CHAPTER VI

SPLINE SOLUTION OF FOURTH-ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

6.1 INTRODUCTION:

We consider the problem of transverse vibrations of a uniform flexible beam of length \( L \), hinged at both ends, which represent the fourth order parabolic differential equation

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0 \quad \ldots (6.1)
\]

with appropriate initial and boundary conditions

\[
\begin{align*}
  u(x,0) &= g_1(x) ; & u_t(x,0) &= g_2(x) & , 0 \leq x \leq L \\
  u(0,t) &= f_1(t) ; & u(L,t) &= f_2(t) \\
  u_x(0,t) &= p_1(t) ; & u_x(L,t) &= p_2(t) & t \geq 0
\end{align*}
\]

or

\[
\begin{align*}
  u_{xx}(0,t) &= p_1(t) ; & u_{xx}(L,t) &= p_2(t) & \ldots (6.2)
\end{align*}
\]

where \( u \) is transverse displacement of the beam, \( t \) and \( x \) are time and distance variables respectively. We denote \( \frac{\partial^2 u}{\partial t^2} = u_{tt} \) and so on.

Numerical solution of equation (6.1) based on finite difference and reduction of (6.1) to a system of first order equations in \( t \), is given by Evans[34], Collatz[22], Fairweather and Gourlay [35], Richtmyer and Morton [92], Jain et al [58].
But difficulty arises if the bending moment is not prescribed at the end that is $x=0$ and $L$, in such a situation we cannot apply the above procedure.

We need to construct a direct numerical method for solution of equation (6.1). Direct explicit and implicit difference methods have been given by Crandall [25], Todd [111], Albrecht [4], Collatz [22], Jain et al [58], Jain [54]. The three level explicit direct method with order of accuracy $O(k^2 + h^2)$ given by Collatz [22] is stable when the mesh ratio $(k/h)^2 \leq \frac{1}{2}$.

The three level unconditionally stable formulas of accuracy $O(k^2 + h^2)$ and $O(k^2 + h^2 + (\frac{k}{h})^2)$ are given by Todd [111], Crandall [25] and Conte [23] respectively. Five level unconditionally stable explicit method with truncation error of $O(k^2 + h^2 + (\frac{k}{h})^2)$ has been given by Albrecht [4]. Recently direct and splitting approach finite difference methods have been proposed by Jain et al [58] and Jain [54].

We have derived new three level methods based on parametric quintic spline for the solution of fourth order parabolic partial differential equation governing transverse vibrations of uniform flexible beam in one and two space dimensions. In section 2 and 3 we present the formulation of our methods in one space dimension and analysis of stability.
We show that by choosing different values of parameters the previous known methods can be derived from our method. In section 4 we develop the method for two space dimension. Finally in section 5 numerical evidence is included to demonstrate the practical usefulness of our schemes and confirm their theoretical behaviour.

6.2 THE METHODS:

Let the region \( R \) be replaced by a set of points \( R_h \) which are the vertices of a grid of points \((j,m)\), where \( x=a+jh, \) \( j=0(1)N, \) \( Nh=b-a. \) \( j \) being an integer, and \( t=mk, m=0,1,2,3... \) The quantities \( k \) and \( h \) are mesh sizes in the time and space directions respectively.

We next develop an approximation for (6.1) in which the time derivative is replaced by a finite difference approximation and the space derivative by the parametric quintic spline function approximation. The equation (6.1) is then replaced by

\[
-k^2 (1+ \sigma^2 )^{-1} \frac{d^2}{dt^2} u_j^m + F_j^m = 0
\]  

...(6.3)

where \( \sigma \) is a parameter such that the finite difference approximation to the time derivative is \( O(h^2) \) for arbitrary \( \sigma \) and of \( O(h^4) \) for \( \sigma = 1/12 \) and \( \sigma = 1/4, 1/6 \) the finite difference approximations reduce to parametric cubic and
cubic spline relations respectively. Also \( F_j = S^{(4)}(x_j) \) and \( S(x) \) is the parametric quintic spline approximation given in Chapter 2, (equation 2.63). Using (2.63) and (6.3) we obtain

\[
\delta^2_t \left\{ p \left( u^m_{j+2} + u^m_{j-2} \right) + q \left( u^m_{j+1} + u^m_{j-1} \right) + \left( p + r^2 \delta^4_x \right) u^m_j \right\} + r^2 \delta^4_x u^m_j = 0
\]

where \( r = k/h^2 \), \( u^m_j = u(jh,mk) \), \( \delta^2_t u^m_j = u^m_{j+2} - 2u^m_{j+1} + u^m_{j-2} \), and \( p, q, s \) are arbitrary parameters.

