CHAPTER III

OPTIMAL RESERVE OF SEMIFINISHED PRODUCT IN BETWEEN TWO MACHINES IN SERIES

Introduction

In this chapter a production oriented inventory problem in which the determination of the optimal size of the inventory of a semi finished product is discussed. It is assumed that the production system involves a series of stations and it is usual to determine the optimal size of the semi finished or partially finished products. This involves the concept of series of stations model.

In many manufacturing industries the production of products is in stages. The machines are in series. A simple case is one in which there are two machines $M_1$ and $M_2$ in series and the output of $M_1$ is a semi finished product and it happens to be the input for the machine $M_2$. One such model is discussed in Hanssmann (1962). In this model it is assumed that the output of $M_1$ is the input for $M_2$. If $M_1$ goes to the down state then the supply of input for $M_2$ is stopped. Then $M_2$ is forced to be in idle state and the idle time cost of $M_2$ in very high and prohibitive. Hence a
reserve inventory of the semi finished product in between $M_1$ and $M_2$ is suggested. If the inventory is very high compared to the demand then it results in inventory holding cost. The consumption rate of $M_2$ is a constant ‘r’. The repair time of $M_1$ is a random variable. Hence the optimal value of reserve inventory between $M_1$ and $M_2$ is determined.

**Basic model**

**Assumptions:**

1. The machines $M_1$ and $M_2$ are in series and the output of $M_1$ is the input for $M_2$.

2. The down time or the repair time of $M_1$ in a random variable.

3. The idle time cost of $M_2$ in very high.

4. A reserve inventory of semi finished product is kept in between $M_1$ and $M_2$.

**Notation**

\[ \tau = \text{a random variable which denotes the downtime of } M_1, \text{ which has p.d.f. } g(\tau) \text{ and c.d.f. is } G(\tau). \]

\[ h = \text{Inventory holding cost of the reserve inventory per unit of time.} \]
\[ \text{d} = \text{Idle time cost of } M_2 \text{ per unit of time} \]

\[ U_i = \text{A random variable denoting the interarrival times between breakdowns of } M_1 \text{ with p.d.f. } f(u) \text{ and c.d.f } F(u). U_i \text{ are i.i.d random variables.} \]

\[ \mu = \text{Mean of } U_i \]

\[ S = \text{Reserve inventory in between } M_1 \text{ and } M_2 \]

\[ r = \text{Constant rate of processing of the semi finished product by } M_2 \]

**Results:**

The idle time of \( M_2 \) is given by

\[ T = 0 \quad \text{if } \tau \leq S/r \]

\[ = \tau - S/r \quad \text{if } \tau > S/r. \]

Hence the expected cost of inventory holding and shortages is given by

\[ E(c) = hs + \frac{d}{\mu} \int_{S/r}^{\infty} (\tau - S/r) g(\tau) d\tau \]

\[ \text{... (3.1)} \]
The optimal value of ‘S’ is obtained by

\[
\frac{dE(c)}{ds} = 0
\]

It may be noted that the ‘S’ is involved both in the integral and also in the limits. Hence using Leitenitz rule for differentiation of the integral given as

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t) dt = b'(x) f[b(x), f[x, b(x)] - a'(x) f[x, a(x)] + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x, t) dt \quad \cdots (3.2)
\]

It can be shown that optional value of ‘S’ satisfies the equation.

\[
G(\frac{S}{r}) = 1 - \frac{r \mu h}{d} \quad \cdots (3.3)
\]

This model has been discussed by Hanssman (1962).

A modified version of the model has been discussed by Ramachandran and Saththiyamoorthi. (1981)

It may be observed that the optimal solution will be a feasible one if \( \frac{r \mu h}{d} \) is less than unity. Hence the model can be reformulated by taking the total expected cost as
\[ E(c) = h r \int_0^{S/r} \left( \frac{S'}{r} - \tau \right) g(\tau) d\tau + \frac{d}{\mu} \int_{S/r}^{\infty} (\tau - S/r) g(\tau) d\tau \] ... (3.4)

Taking \( \frac{dE(C)}{ds} = 0 \) gives the optimal reserve as inventory size as

\[ G\left(\frac{S}{r}\right) = \frac{d}{d + h \mu r} \] ... (3.5)

This solution has no restrictions and is an improved version of the previous solution.

