CHAPTER 5
Cubic Diophantine Equations with three Unknowns

This chapter consists of two sections.

In section (A), non trivial integral solutions of the ternary cubic diophantine equation

\[ 2(x^2 - y^2) + 4xy = (k^2 + 4k - 4)z^3 \]

are obtained.

Non trivial distinct integer solutions of the cubic equation with three unknowns

\[ x^2 - y^2 + xy = (m^2 - 5n^2)z^3 \]

are presented in section (B).

In each of the sections, a few interesting relations among the solutions \(x, y, z\) are exhibited.
SECTION – A

Consider the equation

\[ 2(x^2 - y^2) + 4xy = (k^2 + 4k - 4)z^3 \]  \hspace{1cm} (5.1)

where \( k \) is a non-zero positive integer.

The substitution of the linear transformations

\[ x = u + v \]
\[ y = u - v \]  \hspace{1cm} (5.2)

in (5.1), leads to

\[ (2u + 2v)^2 - 8v^2 = (k^2 + 4k - 4)z^3 \]

Choosing

\[ z = p^2 - 8q^2 \]  \hspace{1cm} (5.3)

The values of \( u \) and \( v \) are given by

\[ u = Kp^3 + (6 - 6K)p^2q + 24Kpq^2 + (16 - 16K)q^3 \]

and

\[ v = p^3 + 6Kp^2q + 24pq^2 + 16q^3 + 6p^2q + 16Kq^3 \]

where \( k = 2K \).

In view of (5.2), the values of \( x \) and \( y \) are given by

\[ x = (K + 1)p^3 + 12p^2q + (24 + 24K)pq^2 + 32q^3 \]  \hspace{1cm} (5.4)

and

\[ y = (K - 1)p^3 - 12Kp^2q + (24K - 24)pq^2 - 32Kq^3. \]

Thus, (5.3) and (5.4) represent the non-zero integral solutions of (5.1).
The properties of the solutions (5.4) for (5.1) are studied when \( k=2 \) and \( 4 \).

When \( k=2 \) (\( K=1 \)), the equation (5.1) is written as

\[
2(x^2 - y^2) + 4xy = 8z^3
\]

and the corresponding solutions are obtained by substituting \( K=1 \) in (5.4), namely,

\[
\begin{align*}
\alpha(p, q) &= 2p^3 + 12p^2q + 48pq^2 + 32q^3 \\
y(p, q) &= -12p^2q - 32q^3 \\
z(p, q) &= p^2 - 8q^2
\end{align*}
\]

Properties:

1. \( x(p, 1) - 96T_{3, q} - 4p^2(p - 1) + 38p^2 \equiv 0 \pmod{32} \)

2. \( x(1, q) - 24T_{3, q} - 64P^5_q \equiv 0 \pmod{2} \)

3. \( x(p, 1) + y(p, 1) + z(p, 1) - 4P^5_p - 96T_{3, p} + 49p^2 \equiv 0 \pmod{8} \)

4. \( x(1, q) + y(1, q) + z(1, q) - 96P^5_q + 40q^3 \equiv 0 \pmod{3} \)

5. \( 4y(p, -q) - 48q.z(p, -q) \) is a cubical integer.

6. \( 24q^2.z(p, q) - 6q.y(p, q) \) is a Nasty number.

7. \( x \) and \( y \) are even for all the values of \( p \) and \( q \).
When $k=4$ ($K=2$), the equation (5.1) is written as

$$2(x^2 - y^2) + 4xy = 28z^3$$

and the corresponding solutions are obtained by substituting $K=2$ in (5.4),

$$x = x(p,q) = 3p^3 + 12p^2q + 72pq^2 + 32q^3$$
$$y = y(p,q) = p^3 - 24p^2q + 24pq^2 - 64q^3$$
$$z = z(p,q) = p^2 - 8q^2$$

Properties:

1. $y(p,1) - P_p^5 + 50T_{3,p-1} + p$ is a square number.
2. $x(1,q) - 48T_{3,q-1} - 1$ is a cubical integer
3. $x(p,1) - 6P_p^5 - 144T_{3,p} + 60p^2 \equiv 0 \pmod{32}$
4. $x(p,1) + y(p,1) + z(p,1) - 8P_p^5 + 30T_{3,p} - 111p \equiv 0 \pmod{40}$
5. $x(1,q) + y(1,q) + z(1,q) + 32P_q^5 - 12T_{3,q-1} - 5 \equiv 0 \pmod{108}$
6. $y(1,q) + 64P_q^5 - 48T_{3,q} - 1 \equiv 0 \pmod{56}$
7. $x(1,q) + y(1,q) + 32P_q^5 - 24T_{3,q-1} - 4 \equiv 0 \pmod{116}$
SECTION – B

Consider the ternary cubic equation

\[ x^2 - y^2 + xy = (m^2 - 5n^2)z^3 \]  
\[ (5.5) \]

where m and n are not both simultaneously zero.

The substitution of the linear transformations

\[ x = u + v \text{ and } y = u - v \]  
\[ (5.6) \]

in (5.5), leads to

\[ (u + 2v)^2 - 5v^2 = (m^2 - 5n^2)z^3 \]

Choosing

\[ z = a^2 - 5b^2 \]  
\[ (5.7) \]

the values of u and v are given by

\[ u = m(a^3 + 15ab^2 - 6a^2b - 10b^3) + n(15a^2b + 25b^3 - 2a^3 - 30ab^2) \]

and

\[ v = m(3a^2b + 5b^3) + n(a^3 + 15ab^2) \]

In view of (5.6), the values of x and y are given by

\[ x(a, b) = m(a^3 + 15ab^2 - 3a^2b - 5b^3) + n(15a^2b + 25b^3 - a^3 - 15ab^2) \]

and

\[ y(a, b) = m(a^3 + 15ab^2 - 9a^2b - 15b^3) + n(15a^2b + 25b^3 - 3a^3 - 45ab^2) \]  
\[ (5.8) \]

Thus, (5.7) and (5.8) represent the non-zero integral solutions of (5.5).
It is worth to mention here that if \((x_0, y_0, z_0)\) is any solution of (5.5), then \((-2x_0 + y_0, -3x_0 + 2y_0, z_0)\) and \((-x_0 - y_0, y_0, z_0)\) also satisfy (5.5).

To analyze various properties among the solutions of (5.5), we have to take particular values of \(m\) and \(n\). In what follows, we study the properties of (5.5) for the choices of \(m\) and \(n\), namely,

(i) \(m=1, n=0\)
(ii) \(m=0, n=1\).

**Choice (i):** Let \(m=1, n=0\).

The equation under consideration is

\[
x^2 - y^2 + xy = z^3
\]  
(5.9)

Then, the corresponding solutions of (5.9) are given by

\[
x(a,b) = a^3 + 15ab^2 - 3a^2b - 5b^3
\]
\[
y(a,b) = a^3 + 15ab^2 - 9a^2b - 15b^3
\]
\[
z(a,b) = a^2 - 5b^2
\]  
(5.10)

**Properties:**

1. \(x(a,a) + y(a,a) = 0\)
2. \(x(5b,b) + y(5b,b) - 4bz(5b,b) = 0\)
3. \(x(a,a) + 2b \ z(a,a) = 0\)
4. \(y(3b,b) + 6b \ z(3b,b) = 0\)
5. \[3x(a,b) - y(a,b) - 2a z(a,b) \equiv 0 \pmod{40}\]

6. \[x(1,b) - y(1,b) - 20I_b^5 + 20T_{3,b} \equiv 0 \pmod{16}\]

7. \[2b[x(a,b) - y(a,b)] + 4b^2 z(a,b)\] is a perfect square.