After simplifying the above equation we obtain

\[
\{(2p+2q+s)+(4p+q)\delta^2_x + (p+r^2)\delta^4_x\} \delta^2_t u^m_j + r^2 \delta^4_x u^m_j = 0 \quad \ldots(6.4)
\]

Equation (6.4) may also be written as:

\[
(p+r^2)u^{m+1}_{j+2} + (q-4\sigma r^2)u^{m+1}_{j-2} + (s+6\sigma r^2)u^{m+1}_{j-1} + (q-4\sigma r^2)u^{m+1}_{j+1} + \\
(p+r^2)u^{m+1}_{j-2} + (-2p-2\sigma r^2 + r^2)u^{m}_{j-2} + (-2q+8\sigma r^2 - 4r^2)u^{m}_{j-1} + \\
(-2s-12\sigma r^2 + 6r^2)u^{m}_{j} + (-2q+8\sigma r^2 - 4r^2)u^{m}_{j+1} + \\
(p+r^2)u^{m-1}_{j-2} + (q-4\sigma r^2)u^{m-1}_{j-1} + (s+6\sigma r^2)u^{m-1}_{j} + (q-4\sigma r^2)u^{m-1}_{j+1} + \\
(p+r^2)u^{m-1}_{j+2} = 0
\]

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Equation (6.4) may be written in schematic form as:

\[
\begin{array}{cccccc}
P_2 & Q_2 & S_2 & Q_2 & P_2 \\
-2P_2+r^2 & -2Q_2-4r^2 & -2S_2+r^2 & -2Q_2-4r^2 & -2P_2+r^2 \quad u_{ij} = 0
\end{array}
\]

where \( P_2 = p + \sigma r^2 \), \( Q_2 = q - 4\sigma r^2 \), \( S_2 = s + 6\sigma r^2 \).

Expanding (6.4) in Taylor series in terms of \( u(x, t) \) and its derivatives and replacing the derivatives involving \( t \) by the relation

\[
\frac{\partial^{i+j} u(x, t)}{\partial x^i \partial t^j} = -\frac{\partial^{i+2j} u(x, t)}{\partial x^{i+2j}}
\]

We obtain the following relations :

\[
\delta_{x}^4 u(x, t, m) = D_6^4 + \frac{1}{6} D_8^6 + \frac{1}{80} D_{10}^8 + \frac{17}{3024} D_{12}^{10} + \frac{62}{10!} D_{12}^{12} + \ldots
\]

\[
\delta_{t}^2 u(x, t, m) = -r D_6^2 + \frac{1}{12} r D_{8}^6 - \frac{1}{360} r D_{12}^{8} + \frac{1}{20160} r D_{16}^{10} - \frac{2}{10!} r D_{20}^{12} + \ldots
\]

\[
\delta_{t}^2 \delta_{x}^4 u(x, t, m) = -r D_6^2 - \frac{1}{6} r D_{8}^4 - \frac{1}{80} r D_{10}^{6} + \frac{4}{72} r D_{12}^{12} - \frac{1}{360} r D_{16}^{14} - \frac{1}{10!} r D_{20}^{16} + \ldots
\]
\[ \delta^2_{x,t} u(x,t) = r^2_D^6 x + r^2_D^8 x + (r^4_D^2 x + r^2_D^6 x + r^2_D^10 x + \frac{2}{56} D^12 x + \ldots) \ldots \] (6.5)

where

\[ D^i_x = h^i (\frac{\partial^i u}{\partial x^i})^m_j \]

Using (6.4) and (6.5) we obtain the truncation error

\[ \tau^m_j = r^2 (1-2p-2q)D^4_x + r^2 \frac{1}{4} - 4p-qD^6_x + r^2 [(\frac{1}{80} - \frac{4}{3} - \frac{1}{12} q)]D^8_x + \ldots \ldots \] (6.6)

