CHAPTER - IV
This chapter deals with the variational approach which is rather more versatile and advanced as compared to other methods like those of complex variables, integral transform and other direct methods. This approach relies on the calculus of variations and involves extremizing of a 'functional'. While employing this approach in solid mechanics, the functional term should be the potential energy, the complementary potential energy or some derivative of these viz. Reissener's principle. In finite element method also the variational approach plays the role of an integral part and this method has been very widely employed to solve problems of elastic bodies having complicated shape.

In solid mechanics, the problems are generally different but have equivalent formulations - a differential formulation or a variational formulation. In the differential formulation, the problem is to integrate a
differential equation or a system of differential equations subjected to given boundary conditions. In the process, certain classes of functionals are considered with the view toward establishing necessary conditions for finding functions that extremize the functionals. The results are ordinary or partial differential equations for the extremizing functions (the Euler-Lagrange equations) as well as the establishment of the dualities of kinematic (or rigid) and natural boundary conditions. It is found that the natural boundary conditions are not easily established without the use of the variational approach. Since these conditions are often important for properly posing particular boundary value problems, it is concluded that the natural boundary conditions are valuable products of the variational approach.

A general method for obtaining approximate solution to problems expressed in variational form is the Ritz method. Weighted residual methods (WRM) are also widely used to solve the problems of elasticity. In using the WRM, the field solution is assumed in such a way that it satisfies the boundary conditions exactly but the differential equations approximately. Among these the Galerkin method, Least Square method, Collocation method and Kantorovich’s method are abundantly used.

In this chapter, an attempt has been made to solve a classical problem of a rectangular plate simply supported at sides $x = 0, x = a$ and clamped at $y = 0, y = b$ by

(i) Ritz method, (ii) Galerkin method and (iii) Kantorovich method.
4.1 Basic equations:

An important variational approach is to obtain the maximum total potential energy. The strain energy \( U \) of the plate for linear elastic behaviour is found by evaluating the following integral

\[
U = \frac{1}{2} \iint_R \int_{y_{1/2}}^{y_{1/2}} \tau_{ij} e_{ij} \, dz \, dx \, dy,
\]

where the convention of summation over repeated index is adopted. Using (1.1.6), we get from (4.1.1)

\[
U = \frac{E}{2(1-\nu^2)} \iint_R \int_{y_{1/2}}^{y_{1/2}} \left\{ e_{xx}^2 + 2\nu e_{xx} e_{yy} + e_{yy}^2 + 2(1-\nu) e_{xy}^2 \right\} \, dz \, dx \, dy,
\]

where \( h \) denotes the thickness of the plate and \( R \) is the region of the plate. The potential energy for the external loads is given as follows:

\[
V = -\iiint_R q(x,y) w(x,y) \, dx \, dy,
\]

wherein the loads \( q(x,y) \) are assumed to act on the mid-plane surface of the plate and \( w(x,y) \) is the vertical displacement function of the mid-surface. The components of strain expressed in terms of displacement field \((u_x, v_y, w)\) of the mid surface are given as follows:

\[
e_{xx} = \frac{\partial u_x}{\partial x} - z \frac{\partial^3 w}{\partial x^2}, \quad \text{ ...(4.1.4a)}
\]

\[
e_{yy} = \frac{\partial v_y}{\partial y} - z \frac{\partial^3 w}{\partial y^2}, \quad \text{ ...(4.1.4b)}
\]
\[ e_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) - z \frac{\partial^2 w}{\partial x \partial y}, \] \hspace{1cm} \text{(4.1.4c)}

where

\[ u_x = u_1(x_1, x_2, x_3), \]
\[ v_y = u_2(x_1, x_2, x_3), \]

refer to stretching action of the mid-surface. All other strains are zero.

We note an obvious difficulty in that the transverse shear stresses \((\tau_{xy}, \tau_{yz})\) will be zero for the proposed displacement field.

The stress components in terms of \(w(x, y)\) are given by [20].

\[ \tau_{xx} = - \frac{Ez}{1-v^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \] \hspace{1cm} \text{(4.1.5a)}

\[ \tau_{yy} = - \frac{Ez}{1-v^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \] \hspace{1cm} \text{(4.1.5b)}

\[ \tau_{xy} = - \frac{Ez}{1+v} \left( \frac{\partial^2 w}{\partial x \partial y} \right). \] \hspace{1cm} \text{(4.1.5c)}