8. \[z(5r^2 + s^2, 2rs)\] is a perfect square.

9. \[\frac{3x(1,b_s) - y(1,b_s)}{2}\] is a perfect square, where

\[b_s = \frac{1}{2\sqrt{15}}[(4 + \sqrt{15})^{s+1} - (4 - \sqrt{15})^{s+1}], \ s = 0, 1, 2, \ldots\]

10. When \(b=1\) and for the values of ‘\(a\)’ given by

\[a_0 = 1 \quad \& \quad a_{s+1} = \tilde{Y}_s + 4\tilde{a}_s, \ s = -1, 0, 1, 2, \ldots\]

\(x-y\) is a perfect square, where

\[\tilde{Y}_s = \frac{1}{2}[(5 + 2\sqrt{6})^{s+1} + (5 - 2\sqrt{6})^{s+1}]\]

\[\& \quad \tilde{a}_s = \frac{1}{2\sqrt{6}}[(5 + 2\sqrt{6})^{s+1} - (5 - 2\sqrt{6})^{s+1}]\]

11. Each of the following expressions represent a nasty numbers:

(i) \[\frac{30[x(a,b) + (3b-a)z(a,b)]}{a-b}, \ a \neq b\]

(ii) \[3b[x(a,b) - y(a,b) + 2bz(a,b)]\]

(iii) \[\frac{30[y(a,b) + (9b-a)z(a,b)]}{(a-3b)}, \ a \neq 3b\]

(iv) \[3x(15,b) - y(15,b) - 30z(15,b)\]

(v) \[6z(5r^2 + s^2, 2rs)\]
11. Each of the following expressions represents a cubic integer:

\begin{align*}
(i) & \quad \frac{x(a,b) - y(a,b) - 6bz(a,b)}{40} \\
(ii) & \quad x(2a,b) + y(2b,b) \\
(iii) & \quad 9[y(3b,b)] \\
(iv) & \quad 4[x(3b,b) + y(3b,b)] \\
(v) & \quad 6[x(4b,b) + y(4b,b)] \\
(vi) & \quad 100[x(5b,b) + y(5b,b)] \\
(vii) & \quad 50[x(6b,b) + y(6b,b)] \\
(viii) & \quad 3x(a,b) - y(a,b) + 6az(a,b)
\end{align*}

**Choice (ii):** Let $m = 0$, $n = 1$.

The equation to be solved is

$$x^2 - y^2 + xy = -5z^3$$  \hspace{1cm} (5.11)

The solutions of equation (5.11) are obtained as

\begin{align*}
x(a,b) &= 15a^2b + 25b^3 - a^3 - 15ab^2 \\
y(a,b) &= 15a^2b + 25b^3 - 3a^3 - 45ab^2 \\
z(a,b) &= a^2 - 5b^2
\end{align*}

**Properties:**

1. $x(a,1) + y(a,1) - 6T_{24,a} + 72P_a^5 - 32a^3 \equiv 0 \pmod{50}$

2. $x(a,a) - y(a,a) - z(a,a) - 64P_a^5 \equiv 0 \pmod{28}$

3. $\frac{y(a,b) + z(a,b)(3a - 15b)}{25b - 15a}$ is a square number
4. Each of the following expressions represent a Nasty numbers:

\[(i)\ 42[y(a,a) + z(a,a) - x(a,a) + 64p_a^5]\]

\[(ii)\ \frac{60[x(a,b) + z(a,b)(a-15b)]}{5b-2a}\]

\[(iii)\ \frac{30[y(a,b) + z(a,b)(3a-5b)]}{5b-3a}\]

5. Each of the following expressions represents a cubic integer:

\[(i)\ 9x(a,a)\]

\[(ii)\ 2[x(a,a)-y(a,a)]\]

\[(iii)\ y(a,a)\]

\[(iv)\ 4[x(a,a) + y(a,a)]\]

\[(v)\ 12[x(2b,b) + y(2b,b)]\]