CHAPTER III

SPLINE METHODS FOR SOLUTION OF A LINEAR FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEM

3.1 INTRODUCTION:

We consider the two-point boundary value problem of the form

\[ \frac{d^4 u}{dx^4} + f(x)u = g(x) , f(x) \geq 0 , \text{ for } x \in [a,b] \]

with the boundary conditions:

\[ u(a) = \alpha_1 , \quad u(b) = \alpha_2 , \]
\[ u''(a) = \beta_1 , \quad u''(b) = \beta_2 \]

\[ \ldots (3.1) \]

A particular case of this differential equation, often occurs in plate deflection theory such as the problem of bending of a uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges [110, p. 30].

It may not be possible to find the analytical solution of the system (3.1) for all \( f(x) \) and \( g(x) \). In such difficult
situations we have to approximate the solutions numerically.

The numerical methods based on finite differences by which the solution of ordinary differential equations are approximated over a finite set of grid points \( x_j \in [a, b] \), have been developed by many authors [60, 112]. Usmani and Marsden [112] derived a difference scheme of order two for the solution of system (3.1), which leads to a five diagonal linear system, Jain et al [60] have developed methods of order two, four and six, making use of quadrature.


Second order nodal collocation methods based on cubic splines are developed in [42]. In [97] second order collocation methods are considered which are based on quintic spline for solving a subclass of linear and non-linear fourth order problems.

In this Chapter we have obtained a class of methods of order two, four, and six, by using a non-polynomial parametric quintic spline, which reduces into quintic spline as the parameter tends to zero. In section 2 we formulate our method which lead to five diagonal linear system. In section 3 we
develop boundary formula of order six. In Section 4 a class of methods are given. In section 5 truncation error are given. In section 6 convergence analysis of the six order method is given. Finally in Section 7 numerical results are tabulated to show the superiority of our method.

3.2 THE METHOD:

We introduce the set \( \{ x_j \} \), so that \( x_j = a + jh \), \( h = (b-a)/N \), \( j = 0(1)N \).

The differential equation in (3.1) can be discretized by using the spline relations (2.63-iii). We obtain

\[
(1 + ph \frac{f_{j+2}}{f_{j+1}}) u_{j+2} + (-4 + gh \frac{f_{j+1}}{f_{j+2}}) u_{j+1} + (6 + sh \frac{f_j}{f_{j+1}}) u_j +
\]

\[
(-4 + gh \frac{f_{j-1}}{f_{j-2}}) u_{j-1} + (1 + ph \frac{f_{j-2}}{f_{j-1}}) u_{j-2} = h^4 \left[ p(g_{j+2} + g_{j-2}) + q(g_{j+1} + g_{j-1}) + sg_j \right], \quad j = 2(1)N-2 \quad \ldots (3.2)
\]

For discretization of boundary conditions, we define
where $b_k$, $c$ and $d_k$ are arbitrary parameters to be determined.

Now for $(b_0, b_1, b_2, b_3) = (-2, 5, -4, 1)$, $c = 1$ and
$(d_0, d_1, d_2, d_3) = (-1/360)(28, 245, 56, 1)$ the difference equations (3.3) become

(i) \[ -2u_0 + 5u_1 - 4u_2 + u_3 + h^2 u'' + h^4 \sum_{k=0}^{3} d_k u_{N-k} = 0 \]

(ii) \[ -2u_N + 5u_{N-1} - 4u_{N-2} + u_{N-3} + h^2 u'' + h^4 \sum_{k=0}^{3} d_k u_{N-k} = 0 \]

With truncation error $t_1 = (241/60480)h^8 u^{(8)}(\zeta)$

By replacing $u_0''$ by $\bar{u}_1$, $u_N''$ by $\bar{u}_2$ and eliminating the fourth derivatives of $u$ by using system (3.1) we obtain
The system (3.2) gives us N-3 equations for the N-1 unknowns \( u_j \), \( j=1(1)N-1 \). From equation (i) and (ii) in (3.5) we obtain two more relations. Therefore the equations (3.2) and [(3.5)(i,ii)] form an \((N-1)\times(N-1)\) linear system to be solved.

To retain the band width of the coefficient matrix as five, when we use the sixth order method we have to develop a formula with truncation error \( O(h^{10}) \). The following identities are developed. See Henrici [47] chapter 5 and 6.