Using Von Newmann's method the characteristic equation of the scheme (6.4) is obtained as:

\[ \xi^2 + 2\gamma \xi + 1 = 0 \] ....(6.7)

where \[ \gamma = \frac{8r^2 \sin^4 \omega}{16(p+ar^2) \sin^4 \omega - 4(4p+q) \sin^2 \omega + (2p+2q+s)} \]

\[ \omega = \frac{1}{2} qh, \text{ where } q \text{ is the variable in the Fourier expansion.} \]
Applying the Routh-Hurwitz Criterion to (6.7) we get the necessary and sufficient conditions for (6.4) to be stable as:

\[-1 \leq 1 - \frac{8r^2 \sin^4 \omega}{16(p+\sigma r^2)\sin^4 \omega - 4(4p+q)\sin^2 \omega + (2p+2q+s)} \leq 1\]

Simplifying and putting \(2p+2q+s=1\) we obtain from the left inequality

\[4[4p(4\alpha-1)r^2] \sin^4 \omega - 4(4p+q)\sin^2 \omega + 1 \geq 0 \quad \ldots (6.8)\]

We deduce that the scheme (6.4) is unconditionally stable if \(\alpha \geq 1/4, q < 1/4\) and \(p=0\) or \(p \geq (4p+q)^2/4\)

and conditionally stable if

(i) \(\alpha < 1/4, q < 1/4, p=0, 0 < r^2 \leq \frac{1 - 4q}{4(1-4\alpha)}\) or

(ii) \(\alpha < 1/4, (4p+q)< 1/4, p = (4p+q)^2/4, \sigma < r^2 \leq \frac{[1-2(4p+q)]^2}{4(1-4\sigma)}\)

6.3 CLASS OF METHODS:

By choosing different values of parameters \(p, q, s\) and \(\sigma\) we obtain various classes of methods.

(1) If we choose \(p=q=0\) and \(s=1\) in (6.4) we get the scheme with truncation error \(O(h^2+k^2)\) which is unconditionally stable when \(\alpha \geq 1/4\).
If we put $\sigma = 1/4$ and $\sigma = 1/2$ in (6.9) we obtain the unconditionally stable formulas of Todd and Crandall respectively.

(2) For $p=0, q=-r^2, s=1+2r^2$ and taking $\sigma=0$ in (6.4), we get the unconditionally stable Conte formula with truncation error $O(k^2+h^2+(\frac{k}{h})^2)$.

\[(1-r^2)\frac{\partial^2}{\partial t^2}u_j^m + r^2\frac{\partial^4}{\partial x^4}u_j^m = 0 \quad \ldots (6.10)\]

(3) If we put $p=\frac{5}{12}r^2, q=(\frac{1}{6} - \frac{9}{3}r^2)$ and $s=\frac{2}{3} + \frac{9}{2}r^2$ we get unconditionally stable method with accuracy $O(k^2+h^2+(\frac{k}{h})^2)$.

\[\left[1-(\frac{1}{6}r^2)\frac{\partial^2}{\partial x^2}+(\sigma+\frac{5}{12})r^2\frac{\partial^4}{\partial x^4}\right]\frac{\partial^2}{\partial t^2}u_j^m + r^2\frac{\partial^4}{\partial x^4}u_j^m = 0 \quad \ldots (6.11)\]

For $\sigma = 1/12$, we obtain Jain’s formula [58]

(4) For $p=0, q=1/6$ and $s=2/3$ we get the formula with minimum truncation error among the class of formulas with truncation error $O(k^2+h^4)$.

\[\left[1+\frac{1}{6}\frac{\partial^2}{\partial x^2}+\sigma r^2\frac{\partial^4}{\partial x^4}\right]\frac{\partial^2}{\partial t^2}u_j^m + r^2\frac{\partial^4}{\partial x^4}u_j^m = 0 \quad \ldots (6.12)\]
If we put \( \sigma = 1/4 \) in (6.12) we obtain unconditionally stable formula of Jain et al [58] and for \( \sigma = 1/12 \) we get the conditionally stable, formula \( O(k^4 + h^4) \) with condition \( r^2 = \frac{1}{8} \) (Jain et al [58]).

