CHAPTER 9

Quintic Diophantine equations with five unknowns

This chapter consists of two sections.

In section (A), the quintic diophantine equation with five unknowns

\[ (x^2 - y^2)(3x^2 + 3y^2 - 2xy) = 2(z^2 - w^2)p^3 \]

has been studied for its non-trivial distinct integral solutions.

Section (B) deals with the study of the quintic diophantine equation with five unknowns

\[ x^4 - y^4 = z^3 (p^2 - q^2) \]

for its non-zero integral solutions.

In each of the sections, a few interesting relations among the solutions are exhibited. In addition, some interesting special polygonal numbers are presented.
SECTION A:

Consider the fifth degree Diophantine equation with five unknowns

\[(x^2 - y^2)(3x^2 + 3y^2 - 2xy) = 2(z^2 - w^2)p^3 \quad (9.1)\]

The substitution of the linear transformations

\[x = u + v \quad \text{and} \quad y = u - v \quad (9.2)\]

in (9.1), leads to

\[(16uv)(u^2 + 2v^2) = 2(z^2 - w^2)p^3 \quad (9.3)\]

**Choice (i):**

Choose

\[z = 2uv + 1, \quad w = 2uv - 1 \quad \text{and} \quad p = a^2 + 2b^2 \quad (9.4)\]

(9.3) becomes

\[(u + i\sqrt{2}v) = (a^3 - 6ab^2) + i\sqrt{2}(3a^2b - 2b^3)\]

Equating the real and imaginary parts on both sides,

\[u = a^3 - 6ab^2\]

and

\[v = 3a^2b - 2b^3\]
Employing the values of \( u \) and \( v \) in (9.2) and (9.4), the solutions of (9.1) are,

\[
x(a, b) = a^3 + 3a^2b - 6ab^2 - 2b^3
\]
\[
y(a, b) = a^3 - 3a^2b - 6ab^2 + 2b^3
\]
\[
z(a, b) = 6a^5b - 40a^3b^3 + 24ab^5 + 1
\]
\[
w(a, b) = 6a^5b - 40a^3b^3 + 24ab^5 - 1
\]

and

\[
p(a, b) = a^2 + 2b^2
\]

**Observations:**

1. \( p(a, a) \equiv 0 \pmod{3} \)

2. \( x(a, 1) + y(a, 1) - 12P_{a-1}^3 \equiv 0 \pmod{10} \)

3. \( x(a, 1) - 2P_a^5 - 4T_{3,a-1} + 2 \equiv 0 \pmod{4} \)

4. \( x(a, 1) - 6P_a^4 + 3O_a + 18P_a^3 + 2 \equiv 0 \pmod{11} \)

5. \( x(a, 1) + y(a, 1) + p(a, 1) - 2P_a^5 - 6P_a^3 - 2 \equiv 0 \pmod{11} \)

6. \( z(a, 1) + w(a, 1) - 144(P_a^3T_{3,a}) + 48P_a^5 - 96T_{3,a} + 96T_{3,a^2} \equiv 0 \pmod{116} \)

7. \( z(a, 1) + w(a, 1) - 48(P_a^5T_{3,a}) \equiv 0 \pmod{4} \)

8. \( b[x(a, b) + y(a, b)] + a[x(a, b) - y(a, b)] \equiv 0 \pmod{8} \)

9. \( x(a, 1) + y(a, 1) + a[x(a, 1) - y(a, 1)] - 48P_{a-1}^3 \equiv 0 \pmod{8} \)

10. \( x(a, 1) + y(a, 1) + a[x(a, 1) - y(a, 1)] - 16P_a^5 + 16T_{3,a} \equiv 0 \pmod{8} \)

11. \( 3[z(a, b) - w(a, b)] \) is a Nasty number
12. The following expressions are cubical integers:

   (i) \( x(a,b) - y(a,b) \)

   (ii) \( 2x(a,a) \)

   (iii) \( 36y(a,a) \)

   (iv) \( 4[z(a,a) - w(a,a)] \)

   (v) \( x(a,b) + y(a,b) + 6ap \)

**Choice (ii):**

Choose

\[ z = uv + 1 \quad \text{and} \quad w = uv - 1 \]  \hspace{1cm} (9.5)

(9.3) becomes

\[ 2v + i\sqrt{2}u = (a^3 - 6ab^2) + i\sqrt{2}(3a^2b - 2b^3) \]

Equating the real and imaginary parts on both sides,

\[ u = 3a^2b - 2b^3 \]

\[ \text{and} \]

\[ v = \frac{1}{2}(a^3 - 6ab^2) \]