3.3 DEVELOPMENT OF THE BOUNDARY FORMULAS:

We assume that the solution \( u(x) \) of the given system (3.1) has sufficiently high order derivatives. Then we define the following identities:
\[(i) \sum_{k=0}^{3} b_k u_k + ch^2 u''_0 + h^4 \sum_{k=0}^{3} d_k u_k^{(4)} + e_1 h^5 u_0^{(5)} + e_2 h^6 u_0^{(6)} + t_1 = 0\]

\[(ii) \sum_{k=0}^{3} b_k^* u_k + c h u_0 + h^4 \sum_{k=0}^{4} d_k^* u_k^{(4)} + e_1^* h u_0^{(5)} + e_2^* h^6 u_0^{(6)} + t_1^* = 0\]

...(3.6)

In order that \(t_1\) and \(t_1^*\) is \(O(h^{10})\), we find that,

\[[b_0, b_1, b_2, b_3, c] = [-2, 5, -4, 1, 1],\]

\[[d_0, d_1, d_2, d_3, e_1, e_2] = (\frac{1}{45360})[-3536, -30375, -7722, 53, 300, 279]\]

and also \([b_0^*, b_1^*, b_2^*, b_3^*, c^*] = [-11/2, 9, -9/2, 1, -3]\), and

\[[d_0^*, d_1^*, d_2^*, d_3^*, e_1^*, e_2^*] = (\frac{1}{30240})[529, -18090, -5157, 38, 1074, 288]\]

and truncation error are \(t_1 = \frac{241}{907200} h^9 u^{(10)}(\zeta),\)

\(t_1^* = \frac{29}{100800} h^9 u^{(10)}(\zeta).\)

Eliminating \(u_0\) which is not specified from both the equations in (3.6) and simplifying, we obtain
\[
(5 + \frac{30375}{45360} h^4 f_1^4 - \gamma_0 (9 + \frac{18090}{30240} h^4 f_1^4))u_1 + \left[- \frac{7722}{45360} h^4 f_2^4 - \gamma_0 \right] u_2 + \left[- \frac{9517}{2} \frac{53}{30240} h^4 f_3^4 - \gamma_0 \left(1 - \frac{38}{30242} h^4 f_3^4\right)\right] u_3
\]
\[+ h^4 [G_0 - \gamma_0 G^*_{\gamma_0}] + (\phi_0 - \gamma_0 \phi^*_{\gamma_0}) + \left[\frac{1241}{2} \frac{29}{907200} - \gamma_0 \frac{29}{100800}\right] h^{10} u(10) = 0
\]  

\[(3.7)\]

where

\[\gamma_0 = \frac{h^4 (300f_0^2 + 558hf')}{(1611h)^4 f_0^3 + 864h^5 f_0^3 + 136080}\]

G_0 = (1/45360)(-3536g - 30375g_1 - 7722g_2 + 53g_3),

\[\phi_0 = h^5 \left[300(-f'_{00} + g_{0}) + 279h(-f''_{00} - f''_{00} + g_{0})\right]/45360 +
\]
\[(-2 + \frac{3536}{45360} \frac{h^4 f_0^4}{h^4 f_0^4})\alpha_1 + h^2 \ell_1^2,\]

G^*_{\gamma_0} = (529g_0 - 18090g_1 - 5157g_2 + 38g_3)/30240,

\[\phi^*_{\gamma_0} = h^5 \left[1074(-f'_{00} + g_{0}) + 288h(-f''_{00} - f''_{00} + g_{0})\right]/30240 -
\]
\[\left(11/2 + \frac{529}{30240} \frac{h^4 f_0^4}{h^4 f_0^4}\right)\alpha_1 \right) \]  

\[(3.8)\]

obviously as \(h \to 0\), \(\gamma \to 0\).
For right hand boundary in the same manner we obtain

\[
\begin{align*}
\left[ (5 + \frac{30375}{45360} h f_{N-1}) - \gamma_N \left(9 + \frac{18090}{30240} h f_{N-1}\right) \right] u_{N-1} + \\
\left[ (-4 + \frac{7722}{45360} h f_{N-2}) - \gamma_N \left(-\frac{9}{2} + \frac{5157}{30240} h f_{N-2}\right) \right] u_{N-2} + \\
\left[ (1 - \frac{53}{45360} h f_{N-3}) - \gamma_N \left(1 - \frac{38}{30240} h f_{N-3}\right) \right] u_{N-3} + h^4 \left( G_{jN} - \gamma_N G_{jN} \right) + \\
\left[ \phi_{jN} - \gamma_N \phi_{jN}^* \right] + \left[ \frac{241}{907200} - \gamma_N \frac{29}{100800} \right] h_{10}^0 (10) (\zeta) = 0 \\
\end{align*}
\]

