CHAPTER-IV

QUINTIC SPLINE SOLUTION OF A LINEAR FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEM

4.1 INTRODUCTION:

We consider the two-point boundary value problem of the form:

\[ \frac{d^4u}{dx^4} + f(x)u = g(x), \quad \text{for } x \in [a, b] \]

with the boundary conditions:

\[ u(a) = \alpha_1, \quad u(b) = \alpha_2 \]
\[ u'(a) = \beta_1, \quad u'(b) = \beta_2 \]

...(4.1)

Where the function \( f(x) \) and \( g(x) \) are continuous on \([a, b]\). Two point boundary value problems of the type (4.1) arise in plate deflection theory. The problem of bending a rectangular clamped beam resting on an elastic foundation is a particular case of (4.1). It may not be possible to find analytical solution of the equation (4.1) for all \( f(x) \) and \( g(x) \). In such difficult situations, we have to approximate the solutions numerically.

Finite difference methods for (4.1) are given by Babuska et.al. [8], Katti [65] and Usmani [113]. Babuska et al
have shown that the resulting error is $O(h^{3/2})$, provided $f(x) \geq 0$ on $[a,b]$. Usmani has given methods of order 2, 4 and 6 for the problem (4.1). While his methods of order 2 and 4 lead to five-diagonal linear systems, the sixth order method leads to a nine-diagonal linear system. Usmani has shown that the boundary value problem (4.1), possesses a unique solution provided,

$$\inf_{a \leq x \leq b} f(x) = \eta > -a/(b-a)^4 \text{ with } a = 500.5639...$$

Katti gives a sixth order method for non-linear case of (4.1), which in linear case leads to a five-diagonal linear system.

Spline collocation methods for fourth-order two-point boundary value problems are given by Fyfe [42], Russell and Shampine [97] and Irodotou-Ellina and Houstis [50]. Second order nodal collocation methods based on cubic splines for linear case is given by Fyfe. Second order collocation method based on quintic spline, for solving a subclass of linear and non-linear fourth order two point boundary value problem is given by Russell and Shampine. Sixth order collocation method based on quintic spline for general fourth order linear two-point boundary value problems, have been developed by Irodotou-Ellina and Houstis.
Recently Chawla and Subramanian [17] described a sixth order method by using quintic splines for the solution of equation (4.1). But this method is based on Bickley's [11] idea of using the continuity conditions to construct a cubic spline approximation, here it is used only after some other methods (e.g. finite difference methods) has been used to obtained accurate nodal values.

In this chapter we have obtained a class of methods of order two, four and six, by using a non-polynomial parametric spline I. In section 2, we present the formulation of this method. To retain the band width of the coefficient matrix as five, we develop the boundary equations with truncation error $O(h^{10})$ in section 3. In section 4 and 5 we give a class of methods with truncation errors respectively. In section 6, we prove the convergence of our methods, and in section 7, we present the numerical results that verify the theoretical behaviour of our methods.

4.2 The Method:

At the grid points $x_j$, $j=2(1)N-2$, where $x_j=a+jh$, $j=0(1)N$, the given differential equation in (4.1) can be discretized by using the spline relations in (2.63-iii) so that we obtain
To retain the band width of the coefficient matrix as five when we use the sixth order method we have to develop a formula with truncation error $O(h^{10})$. The following identities are developed.

4.3 DEVELOPMENT OF THE BOUNDARY FORMULAS:

We assume that the solution $u(x)$ of the given system (4.1) has sufficiently high order derivatives. Then we define the following identities:

\[ (i) \sum_{k=0}^{3} b_k u_k + c h^2 u_o + h^4 \sum_{k=0}^{3} d_k u_k^{(4)} + e_1 h^5 u_o^{(5)} + e_2 h^6 u_o^{(6)} + t_1 = 0 \]

\[ (ii) \sum_{k=0}^{3} b_k^* u_k + c^* h u_o + h^4 \sum_{k=0}^{4} d_k^* u_k^{(4)} + e_1^* h^5 u_o^{(5)} + e_2^* h^6 u_o^{(6)} + t_1^* = 0 \]