\( v \) is an integer if

   (i) both \( a \) and \( b \) are even

   (ii) \( a \) is even.
**Case (i):** Choose \( a = 2A \) and \( b = 2B \)

Using (9.2) and (9.5), the solutions of (9.1) are given by,

\[
\begin{align*}
x(A, B) &= 24A^2B - 16B^3 + 4A^3 - 24AB^2 \\
y(A, B) &= 24A^2B - 16B^3 - 4A^3 + 24AB^2 \\
z(A, B) &= 96A^5B - 640A^3B^3 + 384AB^5 + 1 \\
w(A, B) &= 96A^5B - 640A^3B^3 + 384AB^5 - 1 \\
p(A, B) &= 4A^2 + 8B^2
\end{align*}
\]

**Observations:**

1. \( x(1, B) + y(1, B) + 192P_B^3 \equiv 0 \pmod{16} \)
2. \( x(A, 1) - y(A, 1) + 48P_A^3 \equiv 0 \pmod{56} \)
3. \( x(A, 1) - 12P_A^4 - 36T_{3,A} + 16 \equiv 0 \pmod{44} \)
4. \( x(1, B) + y(1, B) + 64P_B^5 - 64T_{3,B} \equiv 0 \pmod{16} \)
5. \( x(1, B) + y(1, B) + 96P_{B-1}^4 + 96T_{3,B-1} \equiv 0 \pmod{16} \)
6. \( x(A, 1) - y(A, 1) - 24P_{A-1}^4 - 24T_{3,A-1} \equiv 0 \pmod{40} \)
7. \( x(A, 1) - y(A, 1) - 16P_A^5 + 16T_{3,A} \equiv 0 \pmod{40} \)
8. The following expressions
   
   \( (i) \) \( 60[x(A, 1) - y(A, 1) - 16P_A^5 + 96T_{3,A}] \)
   
   \( (ii) \) \( 2[x(A, 1) + y(A, 1) + 32] \)
   
   \( (iii) \) \( 6[x(1, \alpha^2) + y(1, \alpha^2) + 64P_{\alpha^2}^5 - 64T_{3,\alpha^2}] \)

represent a nasty numbers.
9. $4[x(A, A) + y(A, A)]$ is a cubical integer

10. $32[z(A, A) - w(A, A)]$ is a sextic integer

**Case (ii):** Choose $a = 2A$.

Then from (9.2) and (9.5), the values of $x, y, z, w, p$ are obtained by

\[
x(A, b) = 12A^2b - 2b^3 + 4A^3 - 6Ab^2
\]
\[
y(A, b) = 12A^2b - 2b^3 - 4A^3 + 6Ab^2
\]
\[
z(A, b) = 48A^5b - 80A^3b^3 + 12Ab^5 + 1
\]
\[
w(A, b) = 48A^5b - 80A^3b^3 + 12Ab^5 - 1
\]
\[
p(A, b) = 4A^2 + 2b^2
\]

**Observations:**

1. $z(A, 1) + w(A, 1) - 36[P_A^4 \cdot T_{18, A}] + 120T_{3, A}^2 + 144P_A^3 \equiv 0 \pmod{2}$
2. The following expressions

   (i) $6[x(A, 1) - y(A, 1) - 48P_A^3 + 8T_{3, A}]$,

   (ii) $x(A, 1) + y(A, 1) + 4$

   represent a nasty numbers.

3. $x(A, A)$ is a cubical integer

**Choice (iii):** Choose $z = 6uv + 1$ and $w = 6uv - 1$ \hspace{1cm} (9.6)

Then (9.3) reduces to,

\[
u + i\sqrt{2}v = (a^3 - 6ab^2 - 6a^2b + 4b^3) + i\sqrt{2}(a^3 - 6ab^2 + 3a^2b - 2b^3)
\]
Equating the real and imaginary parts on both sides,

\[ u = a^3 - 6ab^2 - 6a^2b + 4b^3 \]

and

\[ v = a^3 - 6ab^2 + 3a^2b - 2b^3 \]

Employing the values of \( u \) and \( v \) in (9.2) and (9.6), the solutions of (9.1) are obtained by

\[ x(a,b) = 2a^3 - 12ab^2 - 3a^2b + 2b^3 \]
\[ y(a,b) = -9a^2b + 6b^3 \]
\[ z(a,b) = 6a^6 - 180a^4b^2 - 18a^5b + 120a^3b^3 + 360a^2b^4 - 72ab^5 - 48b^6 + 1 \]
\[ w(a,b) = 6a^6 - 180a^4b^2 - 18a^5b + 120a^3b^3 + 360a^2b^4 - 72ab^5 - 48b^6 - 1 \]

and
\[ p = a^2 + 2b^2 \]