Where

\[
G_{jN} = \left( \frac{1}{45360} \right) (-3536 g_N - 30375 g_{N-1} - 7722 g_{N-2} + 53 g_{N-3}),
\]

\[
G_{jN}^* = (1/30240)(529 g_N - 18090 g_{N-1} - 5157 g_{N-2} + 38 g_{N-3}),
\]

\[
\phi_{jN} = \frac{h^5}{45360} \left[ -300 (-f' u_N + g'_N) + 279 h (-f'' u''_N - f''' u'''_N + g''''_N) \right] + \\
\left( -2 + \frac{3536}{45360} h f N^4 \right) x_2 + h^2 l^2,
\]

\[
\phi_{jN}^* = \frac{h^5}{30240} \left[ -1074 (-f' u_N + g'_N) + 288 h (-f'' u''_N - f''' u'''_N + g''''_N) \right] + \\
\left( -\frac{11}{2} + \frac{529}{30240} h f N^4 \right) x_2,
\]

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\[ \gamma_N = \frac{h^4 (300f - 558hf')}{(1611h^4f - 6645f' + 136080)} \] ...(3.10)

3.4 **A Class of Methods:**

All the previous methods are special cases of our method, for different choices of parameters p, q and s.

(i) If we choose \( p=q=0 \) and \( s=1 \), we get the scheme

\[ \delta^4 u_j = h^4 u_j^{(4)}, \quad j=2(1)N-2 \] of second order with truncation error

\[ T_j = \frac{1}{6} h^6 M_6, \quad \text{where} \quad M_6 = \max_{a \leq x \leq b} |u^{(6)}(x)| \] ...(3.11)

Which is developed by Usmani and Marsden [112].

(ii) If we choose \( p=0, \ s=2/3, \ q=1/6 \), we obtain the fourth order finite difference method

\[ \delta^4 u_j = \frac{h^4}{6} \left[ u_{j+1}^{(4)} + 4 u_j^{(4)} + u_{j-1}^{(4)} \right], \quad j=2(1)N-2 \] ...(3.12)

and (iii) If we take \( (p,q,s)=\frac{1}{720}(-1,124,474) \) we get the sixth order method

\[ \delta^4 u_j = \frac{h^4}{720} \left[ -(u_{j+2}^{(4)} + u_{j-2}^{(4)}) + 124(u_{j+1}^{(4)} + u_{j-1}^{(4)}) + 474u_j^{(4)}\right], j=2(1)N-2 \] ...(3.13)
Fourth and sixth order methods have been obtained by Jain et al [60]. Mention that their boundary equations are only fourth order. Using (3.1), (3.5) and (3.13), our sixth order scheme depending on five consecutive mesh point is

(i) \((5+ \frac{245}{360} h^4 f_1)u_1 + (-4+ \frac{56}{360} h^4 f_2)u_2 + (1+ \frac{h^4}{360} f_3)u_3 = \)

\((2- \frac{28}{360} h^4 f_0)\alpha_1 - h^2 f_1 + \frac{h^4}{360}(28g_0 + 245g_1 + 56g_2 + g_3)\)

(ii) \((1- \frac{h^4}{720} f_{j+2})u_{j+2} + (-4+ \frac{124}{720} h^4 f_{j+1})u_{j+1} + (6+ \frac{474}{720} h^4 f_j)u_j + \)

\((-4+ \frac{124}{720} h^4 f_{j-1})u_{j-1} + (1- \frac{h^4}{720} f_{j-2})u_{j-2} = \)

\(\frac{h^4}{720} [474g_j + 124(g_{j+1}g_{j-1}) - (g_{j+2} + g_{j-2})] \quad j=2(1)N-2\)

(iii) \((1+ \frac{1}{360} h^4 f_{N-3})u_{N-3} + (-4+ \frac{56}{360} h^4 f_{N-2})u_{N-2} + (5+ \frac{245}{360} h^4 f_{N-1})u_{N-1} = \)