(5) If we choose \((p,q,s)=-\frac{1}{144}(1,20,102)\), we get

\[
[1+ \frac{1}{6^2} x + \left( \frac{1}{144} + \sigma r^2 \right) \delta^4_x ] \delta^2 t^m_j + r^2 \delta^4 x^m_j = 0 \quad \cdots (6.13)
\]

For \( \sigma = 1/4 \) in (6.13), we get the following unconditionally stable formula of Jain et al [58] which has the minimum truncation error from the class of formulas with order of accuracy \( (k^2 + h^4) \),

\[
[1+ \frac{1}{6^2} x + \left( \frac{1}{4} - \frac{1}{36} + r^2 \right) \delta^4_x ] \delta^2 t^m_j + r^2 \delta^4 x^m_j = 0 \quad \cdots (6.14)
\]

and if \( \sigma = 1/12 \) we get conditionally stable formula of \( O(k^4 + h^4) \) with \( r^2 = 1/6 \) obtained by Jain et al [58].

\[
[1+ \frac{1}{6^2} x + \left( \frac{1}{4} - \frac{1}{36} + r^2 \right) \delta^4_x ] \delta^2 t^m_j + r^2 \delta^4 x^m_j = 0 \quad \cdots (6.15)
\]

(6) If we choose \((p,q,s) = \frac{1}{120}(-1,124,474)\), \( \sigma = 1/12 \) we obtain a new high accuracy method with truncation error \( O(k^4 + h^6) \).
\[ [1 + \frac{1}{6} \delta_x^2 + \frac{1}{12} (r^2 - \frac{1}{60}) \delta_x^4] \delta_t^2 u_j^m + r^2 \delta_x^4 u_j^m = 0 \quad \ldots (6.16) \]

which is conditionally stable for \( r^2 \leq \frac{7}{60} \). If we choose \( r^2 = \frac{1}{84} \) we obtain the scheme of \( O(k^2 + h^8) \) with truncation error \( T = -\frac{11}{50803200} D_x^{12} \), where \( D_x \) is defined in (6.5).

And if we choose \( \alpha = 1/4 \) in (6.16) we obtain unconditionally stable scheme with \( O(k^2 + h^4) \) given by Jain et al [58].

So scheme (6.16) with \( r^2 \leq 7/60 \) and \( r^2 = 1/84 \) are the highest accuracy formulas for system (6.1), so far known.

6.4 TWO SPACE VARIABLES:

We consider the parabolic differential equation

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 \quad \ldots (6.17)
\]

subject to appropriate initial boundary conditions. This type of problem arises in the study of transverse vibrations of a uniform plate. Finite difference scheme for solution of (6.17) has been proposed by Jain et al [58] by using the Richtmyer approach. Here we use spline function approximation for direct solution of (6.17).
We place a uniform square mesh of size $h$ in the $x$-$y$ plane and $t = mK$, $k$ being the mesh spacing in the time direction. We denote the approximate value of $u(x_i, y_j, t_m)$ by $u_{i,j}^m$. Applying the parametric cubic spline in the time direction and finite difference approximation in the space directions we obtain:

$$M_{i,j}^m = -16h^{-4}[(\sinh^{-1}\frac{\delta x}{2})^2 + (\sinh^{-1}\frac{\delta y}{2})^2]u_{i,j}^m \quad ...(6.18)$$

using the cubic spline in compression relation

$$k^2(1+\alpha\delta_t^2)M_{i,j}^m = \delta_t^2 u_{i,j}^m \quad ...(6.19)$$

where $\alpha = 1/4$ and $1/6$ for parametric and cubic splines respectively. Eliminating $M$ from (6.18) and (6.19) we get

$$[[1+\sigma\delta_{xy}^2]\delta_t^2 + r^2 G_{xy}]u_{i,j}^m = 0 \quad ...(6.20)$$

where $G_{xy} = [(\delta_x^2 + \delta_y^2) - \sigma(\delta_x^4 + \delta_y^4)]^2$

which has second order of accuracy for $\alpha=0$ and fourth order of accuracy for $\alpha=1/12$. For different combinations of $\alpha=0, 1/12$ and $\sigma=1/4, 1/6$ we obtain four different schemes.
6.5 NUMERICAL ILLUSTRATIONS:

We consider in this section numerical results obtained by the schemes discussed above by applying them to the following fourth order initial boundary value problems:

**EXAMPLE -I:**

We consider the fourth order initial boundary value problem:

\[ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0 \]  

...(6.21)

with initial condition

\[ u(x,0) = \sin \pi x - \pi x(1-x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \leq x \leq 1 \]  