**Observations:**

1. \( z(a,1) + w(a,1) - 2[T_{3,a}T_{6,a}T_{8,a}] - [T_{3,a}^2P_a^3] + 12[P_a^4T_{18,a}] + 2[T_{25,a}T_{22,a}] - 48P_a^4 - 272T_{3,a}^2 - 218T_{3,a}^2 + 160 \equiv 0 \pmod{412} \)

2. \( x(1,b), y(1,b) - 2[T_{3,b}T_{8,b}T_{10,b}] + 12[T_{25,b}P_b^4] + 12[T_{23,b}P_b^3] + 2[T_{28,b}T_{3,b}] + 2[T_{3,b}T_{10,b}] + 18 \equiv 0 \pmod{5} \)

3. \( x(a,a) - y(a,a) \) is a cubical integer
SECTION B:

The quintic equation with five unknowns to be solved is

\[ x^4 - y^4 = z^3(p^2 - q^2) \]  \hspace{1cm} (9.7)

Introducing the linear transformations

\[ x = u + v, \ y = u - v \quad (u, v \neq 0, u \neq v) \]  \hspace{1cm} (9.8)

(9.7) is written as

\[ 8uv(u^2 + v^2) = z^3(p^2 - q^2) \]  \hspace{1cm} (9.9)

Again, choosing

\[ p = 2u + v, \ q = 2u - v \]  \hspace{1cm} (9.10)

In (9.9), it simplifies to

\[ u^2 - v^2 = z^3 \]  \hspace{1cm} (9.11)

which is satisfied by

\[ u = m(m^2 + n^2), \ v = n(m^2 + n^2) \]  \hspace{1cm} (9.12)

\[ z = z(m,n) = m^2 + n^2 \]  \hspace{1cm} (9.13)

Substituting (9.12) in (9.8) and (9.10), the values of \( x, y, p \) and \( q \) are respectively,

\[ x = x(m,n) = (m + n)(m^2 + n^2) \]
\[ y = y(m,n) = (m - n)(m^2 + n^2) \]  \hspace{1cm} (9.14)
\[ p = p(m,n) = (2m + n)(m^2 + n^2) \]
\[ q = q(m,n) = (2m - n)(m^2 + n^2) \]

Thus, (9.13) and (9.14) represents non-zero integral solution of (9.7).
Observations:

1. \[2[x(m,n) + y(m,n)] = p(m,n) + q(m,n)\]

2. \[x(m,n) - p(m,n)] = y(m,n) - q(m,n)\]

3. \[\frac{x^2(m,n) - y^2(m,n)}{z^2(m,n)} \equiv 0 \pmod{4}\]

4. \[x^2(m,n) + y^2(m,n)] = 2z^3(m,n)\]

5. \[2x^2(m,n) - p^2(m,n)] = 2y^2(m,n) - q^2(m,n)\]

6. \[x(m,n) + y(m,n)] - 3O_m - m = 0\]

7. \[p(m,1) + q(m,1) - 6O_m \equiv 0 \pmod{2}\]

8. \[2q(m,n) + x(m,n)] - y(m,n) = 0\]

9. \[x(1,n) - y(1,n)] + z(1,n) - 4P^n_5 + T_8,n - 1 \equiv 0 \pmod{2}\]

10. \[x(m,1) + y(m,1) + p(m,1) + q(m,1) - 9O_m - 2T_3,m - 1 \equiv 0 \pmod{2}\]

11. \[\frac{x(m,1) \times y(m,1)}{z^2(m,1)} \times \frac{6}{m} P^n_5 = 0\]

12. \[p(m,m) + q(m,m)\] is a cubic integer

13. Each of the following expressions represent a nasty number:

   (i) \[p^2(m,n) + q^2(m,n)] - 2z^3\]

   (ii) \[2[18P^n_m - 2T_7,m - x(m,1) - y(m,1) - p(m,1) - q(m,1)] - 1]\n
   (iii) \[42[x(m,1) \times z(m,1) + p(m,1) \times q(m,1) - (2T_3,m \times T_6,m \times T_8,m + T_{10,m}) + (T_{3,m^2 \times T_6,m}) - (4P^n_m \times T_{3,m}) - (2T_3,m \times T_{26,m}) + 6P^n_{m^4} - 10T_{3,m}]\]
It is to be noted that (9.11) is also satisfied by 

\[ u = m^3 - 3mn^2, \quad v = 3m^2n - n^3. \]

