are essential steps of Operations Research. Operations Research has been developed as a full-fledged science after the Second World War. The primary cause for the evolution of this subject is due to limited availability of the resources and unlimited demand for the same. Therefore the aim of the military personnel is to determine the ways and means of optimum utilization of the limited available resources. Hence they had to utilize the services of the academicians especially the Mathematicians and Statisticians to determine the optimum solution. It also involves the process of conceptualization of the model, for which the optimal solution is derived.

After the Second World War, Operations Research has been taken up as an exclusive area of research and this in turn has resulted in
developing suitable mathematical models and determination of optimal solutions using such model.

“Operations Research is the application of scientific methods to problems arising from operations involving integrated systems of men, machines and materials. It normally utilizes the knowledge and skill of an interdisciplinary of such system with optimum operating solutions” - Fabryeky and Torgersen.

Applications of Operations Research serve as a scientific approach to problem solving for an executive in management. Applications of Operations Research involves,

i) Constructing Mathematical and Statistical models of decision making and controlling the problem in situations of complexity and uncertainty.

ii) Analyzing the relationship that determines the probable future consequences of decision choices and deriving appropriate measures of effectiveness.

The Operations Research – approach to any management problem is obtaining the cost effective solution requires the following.
i) The principal results of the analysis must be with primary focus on
decision making to give an unambiguous idea about the
implications for executive action.

ii) An appraisal testing on economic effectiveness criteria is necessary
so that the comparison of various feasible actions can be done on
the basis of measurable values such as profits, costs, and so on.

iii) Procedures for the construction of the measure of reliability need to
be on a formal mathematical basis.

Inventory control is a branch of Operation Research as indicated
above. The basic problem of inventory control or inventory management
is to determine the optimal stock size, optimal reorder size.
Determination of the time to reorder is also a question. They very famous
formula for Economic reorder quantity (EOQ) by Wilson and Harris is a
typical example of the usefulness of inventory control methods. A very
detailed and application oriented treatment of this subject is seen in
Hanssman(1962).

Arrow, KJ. Karlin, and Scarf (1958) have made significant
contribution to the development of this subject. Hanssmann (1959) has
developed many mathematical models relating to Inventory control.
Using linear programming principles, competitive bidding methods, many models have been developed by Hanssmann and Rivett (1959). Using the dynamic programming principles Bellman (1956, 1957) has developed many Inventory models. Warehousing problems have been developed by Bellman (1956). A generalized model of inventory control is due to Arrow, Harrris and Marschak (1951). This is a very general model which encompasses many inventory situations. A model for the optimal discharge of water from a reservoir has been due to Little J.D.C (1955). A systematic review of such models is seen in Whitin (1953).

In inventory management, the inventory model is first of all conceptualized and the optimal solution, which is cost effective, is derived. So in the conceptualization of inventory model the various costs, the different variables such as control variables and non-control variables are incorporated. It is quite interesting to observe that the inventory model can be either deterministic or probabilistic. If the model is of probabilistic nature, then the probability theory as well as stochastic processes plays a vital role not only in the formulation of the model but also in the determination of optimal solution.

An inventory is an idle resource of any organization or enterprise. The input of the resource is by production and supply. The output is due
to demands which occur at random epochs. If the demand and supply are equal it is called equilibrium. But it does not happen usually. This result is either excess of inventory or shortages.

The inventory on hand at any time ‘t’ is given by

\[ I(t) = I_0 + \int_0^t [a(x) - b(x)]dx \]

Where

\[ a(x) = \text{supply rate/unit time} \]
\[ b(x) = \text{demand rate/unit time} \]
\[ I_0 = \text{initial or starting inventory level}. \]

Any inventory system has three characteristics and only on the basis of these characteristics the classification of the inventory systems is carried out.

i) **Topology of the system:**

   If in an inventory system the supply or input is from a single source and the demand or output is also through a single source, then it is called a single station model. If there are many supply sources and similarly several sources of demand and a number of stations operate
simultaneously then it is called a system of parallel stations. A system of stations is called a series of station model if the output of one station is the input for the next, which are in series. The following chart explains the topology of the system.

(ii) **Time Behaviour**

If the demand and supply namely $a(t)$ and $b(t)$ are constant over time then it is called a static system, otherwise it is called a dynamic one.

(iii) **The determinacy**

If the supply and demand which are $a(x)$ and $b(x)$ respectively in an inventory model are constants or of deterministic character then the model itself is a deterministic model. On the other hand if they are
random variables then they have respective probability distributions. In such a case the model itself is called a probabilistic model.

The solution to any model depends upon these three characteristics.

**Solution of an Inventory Problem**

The inventory problem is real life situation is conceptualized as a mathematical, stochastic model. In doing so, the two costs namely the cost of excess inventory which is also known as the holding cost is incorporated into the model. If the demand is more than the supply the shortage may arise and hence the shortage cost is incorporated. In addition to this the cost of reordering is also incorporated. The objective of obtaining the optimal solution is to determine the solution which minimizes the overall cost. It is known as the optimal policy. Another problem is to determine the optimal reorder size as well the time at which the reordering is to be made.

