CHAPTER 10
System of equations

We search for two non-zero integral pairs \((x, y)\) and \((u, v)\) representing the length and breadth respectively of two rectangles which are related such that

\[
u + v = \alpha(x + y), \quad xy = \beta uv, \quad \alpha, \beta \in \mathbb{Z}^+ - \{0\},
\]

some interesting relations among the breadths and lengths are also presented.

Consider the system of double equations

\[
\begin{align*}
u + v &= \alpha(x + y) \tag{10.1} \\
xy &= \beta uv \tag{10.2}
\end{align*}
\]

where \(\alpha, \beta \in \mathbb{Z}^+ - \{0\}\).

The assumptions

\[
\begin{align*}x &= p + q, \\
y &= p - q \tag{10.3}
\end{align*}
\]

and

\[
\begin{align*}u &= \alpha U, \quad v = \alpha \tag{10.4}
\end{align*}
\]

in (10.1) and (10.2) lead respectively to

\[
\begin{align*}U + 1 &= 2p \tag{10.5} \\
p^2 - q^2 &= \alpha^2 \beta U \tag{10.6}
\end{align*}
\]

Eliminating \(U\) between (10.5)-(10.6), the resulting equation is written as

\[
(p - \alpha^2 \beta)^2 = q^2 + \alpha^2 \beta (\alpha^2 \beta - 1)
\]
which is satisfied by

\[ p = \alpha^2 \beta + \frac{\alpha}{2}(\alpha^2 \beta - 1 + \beta) \] (10.7)

and

\[ q = \frac{\alpha}{2}(\alpha^2 \beta - 1 - \beta) \] (10.8)

From (10.5) and (10.7), we have

\[ U = 2\alpha^2 \beta + \alpha^3 \beta + \alpha \beta - \alpha - 1 \] (10.9)

Using (10.7),(10.8),(10.9) in (10.3) and (10.4) the values of \( x, y, u \) and \( v \)
satisfy the system of equations (10.1) and (10.2) are given by

\[ x(\alpha, \beta) = \alpha^3 \beta + \alpha^2 \beta - \alpha \] (10.10)

\[ y(\alpha, \beta) = \alpha^2 \beta + \alpha \beta \] (10.11)

\[ u(\alpha, \beta) = \alpha^4 \beta + 2\alpha^3 \beta + \alpha^2 \beta - \alpha^2 - \alpha \] (10.12)

\[ v(\alpha) = \alpha \] (10.13)

It is observed that the solutions \( x, y \) and \( u \) when \( \beta = \alpha = n \), say, obtained

from (10.10), (10.11) and (10.12) are different from that of [31 ,115].
Observations:

A few interesting relations among the solutions are presented below:

1. \( v(\alpha).x(\alpha,1) - v(\alpha).y(\alpha,1) - u(\alpha,1) + 4T_{3,\alpha} = 1 \)

2. \( y(\alpha,\beta) = 2T_{3,\alpha} \)

3. \( v(\alpha).x(\alpha,1) + u(\alpha,1) - 4P_{\alpha} - 2T_{3,\alpha-1} + 1 = 0 \)

4. \( u(\alpha,1) - v(\alpha).y(\alpha,1) - 2P_{\alpha} + \alpha + 1 = 0 \)

5. \( u(\alpha,1) - v(\alpha).x(\alpha,1) - 2T_{3,\alpha} + 1 = 0 \)

6. \( u(\alpha,1) - 6v(\alpha).P_{\alpha-1} - 4T_{3,\alpha} + 1 = 0 \)

7. \( v(\alpha).x(\alpha,1) + u(\alpha,1) - 3v(\alpha).O_{\alpha} + 4v(\alpha)T_{3,\alpha-1} - \text{Perfect square} + 1 = 0 \)

8. \( v(\alpha).x(\alpha,\alpha) + v(\alpha).y(\alpha,\alpha) + u(\alpha,\alpha) - 2v(\alpha).T_{3,\alpha} - 24v(\alpha).P_{\alpha} + 1 \equiv 0 \pmod{2} \)

9. \( x(\alpha,\alpha) + y(\alpha,\alpha) - 2T_{3,\alpha} - 12P_{\alpha} - \alpha = 0 \)

10. \( x(\alpha,\beta) + \alpha = 2\beta P_{\alpha}^5 \)

11. \( y(\alpha,\alpha) = 2P_{\alpha}^5 \)

12. \( y(\alpha,\alpha + 2) = 6P_{\alpha}^3 \)

13. \( 6\left[\frac{v^2(\alpha,k^2) - y(\alpha,k^2)}{u(\alpha,k^2)}\right] \) is a Nasty number