Sathiyamoorthi (1980) has given a different version of the above model as given below.

In a single station inventory model with deterministic demands, the problem of determining optimal reserve inventory level between two machines is discussed in Hanssmann (1962). In this chapter we discuss the optimal reserve level between two machines M_1 and M_2 with a reserve inventory in between, when the output of M_1 is the input for M_2, where the input for M_2 is stochastic during a finite period (0,t). We have obtained the limiting behaviour as t tends to \( \infty \), when the working and failure times of M_1 have general distributions.
A system consists of two machines $M_1$ and $M_2$ a reserve inventory in between. $M_1$ can be working or failed. The output of $M_1$ is the input for $M_2$. Similar situations arise when the supply mechanism is independent of the working of costly equipment. For instance the supply mechanism of coal is independent of the working of the thermal power station. Here we obtain the optimal reserve inventory between $M_1$ and $M_2$ under stochastic demand pattern for input of $M_2$ for a finite period $(0,t)$ and the limiting case as $t$ tends to $\infty$.

**Model:**

1. $M_1$ is either up (working) or down (failed and under repair), $M_1$ alternates between up and down states.

2. The up times of $M_1$ after each start are independent identically distributed random variables.

3. The down times of $M_1$ are independent identically distributed random variables.

4. The interarrival times between restarts, the sum of uptime and down time of $M_1$ are independent identically distributed random variables.
5. The consumption rate of $M_2$ is a random variable.

6. The reserve inventory is replenished to the optimal level immediately after each restart.

7. At time $t = 0$, $M_1$ goes down.

**Notation:**

- $S$: reserve inventory between $M_1$ and $M_2$.
- $S^*$: optimal value of $S$.
- $F(.)$: Cdf of interarrival times of breakdowns.
- $\tau_i$: repair times of $M_1$, which are independent identically distributed random variables.
- $G(.)$: cdf $\{\tau_i\}$.
- $G_n(.)$: $n$ – fold Stieltjes convolution of $G(.)$.
- $N_t$: number of breakdowns in $(0, t)$.
- $r$: consumption rate of $M_1$, a random variable.
- $H(.)$: cdf of ‘$r$’.
$X_i$ uptime after each restart, independently identically distributed random variables.

$h$ inventory holding cost rate per unit of the reserve inventory

$T$ idle time of $M_2$. 

d cost rate of idle time $M_2$. 

$F_n(t) = \text{cdf} \left\{ \sum_{i=1}^{n-1} \tau_i + \sum_{i=1}^{n} X_i \right\}$

$\psi(t) = \sum_{n=1}^{\infty} F_n(t)$

© convolution

**Results:**

It can be noted from Sheldon M. Ross (1972)

$$\Pr[N_t = n] = F_n(t) - F_{n+1}(t)$$

The up times after restart $X_i$’s and the down times $\tau_i$’s after breakdowns from an alternating renewal process Cos, D.R (1962).
The idel time $T$ of $M_2$ is

$$T = \begin{cases} \text{O} & \text{if } S \geq \tau r \\ \tau - S/r & \text{if } S < \tau r \end{cases}$$

... (3.6)

From, the breakdown intensity is $\frac{\psi(t)}{t}$. Hence, the $s$ – expected cost rate for inventory holding and idle time cost of $M_2$ is

$$C_s(S) = hS + \frac{d}{t} \int_{S/r}^{\infty} (\tau - S/r) dG(\tau)$$

... (3.7)

Which is a function of $t$ and $S$.