It may be observed that the demand depends upon many factors like market conditions, availability of substitutes etc. Hence it is not under the control of the decision maker. On the other hand the supply is under the control of the decision maker and hence called the control variable. In many problems of inventory control the optimal size of the
supply is a matter of interest. Hence the optimal solution is often the determination of the supply size. A similar approach is to determine the time of reorder and quantity or reorder. If the demands as well as the supply are probabilistic in nature then the probability distributions are taken into account and the expected cost is found out. The solution which minimizes the expected cost is the optimal solution.

It may be noted that the recent approach to find the optimal solution takes into consideration another fact. The demand distribution may undergo a parametric change, after a particular value of the random variable involved in the model. The point at which the change occurs is called the truncation point. Sometimes after the truncation point the distribution of demand which is a random variable can undergo a change of distribution itself. Such facts are also incorporated in the model and the optimal solution is derived.

Another interesting area of research in inventory control has come up recently. It is the so called perishable inventory theory. There are many products such as vegetables, food products, fruits pharmaceutical products in which deterioration occurs. After a certain period the entire lot unsold will deteriorate completely and hence cannot be sold. In such models, the rate of deterioration is in an important aspect of
consideration. Many models taking the rate of deterioration is exponential, Weibull distribution are taken in to consideration. Ghare and Schrader (1963) have discussed a model for exponential decaying inventory. An EOQ model for items with Weibull distribution deterioration has been discussed by George and Philip (1974).

Some preliminary concepts and resulted used.

(a) Setting the Clock Back to Zero (SCBZ) property:

A special property known as Setting the Clock Back to Zero (SCBZ) property is due to Raja Rao and Talwalkar (1990).

This property is given as under:

A family of life distributions \( \{ f(x, \theta), \ x \geq 0, \ \theta \in \Omega \} \) is said to have the ‘Setting the Clock Back to Zero’ (SCBZ) property if the form of \( f(x, \theta) \) remains unchanged except for the value of the parameters, under the three operations,

\[
f(x, \theta) \rightarrow f(x, \theta^*) \quad \text{Where} \quad \theta^* \in \Omega
\]

(i) Truncating the original distribution of some point \( X_0 \geq 0 \)

(ii) Considering the observable distribution for life time \( X \geq X_0 \) and
(iii) Changing the origin by means of the transformation given by

\[ X_1 = X - X_0, \quad X_1 \geq 0. \]

This property can be used to define some random variables which are involved in the models discussed in this thesis.

(b) Change of distribution at a change point:

The Setting the Clock Back to Zero (SCBZ) property is one in which a random variable \( X \) has a parametric change after a certain value of \( x \), say \( x_0 \), which is called the truncation point. But there may be occasions where the random variable \( X \) has a p.d.f \( f(x) \) with cdf \( F(x) \) if \( X \geq x_0 \) and after that it has p.d.f \( h(x) \) with c.d.f \( H(x) \). Here \( x_0 \) is called the change point. The concept of change of distribution is discussed in Stagnl (1995). Sureshkumar (2006) has used this concept in shock model and cumulative damage process, to estimate the expected time to cross the threshold, where the threshold random variable \( Y \) undergoes a change in the distribution itself. This is an extended and improved version of the so called SCBZ property due to Raja Rao and Talwalker (1990).

The concept change of distribution can be explained as follows, a random variable \( Y \) has the p.d.f \( h_1(y) \) before \( \tau \) and if undergoes a change
in the form of p.d.f , such as \( h_2(y) \) after \( \tau \), therefore it is assumed that \( Y \) has p.d.f \( h_1(y) \) with corresponding c.d.f has \( H_1(y) \) Whenever \( Y \leq \tau \), and \( Y \) has p.d.f \( h_2(y) \) and the corresponding c.d.f has \( H_2(y) \) whenever.

We can write

\[
H(Y) = \begin{cases} 
H_1(Y) & \text{if } Y \leq \tau \\
H_1(\tau) + \overline{H}_1(\tau) H_2(Y - \tau) & \text{if } Y > \tau 
\end{cases}
\]

Where \( \overline{H}_1(\tau) = 1 - H_1(\tau) \)

Similarly the p.d.f of \( Y \) is given by

\[
h(Y) = \begin{cases} 
h_1(y) & \text{if } Y \leq \tau \\
\overline{H}_1(\tau) h_2(y - \tau) & \text{if } Y > \tau 
\end{cases}
\]

It can be verified that \( H(0) = 0 \)

\( H(\infty) = 1 \)

and hence \( h(y) \) is a proper p.d.f.