\((2- \frac{28}{360} h^4 f_N)\alpha_2 - h^2 f_2 + \frac{h^4}{360}(28g_N + 245g_{N-1} + 56g_{N-2} + g_{N-3})\)

\(\ldots (3.14)\)
Equation (3.14) forms an \((N-1)\times(N-1)\) system. This system can be written in matrix form as

\[ A U = R \] ... (3.15)

where \(A = (a_{j,k})\) is a five band matrix of order \((N-1)\times(N-1)\) given by

\[
a_{j,k} = \begin{cases} 
6 + \frac{474}{720} h^4 f_j & \text{when } j=k=2(1)N-2 \\
-4 + \frac{124}{720} h^4 f_{j-1} & \text{when } k-j=1, j=2(1)N-2 \\
1 - \frac{h^4}{720} f_{j-2} & \text{when } j-k=2, j=3(1)N-2 \\
1 - \frac{h^4}{720} f_{j+2} & \text{when } k-j=2, j=2(1-)N-3 \\
0 & |j-k| > 2
\end{cases}
\]

and \(a_{1,1} = 5 + \frac{245}{360} h^4 f_1, a_{1,2} = -4 + \frac{56}{360} h^4 f_2, a_{1,3} = 1 + \frac{h^4}{360} f_3 \)

\[ a_{N-1,N-1} = 5 + \frac{245}{360} h^4 f_{N-1}, a_{N-1,N-2} = -4 + \frac{56}{360} h^4 f_{N-2}, \]

\[ a_{N-1,N-3} = 1 + \frac{h^4}{360} f_{N-3} \] ... (3.16)
and \( R \) is the Column vector \([r_1, r_2, ..., r_{N-1}]^T\) given by

(i) \( r_1 = \left(2 - \frac{28}{360} h \int_0^1 f_o, h^4 \beta_1 + \frac{h^4}{360} (28g_o + 245g_1 + 56g_2 + g_3)\right) \)

(ii) \( r_2 = \frac{h^4}{720} [474g_2 + 124(g_3 + g_1) - (g_4 - g_0)] - (1 - \frac{h^4}{720} f_o)\alpha_1 \), \( j = 3(1)N-3 \)

(iii) \( r_j = h^4 [s g_j + q (g_{j+1} + g_{j-1}) + p (g_{j+2} + g_{j-2})] \), \( J = 3(1)N-3 \)

(iv) \( r_{N-2} = \frac{h^4}{720} [474g_{N-2} + 124(g_{N-3} + g_{N-1}) - (g_{N-4} + g_N)] - (1 - \frac{h^4}{720} f_N)\alpha_2 \)

(v) \( r_{N-1} = \left(2 - \frac{28}{360} h \int_0^1 f_o, h^4 \beta_1 + \frac{h^4}{360} (28g_N + 245g_{N-1} + 56g_{N-2} + g_{N-3})\right) \)

where \( u_o = \alpha_1, u_1 = \alpha_2, u''_o = \beta_1, u''_1 = \beta_2 \)

The above system can be solved by Gauss elimination method or any other suitable method.

\[3.5\] TRUNCATION ERRORS:

Truncation errors of various methods are listed below:

(i) For second order scheme the truncation error is

\[
\frac{1}{6} h^6 u^{(6)}(\xi_j), \quad x_{j-2} < \xi_j < x_{j+2}, \quad j = 2(1)N-2
\]
(ii) For fourth order method truncation error is

\[- \frac{1}{720} h^8 u^{(8)}(\xi_j), \quad x_{j-2} < \xi_j < x_{j+2}, \quad j=2(1)N-2\]

and (iii) The truncation error of the sixth order method is

\[- \frac{1}{3024} h^{10} u^{(10)}(\xi_j), \quad x_{j-2} < \xi_j < x_{j+2}, \quad j=2(1)N-2\]

(iv) The truncation error of the boundary formula (3.3) is

\[ t_i = \left(\frac{241}{60480}\right)h^8 u^{(8)}(\xi_j), \quad i=1,N \]

(v) The truncation error of the developed boundary formula (3.6) is

\[ t_i = \left(\frac{241}{907200} + \frac{29}{100800}\right) h^{10} u^{(10)}(\xi), \quad i=1,N \]

as \( h \to 0 \), \( \xi_j \to 0 \)