\[ \ldots (4.3) \]
By using (4.3, ii), in order that $t_1^*$ is $O(h^6)$, we find that

$$[b_o^*, b_1^*, b_2^*, b_3^*, c^*] = [-11/2, 9, -9/2, -3],$$
and

$$[d_o^*, d_1^*, d_2^*, d_3^*, e_1^*, e_2^*] = [3, -9/10, 0, 0, 0, 0].$$

We obtain the second order boundary formulas as follows

(i) $\frac{-11}{2}u_0 + 9u_1 - \frac{9}{2}u_2 + u_3 - 3h\beta_1 = \frac{h^4}{20} \left[ -3u_o^{(4)} + 18u_1^{(4)} \right] + \frac{7}{40} h^6 u_1^{(6)}(\zeta_1)$

(ii) $u_{N-3} + \frac{9}{2}u_{N-2} + 9u_{N-1} - \frac{11}{2}u_N + 3h\beta_2$

$$= \frac{h^4}{20} \left[ 18u_N^{(4)} - 3u_{N-1}^{(4)} \right] + \frac{7}{40} h^6 u_N^{(6)}(\zeta_N) \quad \ldots(4.4)$$

where $x_0 < \zeta_1 < x_3$, and $x_{N-3} < \zeta_N < x_N$.

Using (4.3, ii), in order that $t_1^*$ is $O(h^8)$, we find that,

$$[b_o^*, b_1^*, b_2^*, b_3^*, c^*] = [-11/2, 9, -9/2, 1, -3],$$
and

$$[d_o^*, d_1^*, d_2^*, d_3^*, e_1^*, e_2^*] = -\frac{h^4}{280} [8, 151, 52, -1, 0, 0].$$

We obtain the fourth order boundary formulas as follows

(i) $\frac{-11}{2}u_0 + 9u_1 - \frac{9}{2}u_2 + u_3 - 3h\beta_1$

$$= \frac{h^4}{280} \left[ 8u_o^{(4)} + 151u_1^{(4)} + 52u_2^{(4)} - u_3^{(4)} \right] + \frac{8}{6720} h^8 u_1^{(8)}(\zeta_1)$
(ii) $u_{N-3} - \frac{9}{2} u_{N-2} + 9u_{N-1} - \frac{11}{2} \alpha - N + 3h \beta$  

$$ = \frac{h^4}{280} [-u_{N-3}^{(4)} + 52 u_{N-2}^{(4)} + 151 u_{N-1}^{(4)} + 8u_{N}^{(4)}] + \frac{h^8}{8720} u^{(8)}(\zeta_{N})$$  

...(4.5)

In order that $t_1$ and $t^*$ is $O(h^{10})$, we find that,

$$[b_0, b_1, b_2, b_3, c] = [-2.5, -4.1, 1],$$  

and also $$[b^*, b_1^*, b_2^*, b_3, c^* ]= [-11/2, 9,-9/2, 1,-3],$$  

$$[d_0^*, d_1^*, d_2^*, d_3, e_1, e_2^* ]= (\frac{1}{30240}) [529, -18090, -5157, 38, 1074, 288]$$  

and truncation error are $t_1 = \frac{241}{100800} \cdot h^{10} u^{(10)}(\zeta)$

$$t^* = \frac{29}{100800} \cdot h^{10} u^{(10)}(\zeta), \text{ where } x_{j-2} < \zeta < x_{j+2}.$$  

Eliminating $u_0^*$ which is not specified from both the equations in (4.3) and simplifying, we obtain

(i) $\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3 = \gamma_0$

(ii) $\gamma_{N-3} u_{N-3} + \gamma_{N-2} u_{N-2} + \gamma_{N-1} u_{N-1} = \gamma_N$  