$S^*$ is obtained by usual calculus methods. If the solution in the interior region then from Hanssmann. F (1962).

$$G\left(\frac{S^*}{r}\right) = 1 - \frac{hrt}{d.\psi(t)}.$$  

... (3.8)

In obtaining equations (3.6), (3.7) and (3.8), ‘$r$’ is taken as a constant (Hanssmann. F (1962)).
But our purpose is to obtain $S^*$ when ‘r’ is a random variable. So, $S^*$ is obtained as

$$E_t\left[G\left(S^*/r\right)\right] = 1 - \frac{\text{hrt}}{d\psi(t)} \cdot E(r) \quad \ldots \ (3.9)$$

$S^*$ satisfies the above equation and cannot be calculated explicitly in all cases. It is obtained below for some special cases.

**Case (i):**

If ‘τ’ is uniformly distributed over (0, a) then $G\left(S^*/r\right) = S/ra$.

Hence

$$E\left(S/ra\right) = 1 - \frac{\text{ht}}{d\psi(t)} \cdot E(r)$$

$$E\left(S'/a\right) E(1/r) = 1 - \frac{\text{ht}}{d\psi(t)} \cdot E(r) \quad \ldots \ (3.10)$$

The solution can be obtained explicitly for all distributions of the consumption rate ‘r’ which possess finite first order moment and finite non zero for $E(1/r)$. Here we discuss two special cases. (log = natural log).
Revised model - I

As a modification of the above model an assumption that the random variable $U_i$, denoting the interarrival times between the breakdowns of $M_1$ has a probability distribution that undergoes a parametric change is introduced. This is due to the fact that the ageing of machine $M_1$ influences the interarrival times between successive breakdowns. This is because of the fact that the wear and tear of the machines in constant use results in the possibility of the breakdown of the machine and it is due to the ageing of the machine.

It is assumed that the probability density function of random variable $U_i$ successive which denotes the interarrival times between breakdowns of the machine $\mu_1$ follows exponential distribution which possesses the so called Lack of Memory Property (LMP).

Now it is assumed that the random variable $U_i$, has the same distribution but there is only a parametric change from $\theta_1$ to $\theta_2$ after a particular value of $U_i$ say $U_0$. 
Hence

\[ f(u) = \theta_1 e^{-\theta_1 u} \text{ if } u \leq u_0 \]

\[ = \theta_2 e^{-\theta_2 u} e^{u_0(\theta_2 - \theta_1)} \text{ if } u > u_0 \]

where \( u_0 \) is called the truncation point and is also a random variable which follows \( \text{exp}(\lambda) \).

This satisfies the so called Setting the Clock Back to Zero (SCBZ) property, which is due to Raja Rao at Talwalker (1990). Srinivasan et. al (2007) have discussed the method of finding the optimal size of the reserve inventory under the assumption that the interarrival times between the breakdowns of \( M_1 \) is a random variable satisfying the SCBZ property where this truncation point is a fixed constant.

The mean inter arrival times between break downs is obtained as follows, when the truncation point is a random variable which follows \( \text{exp}(\lambda) \)

Now the p.d.f of \( u \) can be written as

\[ f(u) = \theta_1 e^{-\theta_1 u} P[u \leq h\lambda] + \theta_2 e^{-\theta_2 u} e^{u_0(\theta_2 - \theta_1)} P[u_0 < u] \quad \ldots \quad (3.11) \]
Since $u_0 \sim \exp(\lambda)$, equation (3.11) can be written as

$$f(u) = \theta_1 e^{-\theta_1 u} e^{-\lambda u} + \theta_2 e^{-\theta_2 u} \int_0^u e^{u_0(\theta_2 - \theta_1)} \lambda e^{-\lambda u_0} \, du_0$$

$$f(u) = \frac{(\theta_1 - \theta_2)(\lambda + \theta_1)}{(\lambda + \theta_1 - \theta_2)} e^{-u(\theta_1 + \lambda)} + \frac{\lambda \theta_2 e^{-\theta_2 u}}{(\lambda + \theta_1 - \theta_2)} \int_0^u e^{-\theta_2 u} \, du$$

... (3.12)