3.6 CONVERGENCE OF THE METHOD:

The proof of convergence for second and fourth order methods have been given by Usmani and Marsden [112] and Jain et al [60] respectively. Here we prove the convergence of sixth order scheme. Let us write the error equation of sixth
order method as

\[ \text{AE} = \text{T} \quad \ldots(3.17) \]

where \( E = (e_j) \), is the \((N-1)\)-dimensional column vector with \( e_j \), the error of discretization defined by \( e_j = u(x_j) - u_j \). In other words \( e_j \) is the amount by which computed solution \( u_j \) deviates from the actual solution \( u(x_j) \) at \( x = x_j \) and \( A \) is a five-band matrix which can be described as

\[ A = M + B F, \quad F = h^4 \text{diag}(f_j) \quad \ldots(3.18) \]

Elements of \( A = (a_{j,k}) \) are given by equation (3.16). The matrix \( M \) is of the form

\[
m_{j,k} = \begin{cases} 
6 & j=k \\
-4 & |j-k|=1 \\
1 & |j-k|=2 \\
0 & |j-k| > 2 
\end{cases} \quad \ldots(3.19)
\]

and the matrix \( B \) is of the form \( B = B + C \), \( B = (b_{j,k}) \).
\( b_{j,k} = \begin{cases} 
\frac{474}{720}, & j = k = 1(1)N-1 \\
\frac{124}{720}, & |j-k| = 1 \\
-\frac{1}{720}, & |j-k| = 2 \\
0, & |j-k| > 2 
\end{cases} \) \( \ldots(3.20) \)

and \( C_{1,1} = (57-48375 \gamma_j)/5040 \)

\( C_{1,2} = (-40+87282 \gamma_j)/20160 \)

\( C_{1,3} = (10-45303 \gamma_j)/45360 \)

\( C_{N-1,N-1} = (57-48375 \gamma_j)/5040 \)

\( C_{N-1,N-2} = (-40+87282 \gamma_j)/20160 \)

\( C_{N-1,N-3} = (10-45303 \gamma_{jN})/45360 \)

and \( C_{j,k} = 0 \) otherwise

Since \( f(x) \geq 0 \) for \( x \in [a,b] \), we have \( B F = h^4 B \text{ diag}(f_j) \geq 0 \) and this implies that \( A > M \). We have to show that matrix \( M \) and under certain conditions, matrix \( A \) are monotone or to show that the elements of \( M^{-1} \) and \( A^{-1} \) are non-negative. Let us consider the matrix \( P \) such that
\[ P_{j,k} = \begin{cases} 2 & \text{if } j = k \\ -1 & \text{if } |j-k| = 1 \\ 0 & \text{if } |j-k| = 2 \end{cases} \quad \ldots \quad (3.21) \]

we can verify that \( M = P^2 \) (See [112]). Then we have from (3.18) that

\[ A = M + BF = P^2 + D, \quad \text{where } D = BF \]

Usmani and Marsden [112] have shown that

\[ P^2 A^{-1} = [I - DP^{-2}] [I + (DP^{-2})^2 + (DP^{-2})^4 + \ldots] \quad \ldots (3.22) \]

Let \( F = h^4 f_M B \), where \( f_M = \text{Max } f(x) \) on \([a,b]\).

By Gershgorin's theorem eigenvalues of \( B \) lie inside the circle

\[ \left| \lambda - \frac{474}{720} \right| = \frac{25}{72} < 1, \text{ hence} \]

\[ \left| \rho(F) \right| = \left| h^4 f_M \rho(B) \right| \leq \frac{25}{72} h^4 f_M \]

we know that \( \rho(P^{-1}) = \frac{1}{4} \csc^2 \left( \frac{\pi}{2N} \right) \leq \frac{N^2}{8} \leq \), then

\[ \rho(DP^{-2}) \leq \rho(D) \rho(P^{-2}) \leq \rho(F) \rho^2 (P^{-1}) \leq \frac{25}{4608} h^4 f_M N^4 \quad \ldots (3.23) \]
if \( f_M < \frac{4608}{25(b-a)^4} \) the system (3.23) will converge. The matrix \( P \) is monotone (see Henrici [47]) and hence \( M = P^2 \) is also monotone.

Now if \( P^{-2} > P^{-2} F P^{-2} \) ... (3.24)

Then \( G = P^{-2} - P^{-2} F P^{-2} \) is a positive matrix and hence \( A^{-1} = GM \), where \( M = I + a \) positive matrix, will also be a positive matrix.