...(4.6)

where $\gamma_1 = [(e_2^* d_1^* - e_2 d_1)F_0 + c d_1^*]F_1 - [(e_2^* b_1 - e_2 b_1^*)F_0 + cb^*]$,  

$$\gamma_2 = [(e_2^* d_2^* - e_2 d_2)F_0 + c d_2^*]F_2 - [(e_2^* b_2 - e_2 b_2^*)F_0 + cb^*],$$

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\[
\gamma_3 = [(e_{2,3}^* - e_{2,3}^*)F_0 + cd_3^*]F_3 - [(e_{2,3}^* - e_{2,3}^*)F_0 + cb_3^*],
\]

\[
\gamma_0 = h^4 \sum_{k=0}^{3} [(e_{2,2}^* - e_{2,2}^*)F + cd_1^*]g_k + h^5 F_0 [e_{2,1}^* (e_{1,1}^* + e_{1,2}^* z_{2}) - e_{2,1}^* (e_{1,1}^* + e_{1,2}^* z_{2}) + c(b - F_{d,2}^*)] + c(b^* - F_{d,2}^*) \alpha_1,
\]

\[
\gamma_{N-1} = [(e_{2,1}^* - e_{2,1}^*)F + cd_3^*]F_{N-1} - [(e_{2,1}^* - e_{2,1}^*)F_0 + cb_1^*],
\]

\[
\gamma_{N-2} = [(e_{2,2}^* - e_{2,2}^*)F + cd_2^*]F_{N-2} - [(e_{2,2}^* - e_{2,2}^*)F_0 + cb_2^*],
\]

\[
\gamma_{N-3} = [(e_{2,3}^* - e_{2,3}^*)F + cd_3^*]F_{N-3} - [(e_{2,3}^* - e_{2,3}^*)F_0 + cb_3^*],
\]

\[
\gamma_N = h^4 \sum_{k=0}^{3} [(e_{2,2}^* - e_{2,2}^*)F + cd_1^*]g_k + h^5 F_0 [e_{2,1}^* (e_{1,1}^* + e_{1,2}^* z_{2}) - e_{2,1}^* (e_{1,1}^* + e_{1,2}^* z_{2}) + c(b - F_{d,2}^*)] + c(b^* - F_{d,2}^*) \alpha_1,
\]

\[
(c-e_{2,N}^* (-e_{2,N}^* + 2e_{F,N}^* + c^*]) + F_0 \alpha_1 [(e_{b,1}^* - e_{b,1}^*) - F_0 (e_{d,2}^* - e_{d,2}^*)] + c(b^* - F_{d,2}^*) \alpha_2.
\]
and \[ \sum_{k=0}^{3} F_k = h^4 f_k, \quad F_0 = h^5 f, \quad z_1 = g' f, \quad z_2 = g'' f u, \]

\[ \sum_{k=0}^{3} F_{N-k} = h^4 f_{N-k}, \quad F_N = h^5 f, \quad z_{N1} = g' f u_N, \quad z_{N2} = g'' f u_N \]

### 4.4 A Class of Methods:

Most of the known methods are special cases of our method, for different choices of parameters \( p, q \) and \( s \).

(i) If we choose \( p = q = 0 \) and \( s = 1 \), we get the scheme

\[ \delta^4 u_j = h^4 u_j^{(4)}, \quad j = 2(1)N-2 \] of second order with truncation error

\[ T_j = \frac{1}{6} h^6 M_6, \quad \text{where} \quad M_6 = \max_{a \leq x \leq b} |u^{(6)}(x)| \quad \ldots(4.7) \]

(ii) If we choose \( p = 0, \ s = 2/3, \ q = 1/6 \), we obtain the fourth order finite difference method,

\[ \delta^4 u_j = \frac{h^4}{6} \left[ u_j^{(4)} + 4 u_j^{(4)} + u_j^{(4)} \right], \quad j = 2(1)N-2 \quad \ldots(4.8) \]

and (iii) If we take \( (p, q, s) = \frac{1}{720} (-1, 124, 474) \) we get the sixth order method
which has been obtained by Usmani [113].