Thus, an alternative solution to (9.7) is given by

\[
\begin{align*}
x(m,n) &= (m-n)(m^2 + 4mn + n^2) \\
y(m,n) &= (m+n)(m^2 - 4mn + n^2) \\
z(m,n) &= m^2 + n^2 \\
p(m,n) &= 2m^3 - 6mn^2 + 3m^2n - n^3 \\
q(m,n) &= 2m^3 - 6mn^2 - 3m^2n + n^3
\end{align*}
\]

**Observations:**

1. \( [p(m,1) + q(m,1)] \times [x(m,1) + y(m,1)] - 2[T_{3,m} \times T_{6,m} \times T_{10,m}] - 12[T_{3,m} \times P_m^4] + [T_{14,m} \times T_{18,m}] + 264P_m^3 - 84T_{3,m} \equiv 0 \pmod{197} \)

2. \( CC_m + Star_m - p(m,1) - 1 \equiv 0 \pmod{3} \)

3. \( z(m,1) \times q(m,1) - 6(T_{3,m} \times O_m) + Nex_m - 12P_m^3 - 2P_m^5 \equiv 0 \pmod{3} \)

It is worth mentioning here that, instead of (9.10), there are other choices of \( p \) and \( q \) represented as below:

**Choice (i):** \( p = 2uv + 1, \quad q = 2uv - 1 \)

**Choice (ii):** \( p = uv + 2, \quad q = uv - 2 \)

Following the analysis presented above, the different patterns of solutions for each of the above two choices are exhibited below:

**Choice (i):** \( p = 2uv + 1, \quad q = 2uv - 1 \)

The corresponding two patterns of solutions are obtained as follows:
Pattern 1:

\[ x(m,n) = m^3 + n^3 + m^2n + mn^2 \]
\[ y(m,n) = m^3 - n^3 - m^2n + mn^2 \]
\[ z(m,n) = m^2 + n^2 \]
\[ p(m,n) = 2m^5n + 2m^3n^3 + 2mn^5 + 1 \]
\[ q(m,n) = 2m^5n + 2m^3n^3 + 2mn^5 - 1 \]

Observations:

1. \[ \frac{p(m,n) + q(m,n)}{[2x(m,n) + 2y(m,n)] \times z(m,n)} \equiv 0 \pmod{n} \]

2. \[ x(m,1) \times z(m,1) - [6P_m^5][2T_{3,m}] - 6P_m^5 = 1 \]

\[ [x(m,1) \times z(m,1)] - (3O_m)(2P_m^5)(2T_{3,m}) - (2T_{3,m}^2)(2P_m^5) - (2T_{3,m})(T_{6,m}) - 6P_m^4 - 2T_{3,m} - m = 1 \]

Pattern 2:

\[ x(m,n) - m^3 - n^3 + 3m^2n - 3mn^2 \]
\[ y(m,n) = m^3 + n^3 - 3m^2n - 3mn^2 \]
\[ z(m,n) = m^2 + n^2 \]
\[ p(m,n) = 6m^5n - 20m^3n^3 + 6mn^5 + 1 \]
\[ q(m,n) = 6m^5n - 20m^3n^3 + 6mn^5 - 1 \]

Here we observe that,

\[ 42[2(CS_m)(T_{3,m})(T_{6,m})(T_{8,m}) + 8(Gno_m)(T_{9,m})(T_{3,m})^2 - 8(T_{3,m})(T_{7,m})(T_{19,m}) + 4(T_{13,m})(P_m^5) + 2(T_{12,m})(T_{19,m}) - [x(m,1) + y(m,1)] \times p(m,1) + 486P_m^4 - 174T_{3,m}] \]

is a nasty number.
Choice(ii): \( p = uv + 2, \ q = uv - 2. \)

The respective two patterns of solutions are derived as below:

**Pattern 1:**

\[
\begin{align*}
    x(m,n) &= m^3 + n^3 + m^2n + mn^2 \\
    y(m,n) &= m^3 - n^3 + mn^2 - nm^2 \\
    z(m,n) &= m^2 + n^2 \\
    p(m,n) &= m^5n + 2m^3n^3 + mn^5 + 2 \\
    q(m,n) &= m^5n + 2m^3n^3 + mn^5 - 2
\end{align*}
\]

**Pattern 2:**

\[
\begin{align*}
    x(m,n) &= m^3 - n^3 + 3m^2n - 3mn^2 \\
    y(m,n) &= m^3 + n^3 - 3mn^2 - 3nm^2 \\
    z(m,n) &= m^2 + n^2 \\
    p(m,n) &= 3m^5n - 102m^3n^3 + 3mn^5 + 2 \\
    q(m,n) &= 3m^5n - 102m^3n^3 + 3mn^5 - 2
\end{align*}
\]