On simplification

Hence

$$\mu = \mathbb{E}(u) = \int_0^\infty u f(u) \, du$$

$$\mu = \frac{(\theta_1 - \theta_2)(\lambda + \theta_1)}{(\lambda + \theta_1 - \theta_2)} \int_0^\infty e^{-u(\theta_1 + \lambda)} \, du + \frac{\lambda \theta_2}{(\lambda + \theta_1 - \theta_2)} \int_0^\infty u e^{-\theta_2 u} \, du$$

$$= I_1 + I_2$$

Now

$$I_1 = \frac{(\theta_1 - \theta_2)(\lambda + \theta_1)}{(\lambda + \theta_1 - \theta_2)} \int_0^\infty e^{-u(\theta_1 + \lambda)} \, du$$

$$I_1 = \frac{(\theta_1 - \theta_2)(\lambda + \theta_1)}{(\lambda + \theta_1 - \theta_2)(\lambda + \theta_1)^2}$$

$$I_1 = \frac{(\theta_1 - \theta_2)}{(\lambda + \theta_1 - \theta_2)(\lambda + \theta_1)}$$

on simplification
\[ I_2 = \frac{\lambda \theta_2}{(\lambda + \theta_1 - \theta_2) \theta_2^2} \int_{0}^{\infty} u e^{-\theta u} \, du \]

\[ I_2 = \frac{\lambda \theta_2}{(\lambda + \theta_1 - \theta_2) \theta_2^2} \]

Hence

\[ \mu = \frac{(\theta_1 - \theta_2)}{(\lambda + \theta_1 - \theta_2)(\lambda + \theta_1)} + \frac{\lambda}{(\lambda + \theta_1 - \theta_2) \theta_2} \]

\[ = \frac{1}{(\lambda + \theta_1 - \theta_2)} \left[ \frac{(\theta_1 - \theta_2)\theta_2 + \lambda(\lambda + \theta_1)}{(\lambda + \theta_1) \theta_2} \right] \]

\[ = \frac{1}{(\lambda + \theta_1 - \theta_2)} \left[ \frac{\theta_1 \theta_2 - \theta_2^2 + \lambda^2 + \lambda \theta_1}{(\lambda + \theta_1) \theta_2} \right] \]

\[ \mu = \frac{1}{(\lambda + \theta_1 - \theta_2)} \left[ \frac{(\lambda + \theta_2)(\lambda + \theta_1 - \theta_2)}{(\lambda + \theta_1) \theta_2} \right] \]

\[ \mu = \frac{\lambda + \theta_2}{\lambda + \theta_1} \frac{1}{\theta_2} \]

on simplification

\[ \ldots (3.13) \]

Hence substituting (3.13) in (3.10) we get the optimal reserve inventory \( \hat{S} \) as one which satisfies the following equation (3.14)
\[ G(\frac{\hat{s}}{r}) = \frac{d}{d + hr\left(\frac{\lambda + \theta_2}{\lambda + \theta_1}\right)\frac{1}{\theta_2}} \]

\[ G(\frac{\hat{s}}{r}) = \frac{d\theta_2(\lambda + \theta_1)}{d(\lambda + \theta_1)\theta_2 + hr(\lambda + \theta_2)} \quad \cdots (3.14) \]

**Special Case**

The optimal value of \( S \) for the basic model given in equation (3.8) and the revised model given (3.14) by taking \( g(u) \sim \exp(\alpha) \), are as under.

From (3.9) we have

\[ G\left(\frac{\hat{S}}{r}\right) = 1 - e^{-\frac{\hat{s}}{r}} = 1 - \frac{r \mu h}{d} \]

\[ e^{\frac{\hat{s}}{r}} = \frac{r \mu h}{d} \]

Taking logarithm

\[ \therefore \frac{\hat{S}}{r} = \log\left(\frac{d}{r \mu h}\right) \]

\[ \hat{S} = r \log_e\left(\frac{d}{r \mu h}\right) \]
Again from (3.14)

\[ 1 - e^{-\frac{\hat{S}}{r}} = \frac{d\theta_2 (\lambda + \theta_1)}{d(\lambda + \theta_1)\theta_2 + hr(\lambda + \theta_2)} \]