4.5 TRUNCATION ERRORS:

Truncation errors of various methods are listed below:

(i) For second order scheme the truncation error is

\[ \frac{1}{6} h^6 u^{(6)}(\zeta_j), \quad x_{j-2} < \zeta_j < x_{j+2}, \quad j = 2(1)N-2 \]

(ii) For fourth order method truncation error is

\[ - \frac{1}{720} h^8 u^{(8)}(\zeta_j), \quad x_{j-2} < \zeta_j < x_{j+2}, \quad j = 2(1)N-2 \]

and (iii) The truncation error of the sixth order method is

\[ \frac{1}{3024} h^{10} u^{(10)}(\zeta_j), \quad x_{j-2} < \zeta_j < x_{j+2}, \quad j = 2(1)N-2 \]

(iv) The truncation error of the developed boundary formula (4.3) is

\[ t_i = \left( -\frac{29}{100800} \right) h^{10} u^{(10)}(\zeta_j), \quad i = 1, N \]

4.6 CONVERGENCE OF THE METHOD:

The proof of convergence for fourth order methods have been given by Usmani [113] by following Usmani here we prove the convergence of sixth order method (equations (4.4) (4.5) and (4.8)). Let us write the error equation of our sixth
order method in matrix form:

\[ AE = T \] \hspace{1cm} \ldots(4.10)\]

where \( E = (e_j) \) is the \((N-1)\)-dimensional column vector which is the amount of deviations of computed solution from the actual solution, and \( A \) is a five-band matrix, described as follows

\[ A = M + B h^4 \text{diag}(f_j) \] \hspace{1cm} \ldots(4.11)\]

where Matrix \( M \) is defined below:

\[
m_{1,1} = m_{N-1,N-1} = (-9 + \frac{91}{11760} F_i),
\]

\[
m_{1,2} = m_{N-1,N-2} = (-9/2 + \frac{7}{672} F_i),
\]

\[
m_{1,3} = m_{N-1,N-3} = \frac{17}{5040} F_i, \quad i=0,N
\]

otherwise,

\[
m_{j,k} = \begin{cases} 
6 & j=k \\
-4 & |j-k| = 1 \\
1 & |j-k| = 2 \\
0 & |j-k| > 2 
\end{cases} \quad \ldots(4.12)\]

and the Matrix \( B \) is of the form \( B = (b_{j,k}) \) with

\[
b_{1,1} = b_{N-1,N-1} = (-\frac{67}{112} + F_i \frac{811}{80640}),
\]

\[
b_{1,2} = b_{N-1,N-2} = (-\frac{191}{1120} + F_i \frac{359}{627200}),
\]

\[
b_{1,3} = b_{N-1,N-3} = (\frac{19}{15120} + F_i \frac{37}{10886400}), \quad i=0,N
\]
otherwise,

\[
b_{j,k} = \begin{cases} 
\frac{474}{720}, & j=k = 1(1)N-2 \\
\frac{124}{720}, & |j-k| = 1 \\
- \frac{1}{720}, & |j-k| = 2 \\
0, & |j-k| > 2
\end{cases} \quad \ldots(4.13)
\]

Usmani [113] has shown that \( \mathbf{M} \) is a monotone matrix and

\[
\| \mathbf{M}^{-1} \| = \max_j \sum m_{j,k}^* \leq N^4 (1+8/N^3)/384 \quad \ldots(4.14)
\]

where \( \mathbf{M}^{-1} = (m_{j,k}^*) \).