...(6.22)

and the boundary conditions

\[ u(0,t) = u(1,t) = 0; \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \geq 0 \]  

...(6.23)

The exact solution of (6.21) is obtained as

\[ u(x,t) = \sum_{n=1}^{\infty} Q_n \cos^2 t [\cosh k_n x - p_n \sinh k_n x - \cos k_n x + p_n \sin k_n x] \]

where \( k_n \) are the roots of the equation \( \cos k_n \cosh k_n = 1 \) and

\[ p_n = (\cosh k_n - \cos k_n) / (\sinh k_n - \sin k_n) = \cot k_n / 2 \quad \text{if } n \text{ even} \]

\[ = \tan k_n / 2 \quad \text{if } n \text{ odd} \]

The coefficients \( Q_n \) may be obtained by the method of least
squares, which gives

\[
\int_0^1 [\sin \pi x - \pi x (1-x) - \sum_{i=1}^{M} Q_i f_i] f_i \, dx = 0 \quad i = 1(1)M
\]

where \( f_i = \cosh k_i x - p_i \sinh k_i x - \cos k_i x + p_i \sin k_i x \).

Integrating, we obtain

\[
C_n - \pi D_n + \pi E_n = \sum_{i=1}^{M} Q_i f_i, \quad n=1(1)M
\]

where

\[
C_n = 0, \quad D_n = E_n = \frac{2(2-k_n \cot k_n/2)}{k_n^2}, \quad \text{for } n \text{ even}
\]

\[
C_n = -\frac{4\pi k_n^2}{(\pi^2 + k_n^2)(\pi^2 - k_n^2)},
\]

\[
D_n = -\frac{2\tan k_n/2, E_n = \frac{2}{k_n^2}(2+k_n \tan k_n/2)}{k_n} \quad \text{for } n \text{ odd}
\]

Also \( F_i = \delta_{i,n} \) where \( \delta_{i,n} \) is Kronecker's delta.

Hence the solution for \( Q_n \) is given as

\[
Q_n = 0, \quad \text{n even}
\]

\[
= C_n - \pi D_n + \pi E_n, \quad \text{n odd}
\]

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For solving (6.21) we use scheme (6.4). Approximating the initial and boundary conditions (6.22) and (6.23) we obtain:

(i) \( \frac{u_{j}^{-1} - u_{j}^{1}}{2h} = 0 \) or \( u_{j}^{1} = u_{j}^{-1} \), \( 0 \leq x \leq 1 \)

(ii) \( u_{0}^{m} = 0 \), for \( t \geq 0 \) \( \ldots (6.24) \)

(iii) \( 18u_{1}^{m} - 9u_{2}^{m} + 2u_{2}^{m} = 0 \)

(iv) \( 18u_{N-1}^{m} - 9u_{N-2}^{m} + 2u_{N-1}^{m} = 0 \) \( \ldots (6.25) \)

For high accuracy methods of \( O(k^{4} + h^{6}) \) and \( O(k^{4} + h^{8}) \) we use the following formulas for approximating the boundary conditions:

(iii) \( 300u_{1}^{m} - 300u_{2}^{m} + 200u_{3}^{m} - 75u_{4}^{m} + 12u_{5}^{m} = 0, \)

(iv) \( 12u_{N-5}^{m} - 75u_{N-4}^{m} + 200u_{N-3}^{m} - 300u_{N-2}^{m} + 300u_{N-1}^{m} = 0 \) \( \ldots (6.26) \)

We solved example 1 with \( h = 0.1 \) and \( k = 0.02 \) giving \( r = 2 \) and by choosing \( \sigma = 1/4 \) and various values of parameters \( (p, q, s) \) stated in Table I. Computations were carried over 50 time steps and then repeated for \( h = 0.1, r^{2} = 1/6 \) over 100 time steps. The results thus obtained are compared with those given in Jain et al [58]. These results tabulated in Table I, show the superiority of our methods. Moreover we solved the same problem with various values of \( h \), carrying the computations to 50 time steps. The errors in displacement function \( u(x, t) \) at mid-point of the interval \([0,1] \) in
different time steps are given in table II.