Hence

\[ e^{-\frac{\hat{S}}{r}} = \frac{d(\lambda + \theta_1)\theta_2 + hr(\lambda + \theta_2) - d\theta_2 (\lambda + \theta_1)}{d(\lambda + \theta_1)\theta_2 + hr(\lambda + \theta_2)} \]

\[ = \frac{hr(\lambda + \theta_2)}{d(\lambda + \theta_1)\theta_2 + hr(\lambda + \theta_2)} \quad \text{... (3.15)} \]

Taking log both sides,

\[ + \hat{S} = r \log e \left[ \frac{d\theta_2 (\lambda + \theta_1) + hr(\lambda + \theta_2)}{hr(\lambda + \theta_2)} \right] \]

\[ \hat{S} = r \log e \left[ 1 + \frac{\theta_2 d(\lambda + \theta_1)}{hr(\lambda + \theta_2)} \right] \quad \text{on simplification} \]
Variation in $\hat{S}$ when $h$ changes.

$r = 50$, $d = 500$, $h = 10$, $\mu = 1.5$, $\theta_1 = 1.0$, $\theta_2 = 1.5$, $\lambda = 2.0$ and $\alpha = 1.2$ are all fixed.

Table 3.1

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\hat{S}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>87.67</td>
</tr>
<tr>
<td>20</td>
<td>72.59</td>
</tr>
<tr>
<td>30</td>
<td>64.88</td>
</tr>
<tr>
<td>40</td>
<td>59.43</td>
</tr>
<tr>
<td>50</td>
<td>48.46</td>
</tr>
</tbody>
</table>

Fig. 3.1
Variation in $\hat{S}$ when $d$ increases

$r = 50, \ h = 10, \ \mu = 1.5, \ \theta_1 = 1.0, \ \theta_2 = 1.5, \ \lambda = 2.0$ and $\alpha = 1.2$ are all fixed

Table 3.2

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\hat{S}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>87.67</td>
</tr>
<tr>
<td>600</td>
<td>94.28</td>
</tr>
<tr>
<td>700</td>
<td>102.77</td>
</tr>
<tr>
<td>800</td>
<td>120.68</td>
</tr>
<tr>
<td>900</td>
<td>145.44</td>
</tr>
</tbody>
</table>

Fig. 3.2
It can be verified that the optimal solution for basic model \( \hat{S} = 19.58 \) where as the revised model it is \( \hat{S} = 87.67 \), given the fixed values of costs and other parameters.

**Revised Model - II**

Now it is assumed that the consumption rate of \( M_2 \) is not a fixed one but it is a random variable denoted as ‘r’. This assumption is justified in the sense that the demand for the finished product namely the output of \( M_2 \) need not be always the same. Due to changes in seasons, the availability of alternative substitutes, changes in consumer choice the demand for the output of \( M_2 \) namely the finished product can vary from time to time and hence it is appropriate to assume ‘r’ to be a random variable, since the consumption rate of \( M_2 \) depends upon the above conditions stated.

To make the model feasible it is appropriate to take the expected value of ‘r’. Hence using equation (3.14) namely

\[
G(\hat{S}/r) = \frac{d\theta_2(\lambda + \theta_1)}{d(\lambda + \theta_1)\theta_2 + hr(\lambda + \theta_2)}
\]

‘r’ and taking expectation on both sides
Given the distribution of ‘r’ the optimal $\hat{S}$ can be obtained.

Some explicit cases of distribution of ‘r’ are taken and the optimal $\hat{S}$ is derived as follows.