Now from (4.10) and (4.11), we obtain

\[
\mathbf{E} = \mathbf{A}^{-1} \mathbf{T} = (\mathbf{M} + \mathbf{BF})^{-1} \mathbf{T} = (\mathbf{I} + \mathbf{M}^{-1} \mathbf{BF})^{-1} \mathbf{M}^{-1} \mathbf{T}, \quad \text{where } \mathbf{F} = h^4 \text{diag}(f_j).
\]

\[
\| \mathbf{E} \| = \| (\mathbf{I} + \mathbf{M}^{-1} \mathbf{BF})^{-1} \mathbf{M}^{-1} \mathbf{T} \| 
\leq \| (\mathbf{I} + \mathbf{M}^{-1} \mathbf{BF})^{-1} \| \cdot \| \mathbf{M}^{-1} \| \cdot \| \mathbf{T} \|
\leq \frac{\| \mathbf{M}^{-1} \| \cdot \| \mathbf{T} \|}{1 - \| \mathbf{M} \| \cdot \| \mathbf{B} \| \cdot \| \mathbf{F} \|} \quad \ldots(4.15)
\]

provided \( \| \mathbf{M}^{-1} \| \cdot \| \mathbf{B} \| \cdot \| \mathbf{F} \| < 1 \). For proof see Froberg [40].

By Gershgorin's theorem, eigenvalues of \( \mathbf{B} \) given in (4.13) lie inside the circle.

\[
| \lambda - \frac{474}{720} | = 25/72 < 1, \quad \text{hence } \| \mathbf{B} \| < 1 \quad \ldots(4.16)
\]

Also \( \| \mathbf{T}_i \| = \text{Max} \ | t_{i} | = \text{Max} \ \left\{ \frac{1}{3024} h^{10} u^{(10)}(\zeta), \frac{29}{100800} h^{10} u^{(10)}(\zeta) \right\} \)

\[
\leq \frac{1}{3024} h^{10} M_{10} \quad \ldots(4.17)
\]
where \( M_{10} = \max_{\zeta} \left| u^{(10)}(\zeta) \right| \)

also we have \( \| F \| = h^4 \max_{x} |f(x)| = h^4 f_M \) \( \cdots \) (4.18)

Using (4.14), (4.16), (4.17) and (4.18) in equation (4.15) we obtain

\[
\| E \| \leq K h^6 \cdots \) (4.19)
\]

where \( K = \frac{1}{3024} M_{10} \lambda (1 - f_M \lambda)^{-1} \), \( \lambda = \frac{(b-a)^4 + 8(b-a)h^3}{384} \)

provided that \( f_M < 1/\lambda \).

Thus \( \| E \| = O(h^6) \), as \( h \to 0 \)

Therefore the method defined by (4.8) for the numerical solution of the system (4.1) is convergent. We summarize the above results in the following theorem.

**THEOREM 4.1:**

Let \( u(x) \) be the exact solution of the system (4.1) and let \( u_j, j=1(1)N \), be the computed solution. If \( E \) is given by (4.10) and \( f(x) \) satisfies

\[
\max_{x \in [a,b]} |f(x)| < \frac{384}{(b-a)^4 + 8h^3(b-a)}
\]

and \( \| M \| \cdot \| B \| \cdot \| F \| < 1 \),

where \( M \) and \( B \) defined in (4.12), (4.13) and \( F = h^4 \text{diag}(f_j) \)

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then $\|E\|=O(h^6)$ and satisfies (4.19), neglecting all round-off errors.

4.7 NUMERICAL RESULTS:

In this section we present the results obtained by the three numerical methods discussed in section 2 to the following two-point boundary value problems.

(i) $u^{(4)}+xu=-(8+7x+x^3)e^x$ ...(4.20)

subject to the boundary conditions

$u(0)=u(1)=0,$ \quad $u'(0)=1,$ \quad $u'(1)=-e$

the analytical solution of (4.20) is $u(x) = x(1-x)e^x$

(ii) $u^{(4)}+4u=1$ ...(4.21)

subject to the boundary conditions

$u(\pm 1)=0,$ \quad $u'(-1)=-u'(1)=(\sinh 2-\sin 2)/4(\cosh 2+\cos 2)$

the analytical solution of (4.21) is

$u(x)=.25[1-2(\sin 1 \sinh 1 \sin x \sinh x + \cos 1 \cosh 1 \cos x$

$cosh x )/(\cos 2 +cosh 2)]$

and with $f(x)$ changing sign on $[a,b]$ we consider

(iii) $u^{(4)}-xu=-(11+9x+x^2-x^3)e^x,$ $x \in [-1,1]$ ...(4.22)

$u(\pm 1)=0,$ \quad $u'(-1)=2e^{-1},$ \quad $u'(1)=-2e$

the analytical solution of (4.22) is $u(x)=(1-x^2)e^x$. 