Example II:

We consider equation (6.21) together with the initial conditions
\[ u(x,0) = \frac{x}{12}(2x^2-x-1); \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \leq x \leq 1 \]  
and the boundary conditions
\[ u(0,t) = u(1,t) = 0; \quad \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(1,t) = 0, \quad t > 0 \]  

The exact solution is given as
\[ u(x,t) = \sum_{s=0}^{\infty} d_s \sin(2s+1)\pi x \cos(2s+1)\pi^2 t \]
where \( d_s = -8/(2s+1)^5 \)

We discretized the boundary conditions by following formulas

(i) \[-5u^m_1 + 4u^m_2 - u^m_3 = 0, \]
(ii) \[-u^m_{N-3} + 4u^m_{N-2} - 5u^m_{N-1} = 0 \]  

For high accuracy formulas \( O(k^4 + h^6) \) and \( O(k^4 + h^8) \) we use the following formula for boundary conditions.
(i) \[-154u_1^m + 214u_2^m - 156u_3^m + 64u_4^m - 10u_5^m = 0\ ,
(ii) \[-10u_{N-5}^m + 61u_{N-4}^m - 156u_{N-3}^m + 214u_{N-2}^m - 154u_{N-1}^m = 0 \quad \ldots (6.30)\]

We solved example II by using scheme (6.4) together with equations (i), (6.24), (6.29) and (ii), (6.24), (6.30). By choosing various values of \((p,q,s)\) stated in table III with \(\sigma = 1/4\), we carried out the computations over 50 time steps with \(h = .1\) and \(k = .02\) giving \(r = 2\). We repeated the computations for 100 time steps with \(r^2 = 1/6\). We also include results given by conditionally stable method obtained by \((p,q,s) = \frac{1}{144} (1,20,102)\) and high accuracy schemes obtained by \((p,q,s) = \frac{1}{720} (-1,124,474)\), \(\sigma = \frac{1}{12}\), with \(r^2 = \frac{7}{60}\) and \(r^2 = 1/84\). The results are shown in Table IV.
TABLE -I: (example I)

Absolute error x10^2 in displacement function u(x,t), h=.1

<table>
<thead>
<tr>
<th>(p,q,s,a)</th>
<th>r^2</th>
<th>Time steps</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
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<td>(\frac{1}{144},\frac{5}{36},\frac{17}{24},\frac{1}{4})</td>
<td>4</td>
<td>50</td>
<td>.176</td>
<td>.626</td>
<td>1.053</td>
<td>1.319</td>
<td>1.404</td>
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<td>.040</td>
<td>.044</td>
<td>.050</td>
<td>.138</td>
<td>.170</td>
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<tr>
<td>Jain et al</td>
<td>4</td>
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<td>.17</td>
<td>.88</td>
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<td>.65</td>
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<td>2.80</td>
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<td>(0,\frac{1}{6},\frac{2}{3},\frac{1}{4})</td>
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<td>.88</td>
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<td>.63</td>
<td>1.69</td>
<td>2.43</td>
<td>2.63*</td>
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</tr>
<tr>
<td>(0,0,\frac{1}{4})</td>
<td>4</td>
<td>50</td>
<td>.145</td>
<td>.593</td>
<td>.940</td>
<td>1.176</td>
<td>1.24</td>
</tr>
<tr>
<td>1/6</td>
<td>100</td>
<td>.019</td>
<td>.078</td>
<td>.292</td>
<td>.559</td>
<td>.673</td>
<td></td>
</tr>
<tr>
<td>Todd</td>
<td>4</td>
<td>50</td>
<td>.40</td>
<td>1.83</td>
<td>3.57</td>
<td>4.94</td>
<td>5.41</td>
</tr>
<tr>
<td>1/6</td>
<td>100</td>
<td>.30</td>
<td>1.21</td>
<td>2.33</td>
<td>3.32</td>
<td>3.78</td>
<td></td>
</tr>
<tr>
<td>(0,0,\frac{1}{2})</td>
<td>4</td>
<td>50</td>
<td>.361</td>
<td>1.121</td>
<td>1.968</td>
<td>2.65</td>
<td>2.917</td>
</tr>
<tr>
<td>1/6</td>
<td>100</td>
<td>.097</td>
<td>.264</td>
<td>.419</td>
<td>.563</td>
<td>.640</td>
<td></td>
</tr>
<tr>
<td>Crandall</td>
<td>4</td>
<td>50</td>
<td>1.50</td>
<td>5.21</td>
<td>9.41</td>
<td>12.68</td>
<td>13.95</td>
</tr>
<tr>
<td>1/6</td>
<td>100</td>
<td>.31</td>
<td>1.34</td>
<td>2.45</td>
<td>3.39</td>
<td>3.82</td>
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</tr>
<tr>
<td>(\frac{1}{144},\frac{5}{36},\frac{17}{24},\frac{1}{12})</td>
<td>1</td>
<td>100</td>
<td>.013</td>
<td>.078</td>
<td>.123</td>
<td>.124</td>
<td>.142</td>
</tr>
<tr>
<td>Jain et al</td>
<td>6</td>
<td>100</td>
<td>.03</td>
<td>.70</td>
<td>1.68</td>
<td>2.36</td>
<td>2.51</td>
</tr>
<tr>
<td>(\frac{-1}{720},\frac{124}{720},\frac{474}{720},\frac{1}{12})</td>
<td>60</td>
<td>100</td>
<td>.028</td>
<td>.057</td>
<td>.021</td>
<td>.056</td>
<td>.092</td>
</tr>
<tr>
<td>84</td>
<td>100</td>
<td>.019</td>
<td>.018</td>
<td>.001</td>
<td>.018</td>
<td>.027</td>
<td></td>
</tr>
</tbody>
</table>
TABLE II: (example I)