Let \( B = B + C \), where \( b_j,k \) and \( c_j,k \) are given in (3.20) and \( P^{-2} = (a^*_{j,k}), \ P^{-2} B P^{-2} = (b^*_{j,k}) \) and \( P^{-2} C P^{-2} = (c^*_{j,k}) \) we know that \([112]\)

\[
\frac{k(N-j)}{6} \left[ 2j + \frac{1}{N} - \frac{j^2+k^2}{N} \right] > 0, \quad j \geq k
\]

\[
\frac{j(N-k)}{6} \left[ 2k + \frac{1}{N} - \frac{j^2+k^2}{N} \right] > 0, \quad j \leq k
\]

We know that \( (b^*_{j,k}) \) is symmetric

\[
b^*_{j,k} = \frac{474}{720} \sum_{i=1}^{N-1} a^*_{j,i} a^*_{i,k} + \frac{124}{720} \sum_{i=1}^{N-2} a^*_{j,i} a^*_{i+1,k} + \frac{1}{720} \sum_{i=2}^{N-1} a^*_{j,i} a^*_{i-1,k}, \quad j \geq k
\]
and  \( C_{j,k}^* = a_{j,1}^* (c_{1,1}, a_{1,k}^* + c_{1,2}^* a_{2,k} + c_{1,3}^* a_{3,k}) + \)

\[
\begin{align*}
&- a_{j,N-1}^* (c_{N-1,1}, a_{N-3,k}^* + c_{N-1,2} a_{N-2,k} + c_{N-1,3} a_{N-1,k}) \\
&\text{Since all } a_{j,k}^* > 0, \text{ it implies that } C_{j,k}^* \geq 0
\end{align*}
\]

From (3.20) and (3.24) it follows that \( A \) will be monotone if

\[
a_{j,k}^* \geq h^4 f \left( b_{j,k}^* + C_{j,k}^* \right)
\]

or \( f \leq \frac{a_{j,k}^*}{h^4 (b_{j,k}^* + C_{j,k}^*)} \leq \frac{a_{j,k}^*}{h^4 b_{j,k}^*} \), since \( C_{j,k}^* \geq 0 \)

We can conclude that \( A \) is monotone if

\[
f_M < \min \left[ \frac{a_{j,k}^*}{h^4 b_{j,k}^*}, \frac{4608}{25 (b-a)^4} \right]
\]

From (3.17) it follows that \( \| E \| \leq A^{-1} \| T \| \). Since the matrices \( A \) and \( M \) are both monotone and \( A > M \) it follows[47] that \( A^{-1} < M^{-1} \)

\[
\| E \| \leq M^{-1} \| T \|
\]

In order to derive a bound on \( |e_j| \) and a bound on \( \| E \| = \max |e_j| \), by following [112] we obtain
\[ |e_j| \leq \max \left| \sum_{i=1}^{N-1} a_{j,k} \right| \cdot h^{10} \sum_{k=1}^{N-1} a_{j,k} \leq \max \left\{ \frac{1}{3024} h^{10} u^{(10)}(\xi) \right\} \cdot \left( \frac{241}{207200} + \frac{29}{100800} \right) h^{10} u^{(10)}(\xi) \cdot \sum_{k=1}^{N-1} a_{j,k} \]

But as \( h \to 0 \), \( \gamma_j \to 0 \), then

\[ |e_j| \leq \frac{1}{3024} h^{10} M_{10} \sum_{k=1}^{N-1} a_{j,k} \leq \frac{1}{3024} \times \frac{25}{4608} (b-a)^4 h^6 M_{10}^6 \]

\[ \leq \frac{25}{13934592} h^6 M_{10}^6 (b-a)^4 \quad \cdots (3.25) \]

Hence

\[ \| E \| = O(h^6) \quad \text{as} \quad h \to 0 \]

Therefore the method defined by (3.8) for the numerical solution of the system (3.1) is convergent.

We summarize the above results in the following theorem.

**THEOREM 3.1:**

Let \( u(x) \) be the exact solution of the system (3.1) and let \( u_j, j=1(1)N \), be the computed solution. If \( E \) is given by (3.17) and \( f(x) \) satisfies
\[ f_M < \text{Min.} \left[ \frac{a_{j,k}^*}{h^4 b_{j,k}^*}, \frac{4608}{25(b-a)^4} \right] \]

then \( \| E \| = 0(h^6) \) and satisfies (3.25), neglecting all errors due to round-off.