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We solve the problem (4.20) over interval \([0,1]\), and both the problems (4.21) and (4.22) over interval \([-1,1]\) with step lengths \(h=2^{-m}, m=2(1)6\) and the maximum absolute errors are listed in tables(I,II).

The results of our sixth order method (five-diagonal scheme) are nearly same as those of Usmani's sixth order method (nine-diagonal scheme). And superior to sixth order collocation method of Irodmtou-Ellina and Houstis [50].

It is verified from table that on reducing the step-size from \(h\) to \(h/2\), the maximum observed error \(|E|\) is approximately reduced by a factor \((1/2)^n\), where \(n\) is the theoretical order of numerical method.

All computations were carried out on a PC-386 in double precision.
Table I

Maximum errors in solution of problems for $h=2^{-m}$, $m=1(1)6$

Variable coefficients problem (4.20)

<table>
<thead>
<tr>
<th>m</th>
<th>2nd order</th>
<th>4th order</th>
<th>our 6th order (five-diagonal)</th>
<th>6th order in [50]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.158(-2)</td>
<td>0.764(-6)</td>
<td>........</td>
<td>........</td>
</tr>
<tr>
<td>3</td>
<td>0.349(-3)</td>
<td>0.783(-7)</td>
<td>0.502(-9)</td>
<td>0.331(-7)</td>
</tr>
<tr>
<td>4</td>
<td>0.859(-4)</td>
<td>0.516(-8)</td>
<td>0.787(-11)</td>
<td>0.213(-9)</td>
</tr>
<tr>
<td>5</td>
<td>0.214(-4)</td>
<td>0.325(-9)</td>
<td>0.384(-13)</td>
<td>0.334(-11)</td>
</tr>
<tr>
<td>6</td>
<td>0.537(-5)</td>
<td>0.203(-10)</td>
<td>0.453(-12)*</td>
<td>0.333(-12)*</td>
</tr>
</tbody>
</table>

Constant coefficient problem (4.21)

<table>
<thead>
<tr>
<th>m</th>
<th>2nd order</th>
<th>4th order</th>
<th>6th order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.177(-2)</td>
<td>0.428(-5)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.406(-3)</td>
<td>0.418(-6)</td>
<td>0.143(-7)</td>
</tr>
<tr>
<td>4</td>
<td>0.100(-3)</td>
<td>0.265(-7)</td>
<td>0.187(-9)</td>
</tr>
<tr>
<td>5</td>
<td>0.250(-4)</td>
<td>0.298(-9)</td>
<td>0.168(-9)</td>
</tr>
<tr>
<td>6</td>
<td>0.622(-5)</td>
<td>0.157(-8)</td>
<td>......*</td>
</tr>
</tbody>
</table>

* Round off error effect.
Table: II

maximum errors in solution of problem (4.22), with \( f(x) \)
changes sign in \([a,b]\), for \( h=2^{-m} \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>2nd order</th>
<th>4th order</th>
<th>6th order method</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.670(-1)</td>
<td>.128(-3)</td>
<td>........</td>
</tr>
<tr>
<td>3</td>
<td>.145(-1)</td>
<td>.127(-4)</td>
<td>.331(-6)</td>
</tr>
<tr>
<td>4</td>
<td>.358(-2)</td>
<td>.825(-6)</td>
<td>.515(-8)</td>
</tr>
<tr>
<td>5</td>
<td>.900(-3)</td>
<td>.543(-7)</td>
<td>.810(-10)</td>
</tr>
<tr>
<td>6</td>
<td>.225(-3)</td>
<td>.340(-8)</td>
<td>.122(-11)*</td>
</tr>
</tbody>
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