Absolute error in displacement function \( u(x,t) \) at Mid-points

<table>
<thead>
<tr>
<th>( r^2 ), ( \sigma ), ( h )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>.02</td>
<td>.51(-7)</td>
<td>.29(-7)</td>
<td>.90(-7)</td>
<td>.78(-7)</td>
<td>.44(-8)</td>
</tr>
<tr>
<td>.02</td>
<td>.62(-7)</td>
<td>.68(-8)</td>
<td>.10(-6)</td>
<td>.18(-6)</td>
<td>.16(-6)</td>
</tr>
<tr>
<td>.02</td>
<td>.31(-6)</td>
<td>.25(-5)</td>
<td>.47(-5)</td>
<td>.62(-5)</td>
<td>.10(-4)</td>
</tr>
</tbody>
</table>

For parameter \((p,q,s) = (\frac{-1 124 474}{720, 720, 720})\)

| .02 | .93(-7) | .10(-6) | .24(-6) | .18(-6) | .10(-4) |
| .02 | .77(-7) | .41(-7) | .37(-6) | .26(-6) | .54(-6) |
| .02 | .64(-5) | .15(-4) | .19(-4) | .33(-4) | .17(-4) |

For parameters \((p,q,s) = (\frac{1 5 17}{144, 36, 24})\)

| .02 | .89(-7) | .98(-7) | .56(-7) | .46(-8) | .78(-7) |
| .02 | .19(-6) | .18(-6) | .29(-7) | .24(-5) | .22(-5) |
| .02 | .38(-4) | .42(-4) | .15(-3) | .68(-4) | .14(-3) |
| .02 | .89(-7) | .94(-7) | .32(-7) | .21(-7) | .16(-6) |
| .02 | .14(-6) | .60(-6) | .47(-4) | .30(-5) | .25(-5) |
| .02 | .44(-4) | .63(-4) | .18(-3) | .79(-4) | .18(-3) |
| .02 | .19(-5) | .25(-5) | .45(-5) | .75(-5) | .54(-5) |
| .02 | .19(-5) | .14(-4) | .77(-6) | .40(-4) | .71(-4) |
| .02 | .90(-3) | .14(-4) | .43(-2) | .47(-2) | .17(-2) |
### TABLE-III:

Absolute Error $x10^3$ in displacement function $u(x,t)$, $h=.1$

example II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$r^2$</th>
<th>Time</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(p,q,s,\sigma)$</td>
<td>steps</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(0,0,1,\frac{1}{4})$</td>
<td>4</td>
<td>50</td>
<td>.321</td>
<td>.577</td>
<td>.724</td>
<td>.789</td>
<td>.810</td>
</tr>
<tr>
<td>Todd</td>
<td>4</td>
<td>50</td>
<td>.319</td>
<td>.619</td>
<td>.881</td>
<td>1.076</td>
<td>1.149</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{6}$</td>
<td>100</td>
<td>.381</td>
<td>.333</td>
<td>.774</td>
<td>.781</td>
<td>.766</td>
</tr>
<tr>
<td>Todd</td>
<td>$\frac{1}{6}$</td>
<td>100</td>
<td>.261</td>
<td>.443</td>
<td>.547</td>
<td>.608</td>
<td>.633</td>
</tr>
<tr>
<td>$(0,0,1,\frac{1}{2})$</td>
<td>4</td>
<td>50</td>
<td>.010</td>
<td>.050</td>
<td>.173</td>
<td>.333</td>
<td>.410</td>
</tr>
<tr>
<td>Crandall</td>
<td>4</td>
<td>50</td>
<td>.432</td>
<td>.834</td>
<td>1.178</td>
<td>1.426</td>
<td>1.518</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{6}$</td>
<td>100</td>
<td>.352</td>
<td>.630</td>
<td>.777</td>
<td>.772</td>
<td>.738</td>
</tr>
<tr>
<td>Crandall</td>
<td>$\frac{1}{6}$</td>
<td>100</td>
<td>.230</td>
<td>.408</td>
<td>.540</td>
<td>.656</td>
<td>.702</td>
</tr>
<tr>
<td>$(-1 124 474 1)$</td>
<td>$\frac{7}{60}$</td>
<td>100</td>
<td>.138</td>
<td>.174</td>
<td>.092</td>
<td>.034</td>
<td>.096</td>
</tr>
<tr>
<td>$(-1 4 4)$</td>
<td>$\frac{1}{84}$</td>
<td>100</td>
<td>.035</td>
<td>.062</td>
<td>.071</td>
<td>.061</td>
<td>.055</td>
</tr>
</tbody>
</table>

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### TABLE-IV:

Absolute Errors in displacement function $u(x,t)$ at Mid-points of interval, Example II

<table>
<thead>
<tr>
<th>$r^2$, $\sigma$, $h$</th>
<th>Time steps</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>$\frac{1}{84}$, $\frac{1}{12}$</td>
<td>0.02</td>
<td>0.12(-11)</td>
<td>0.35(-10)</td>
<td>0.77(-10)</td>
<td>0.22(-9)</td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td></td>
<td>0.90(-10)</td>
<td>0.24(-10)</td>
<td>0.40(-8)</td>
<td>0.31(-7)</td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td></td>
<td>0.28(-7)</td>
<td>0.67(-7)</td>
<td>0.70(-6)</td>
<td>0.32(-5)</td>
</tr>
</tbody>
</table>

For parameter $(p,q,s) = \left( \frac{-1}{720}, \frac{124}{720}, \frac{474}{720} \right)$

| $\frac{7}{60}$, $\frac{1}{12}$ | 0.02      | 0.21(-9)  | 0.24(-10) | 0.16(-7) | 0.12(-7) | 0.30(-7)  |
| $\frac{1}{32}$      |            | 0.13(-7)  | 0.11(-6)  | 0.23(-6) | 0.22(-6) | 0.27(-7)  |
| $\frac{1}{16}$      |            | 0.27(-6)  | 0.63(-6)  | 0.50(-5) | 0.12(-4) | 0.66(-5)  |

For parameters $(p,q,s) = \left( \frac{1}{144}, \frac{5}{36}, \frac{17}{24} \right)$

| $\frac{1}{6}$, $\frac{1}{12}$ | 0.02      | 0.25(-9)  | 0.62(-8)  | 0.24(-7) | 0.56(-7) | 0.25(-8)  |
| $\frac{1}{32}$      |            | 0.26(-7)  | 0.93(-7)  | 0.38(-6) | 0.74(-6) | 0.13(-5)  |
| $\frac{1}{16}$      |            | 0.23(-5)  | 0.42(-5)  | 0.17(-4) | 0.48(-4) | 0.47(-4)  |
| $\frac{1}{4}$, $\frac{1}{12}$ | 0.02      | 0.40(-7)  | 0.46(-6)  | 0.49(-6) | 0.33(-6) | 0.54(-8)  |
| $\frac{1}{32}$      |            | 0.21(-5)  | 0.27(-5)  | 0.89(-5) | 0.72(-5) | 0.23(-5)  |
| $\frac{1}{16}$      |            | 0.33(-4)  | 0.11(-3)  | 0.81(-4) | 0.11(-3) | 0.39(-3)  |