**Case i:**

Let ‘τ’ the breakdown duration of $M_1$ be distributed as uniform distribution over (0, a)

Then

$$G\left(\frac{S}{r}\right) = \frac{S}{ra}$$

where

$G(.)$ is the cumulative distribution function of $\tau$.

$$\therefore E_t\left[G\left(\frac{\hat{S}}{r}\right)\right] = E\left(\frac{\hat{S}}{ra}\right)$$
Hence

\[
\hat{S}/a \left[ \frac{1}{(\beta - \alpha)} \log \left( \frac{\beta}{\alpha} \right) \right] = \frac{d \theta_2 (\lambda + \theta_1)}{d (\lambda + \theta_1) \theta_2 + h (\lambda + \theta_2) E(r)}
\]  

... (3.17)

Given the distribution of ‘r’, the optimal \( \hat{S} \) can be determined. For example if ‘r’ follows uniform distribution over \([\alpha, \beta]\) and \( \alpha > 0, \beta > 0 \), now

\[
\hat{S}/a \left[ \frac{1}{(\beta - \alpha)} \log \left( \frac{\beta}{\alpha} \right) \right] = \frac{d \theta_2 (\lambda + \theta_1)}{d (\lambda + \theta_1) \theta_2 + h (\lambda + \theta_2) E(r)} \times \frac{(\alpha + \beta)}{2}
\]

**Case ii**

In the model to be discussed here under all the assumptions of the previous model are kept the same except then feet that random variable \( U_i, i = 1,2,...,k \) representing of the inter arrival times between break downs of machine \( M_1 \) is such that.

i) It under goes a parametric change at the truncation point \( U_0 \).

ii) The truncation point \( U_0 \) it self is a random variable which is distributed according to uniform distribution over \((0,1)\).
Hence the mean interarrival time between breakdowns of M is given by

$$\begin{align*}
E(U) &= \int_{0}^{\infty} u.f(u).du \\
&= \int_{0}^{U_0} u.\theta_1 e^{-\theta_1 u} \, du \, P[u \leq u] \\
&+ \int_{0}^{\infty} u.\theta_2 e^{-\theta_2 u} \, e^{u_0(\theta_2-\theta_1)} \, P[u > u_0]
\end{align*}$$

Now when $U_0 \sim$ uniform over $(0, 1)$

$$P[u \leq u_0] = 1 - P[u_0 \leq u] = 1 - u$$ since $P[u_0 \leq u] = u$

$$f(u) = \theta_1 e^{-\theta_1 u} (1 - u) + \theta_2 e^{-\theta_2 u} \int_{0}^{u} e^{u_0(\theta_2-\theta_1)}du_0$$

$$= \theta_1 e^{-\theta_1 u} - \theta_1 u e^{-\theta_1 u} + \frac{\theta_2 e^{-\theta_2 u}}{(\theta_2-\theta_1)}[e^{u(\theta_2-\theta_1)} - 1]$$

$$= \theta_1 e^{-\theta_1 u} - \theta_1 u e^{-\theta_1 u} + \frac{\theta_2}{(\theta_2-\theta_1)}[e^{-\theta_2 u}]$$

$$f(u) = \frac{\theta_2}{(\theta_2-\theta_1)}[e^{-\theta_2 u}] \quad \text{... (3.18)}$$
Hence

\[ E(u) = \int_0^\infty u f(u) \, du \]

\[ = \int_0^\infty u \left[ \theta_1 e^{-\theta_1 u} \right] \, du \]

\[ - \theta_1 \int_0^\infty u^2 e^{-\theta_1 u} \, du \]

\[ + \frac{\theta_2}{(\theta_2 - \theta_1)} \int_0^\infty u e^{-\theta_2 u} \, du \]

\[ - \frac{\theta_2}{(\theta_2 - \theta_1)} \int_0^\infty u e^{-\theta_2 u} \, du. \]

\[ = I_1 + I_2 + I_3 + I_4 \]