### 3.7 Numerical Results:

In this section we present the results obtained by the three numerical methods discussed in section 3 to the following two-point boundary value problems.

**Example (I):**

\[ u^{(4)} + xu = -(8+7x+x^3)e^x \quad \ldots(3.26) \]

subject to the boundary conditions

\[ u(0)=0 \quad , \quad u(1)=0 \]
\[ u''(0)=0 \quad , \quad u''(1)=-4e \quad \ldots(3.27) \]

The analytical solution of (3.26) is \( u(x) = x(1-x)e^x \)

**Example (II):**

\[ u^{(4)} + 4u = 1 \quad \ldots(3.28) \]

Subject to the boundary conditions

\[ u(\pm 1)=0 \quad , \quad u' (\pm 1)=0, \quad \ldots(3.29) \]
The analytical solution is:
\[ u(x) = \frac{.25[1-2(\sin 1 \sinh 1 \sin x + \sinh x + \cos 1 \cosh 1 \cos x \cosh x) / (\cos 2 + \cosh 2)]] }{ \cos^2 x} \]

Example (III):
\[ u^{(4)} - xu = -(11+9x^2-x^3) \exp(x), \quad \ldots(3.30) \]
subject to the boundary conditions
\[ u(-1) = u(1) = 0, \quad u''(-1) = 2/e, \quad u''(1) = -6e \quad \ldots(3.31) \]
with exact solution \[ u(x) = (1-x^2) \exp(x). \]

As the problem 3.28 is symmetrical about the origin, we solve both the problems over interval \([0,1]\) with step lengths \( h = 2^{-n}, m = 3(1)5 \) and the maximum absolute errors are listed in table I. We solved the boundary value problem (3.30) with step lengths mentioned above the results are given in table II, and compare with result in Chawla [13].

It is verified from tables that on reducing the step-size from \( h \) to \( h/2 \), the maximum observed error \( |E| \) is approximately reduced by a factor \( (1/2)^n \), where \( n \) is the theoretical order of numerical method.
### TABLE I:
Maximum errors in solution of problems for $h=2^{-m}, m=3(1)5$

<table>
<thead>
<tr>
<th>Problem with variable coefficient</th>
<th>m</th>
<th>Second order</th>
<th>Fourth order</th>
<th>Sixth order method</th>
<th>Our</th>
<th>In [60]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>method</td>
<td>method</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1.47(-3)</td>
<td>5.49(-7)</td>
<td>2.47(-9)</td>
<td>1.913(-7)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>4.15(-4)</td>
<td>2.83(-8)</td>
<td>3.93(-11)</td>
<td>3.117(-9)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1.07(-4)</td>
<td>1.67(-9)</td>
<td>3.25(-13)</td>
<td>4.983(-11)</td>
<td></td>
</tr>
<tr>
<td>Problem with constant coefficients</td>
<td>3</td>
<td>2.48(-5)</td>
<td>1.47(-8)</td>
<td>1.87(-11)</td>
<td>5.073(-9)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>6.82(-6)</td>
<td>7.57(-10)</td>
<td>2.76(-13)</td>
<td>8.167(-11)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.74(-6)</td>
<td>4.47(-11)</td>
<td>2.99(-13)</td>
<td>1.302(-12)</td>
<td></td>
</tr>
</tbody>
</table>
TABLE II:

Maximum errors in solution of example III for $h=2^{-m}, m=3(1)5$

<table>
<thead>
<tr>
<th>m</th>
<th>Second order</th>
<th>Fourth order</th>
<th>Sixth order method</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4.9(-2)</td>
<td>7.5(-2)</td>
<td>5.8(-5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>9.8(-5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>8.2(-7)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6.9(-7)</td>
</tr>
<tr>
<td>4</td>
<td>1.5(-2)</td>
<td>1.9(-2)</td>
<td>3.8(-6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5.0(-6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.8(-8)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.2(-8)</td>
</tr>
<tr>
<td>5</td>
<td>4.3(-3)</td>
<td>4.7(-3)</td>
<td>2.5(-7)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.9(-7)</td>
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<td></td>
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<td></td>
<td>3.5(-10)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4.0(-10)</td>
</tr>
</tbody>
</table>