In inventory theory, the probabilistic demand can undergo change in the very distribution itself after a change point.
Arrangement of the Chapters

Chapter I of this thesis contains a brief introduction about inventory control and its practical applications to real life problems. Some of the results in stochastic process which are applied for dealing with the model in this thesis are outlined.

In Chapter II a brief summary of the different research papers published by various authors is given and it is the review of literature.

Chapter III is devoted to the discussion of a model using which it is possible to find the optimal reserve of semi finished product in between two machines in series. Two machines $M_1$ and $M_2$ are in series so that the output of machine $M_1$ which is the semi finished product is the input for machine $M_2$.

The configuration is given as follows.

Whenever $M_1$ goes to the break down state the machine $M_2$ will be forced to be idle. The idle time cost of $M_2$ is very high. Hence a reserve inventory of the semi finished product which is the output of $M_1$ is kept,
in between $M_1$ and $M_2$. There is inventory holding cost if the reserve inventory has excess stock pile. The breakdown duration or the repair time of $M_1$ is a random variable. Because of ageing the repair time of $M_1$ undergoes a parametric change and satisfies the so called Setting Clock Back to Zero (SCBZ) property which is due to Raja Rao and Talwalker (1990). Under this assumption the expression for optimal reserve inventory size is determined. Numerical illustrations are also furnished. Another variation of this model has also been considered. In the previous model it has been assumed that the consumption rate of reserve inventory in between $M_1$ and $M_2$ by the machine $M_2$ is a constant $r$. But this need not hold always. Hence the consumption rate of $M_2$ is taken to be a random variable. The optimal solution is derived.

In Chapter IV, a stochastic model to find the optimal size of supply of a product is discussed. In doing so, the cost of inventory holding is taken as ‘$h$’ per unit/per unit time and ‘$d$’ is the shortage cost. The demand for the product is a random variable which follows Erlang 2 distribution, truncated below at ‘$a$’ and truncated above at ‘$b$’. Truncation on both sides is justified by the fact that supply cannot be below ‘$a$’ and above ‘$b$’. The optimal supply size ‘$s$’ is obtained. Numerical illustration is also provided.
In Chapter V, an inventory model is discussed under the following assumptions regarding the model.

(i) There is a one time supply at the start of the period \((0, t)\).

(ii) The demands occur at ‘k’ random epochs in \((0, t)\) and the magnitude of the demands are random variables denoted as \(X_i\), \(i = 1,2,...,k\). If the cumulative demand during \((0, t)\) is less than ‘S’, then average occurs. If \(\sum X_i > S\) then shortages occur. The random variable representing demand namely \(X_i\) has p.d.f \(f(.)\) and c.d.f \(F(.)\). The random variables \(X_1, X_2,...,X_k\) are i.i.d random variables. Assuming there are exactly ‘k’ demand epochs in \((0, t)\), and using renewal theory, the optimal value of S is obtained. Numerical illustration is also provided. This model is compared with the conventional model in which the demand is assumed to be a one time demand represented by the random variable denoting the consumption rate of \(M_2\). It is a random variable which satisfies the so called SCBZ property. The optimal solution has been derived. Another extension is by assuming that the random variable ‘r’ has a distribution initially but there a change of distribution after a truncation point. Using the results by Sureshkumar(2006), the optimal size of reserve inventory is obtained. A comparative study of these two models is also given.
In Chapter VI an extension of the model in which the
determination of the optimal reserve inventory between two machines in
series to the case of three machines in series. In this model there three
machines $M_1$, $M_2$ and $M_3$ are in series. The output of $M_1$ is the input for
$M_2$. Similarly the output of $M_2$ is the input for $M_3$. It is assumed that the
machine $M_1$ goes to the down state. In other words it is the break down
of machine $M_1$. Then the supply of semi finished product which is the
raw material for $M_2$ is stopped. Reserve inventory is maintained in
between $M_1$ and $M_2$ and similarly between $M_2$ and $M_3$. If the machine $M_1$
is brought to upstate before the reserve inventory between $M_1$ and $M_2$ is
exhausted, then the process will not suffer a breakdown since $M_2$ and $M_3$
will not stop for want of raw material. If the repair time $M_1$ is more than
the time taken to exhaust the reserve inventory between $M_1$ and $M_3$ then
$M_2$ will have to stop. In this chapter only one case is considered, namely
the determination of the optimal reserve inventory between $M_1$ and $M_2$
which is denoted by $\hat{S}_1$ and the optimal reserve between $M_2$ and $M_3$ which
is denoted as $\hat{S}_2$. In doing so it is also assumed that the repair time of
$M_1$ undergoes a parametric change and satisfies the so called Setting the
Clock Back to Zero property as discussed by Raja Rao and
Talwalker(1990). This is due to the fact that the repair time undergoes a
parametric change due to the ageing of the machine. Numerical illustrations are also given.

In Chapter VII a brief summary of the results and conclusions drawn hereby are furnished.