It can be seen that

\[ I_1 = \theta_1 \int_0^\infty u e^{-\theta_1 u} \, du = \frac{1}{\theta_1} \]

\[ I_2 = -\theta_1 \int_0^\infty u^2 e^{-\theta_1 u} \, du = -\frac{2}{\theta_1^2} \]

\[ I_3 = \frac{\theta_2}{(\theta_2 - \theta_1)} \int_0^\infty u e^{-\theta_2 u} \, du = -\frac{\theta_2}{\theta_1^2(\theta_2 - \theta_1)} \]

\[ I_4 = -\frac{\theta_2}{(\theta_2 - \theta_1)} \int_0^\infty u e^{-\theta_2 u} \, du = \frac{1}{\theta_2} (\theta_2 - \theta_1) \]
Hence

\[ E(u) = I_1 + I_2 + I_3 + I_4. \]

\[ = \frac{1}{\theta_1} - \frac{2}{\theta_1^2} - \frac{\theta_2}{\theta_1^2(\theta_2 - \theta_1)} + \frac{1}{\theta_2(\theta_2 - \theta_1)} \]

\[ = \frac{\theta_2\theta_1(\theta_2 - \theta_1) - 2\theta_2(\theta_2 - \theta_1) - \theta_2^2 + \theta_1^2}{\theta_1^2(\theta_2 - \theta_1)\theta_2} \]

\[ = \frac{\theta_2\theta_1 - 2\theta_2 - \theta_2 - \theta_1}{\theta_1^2\theta_2} \]

\[ \mu = \frac{\theta_1(\theta_2 - 1) - 3\theta_2}{\theta_1^2\theta_2} \]

Hence

\[ G\left(\frac{\dot{S}}{r}\right) = \frac{d}{d + r \mu h} \]

\[ \Rightarrow \frac{d}{d + rh\left[\frac{\theta_1(\theta_2 - 1) - 3\theta_2}{\theta_1^2\theta_2}\right]} \]
If \( g(u) \) follows \( \exp(\alpha) \) then

\[
1 - e^{-\hat{S}/\alpha} = \frac{d}{d + r h \left[ \frac{\theta_1 (\theta_2 - 1) - 3 \theta_2}{\theta_1^2 \theta_2} \right]}
\]

\[
\therefore e^{-\hat{S}/\alpha} = 1 - \frac{d}{d + r h \left[ \frac{\theta_1 (\theta_2 - 1) - 3 \theta_2}{\theta_1^2 \theta_2} \right]} \quad \ldots (3.20)
\]

From equation (3.20) in the optimal \( \hat{S} \) can be obtained given the values of \( \alpha, r, d, h, \theta_1 \) and \( \theta_2 \).

**Numerical Illustration**

For example when \( \alpha = 1.0, r = 10, d = 100, h = 5, \theta_1 = 1.2, \theta_2 = 1.8 \)

We have

\( \hat{S} = 23.126 \)
Case i

Variations in $\hat{S}$ for changes in $h$.

$\alpha = 1.0$, $r = 10$, $d = 100$, $h = 5$, $\theta_1 = 1.2$, $\theta_2 = 1.8$ are all fixed.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\hat{S}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>23.126</td>
</tr>
<tr>
<td>10</td>
<td>9.101</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
</tr>
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It may be noted that the optimal $\hat{S}$ is not feasible if $h$ is greater than 15.
Case ii

Variations in $\hat{S}$ for changes in d.

Table 3.4

<table>
<thead>
<tr>
<th>d</th>
<th>$\hat{S}$</th>
</tr>
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<tbody>
<tr>
<td>100</td>
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</tr>
<tr>
<td>200</td>
<td>37.668</td>
</tr>
<tr>
<td>300</td>
<td>68.92</td>
</tr>
<tr>
<td>400</td>
<td>105.65</td>
</tr>
</tbody>
</table>

Fig. 3.4
Conclusion:

1) From Table 3.1 and the corresponding graph (3.1) it could be seen that as $h$ namely the inventory holding cost increases then smaller size of inventory is suggested. Similarly if the shortage cost increases then it is desirable to have a larger stock size. This is indicated in table (3.2) and the graph (3.2).

2) Even in the case of the revised model where the random variable representing the interarrival times undergoes a parametric change, a smaller inventory size is suggested as the inventory holding cost ‘$h$’ increases. This is indicated in table (3.3) and graph (3.3) similarly a larger inventory size is suggested when the shortage cost ‘$d$’ increases as indicated in table (3.4) and graph (3.4)