Using the equations (4.1.2) and (4.1.4), the expression for the total potential energy functional \(\pi\) is given as follows [20] :

\[ \pi = \frac{D}{2} \iint_K \left[ (\nabla^2 w)^2 + 2(1-\nu) \left( \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 w}{\partial x^2} \right) \left( \frac{\partial^2 w}{\partial y^2} \right) \right) - \iint_K q w \right] \, dx \, dy \] \hspace{1cm} \text{(4.1.6)}

where \(D = \frac{Eh^3}{12(1-v^2)}\) is a constant called the bending rigidity.
4.2 **Formulation of the problem**:

A rectangular plate has been considered to be simply supported at its sides \( x = 0 \) and \( x = a \), while clamped at \( y = 0 \) and \( y = b \). The plate is assumed to be under the action of a triangular load \( (q = q_0 x / a) \) at its centre. The boundary conditions for this problem are given by:

\[
\frac{\partial w}{\partial x} = 0, \quad \text{at} \quad x = 0, \ a, \quad \text{(4.2.1a)}
\]

\[
\frac{\partial w}{\partial y} = 0, \quad \text{at} \quad y = 0, \ b. \quad \text{(4.2.1b)}
\]

Under the above boundary conditions the Gaussian curvature is zero and thus equation (4.1.6) reduces to

\[
\pi = \frac{D}{2} \iint_{\mathbb{R}} \left\{ (\nabla^2 w)^2 - \frac{2q}{D} w \right\} \, dx \, dy, \quad \text{(4.2.2)}
\]

since the plate is under the action of a triangular load at its centre \( (x = a/2, \ y = b/2) \) we have from (4.2.2)

\[
\pi = \frac{D}{2} \iint_{\mathbb{R}} \left\{ (\nabla^2 w)^2 - \frac{2q_0 x}{aD} w(x,y) \right\} \, dx \, dy. \quad \text{(4.2.3)}
\]

**Solution of the problem by Ritz method**:

One general method for obtaining an approximate solution to problem expressed in variational form is the Ritz method. It consists of assuming the form of the unknown solution in terms of known trial functions with unknown adjustable parameters. From the set of trial functions, the function
that renders the functional stationary is selected. The trial functions are substituted into the functional and thereby the functional is expressed in terms of adjustable parameters. The functional is then differentiated with respect to each parameter and resulting equation is set equal to zero. Solving the set of simultaneous equations, the unknown parameters are determined and the approximate solution is found out.

Let the approximate solution be

\[ w = c_1 xy \left( x^2 - a^2 \right)^2 \left( y^2 - b^2 \right)^2 , \quad \ldots(4.2.4) \]

which satisfies the boundary conditions exactly. Using equation (4.2.4) in equation (4.2.3) and extremizing with respect to parameter \( c_1 \) we get

\[ c_1 = \frac{\left( q/45 \right) \left( \frac{q}{D} \right) \left( \frac{q}{D} \right)}{\left[ 0.67a^4 + 0.67b^4 + 0.66a^2b^2 \right]ab} \quad \ldots(4.2.5) \]

The displacement function \( w(x,y) \) is thus known. Then the approximate solution of the problem will be

\[ w = \frac{0.088 \left( \frac{q}{D} \right) xy \left( x^2 - a^2 \right)^2 \left( y^2 - b^2 \right)^2}{\left[ 0.67a^4 + 0.67b^4 + 0.66a^2b^2 \right]ab} \quad \ldots(4.2.6) \]

The stress components can be found out in terms of displacement function \( w(x,y) \) from the equation (4.1.5). The maximum deflection \( w_{\text{max}} \) occurs at the centre \((a/2, b/2)\) of the plate and is given as follows:
\[ w_{\text{max}} = w\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{7.128 \left(\frac{q_o}{D}\right) a^4 b^4}{1024 \left[ 0.67a^4 + 0.67b^4 + 0.66a^2 b^2 \right]} \] \quad \text{...(4.2.7)}

**Stress components:**

The stress components are given as follows:

\[
\tau_{xx} = -\frac{c_1 E z}{1-\nu^2} \left[ \frac{(20x^3 - 12a^2 x)(y^5 + b^4 y - 2b^2 y^3) +}{+ v(20y^3 - 12b^2 y)(x^5 + a^4 x - 2a^2 x^3)} \right], \quad \text{...(4.2.8a)}
\]

\[
\tau_{yy} = -\frac{c_1 E z}{1-\nu^2} \left[ \frac{(20y^3 - 12b^2 y)(x^5 + a^4 x - 2a^2 x^3) +}{+ v(20x^3 - 12a^2 x)(y^5 + b^4 y - 2b^2 y^3)} \right], \quad \text{...(4.2.8b)}
\]

\[
\tau_{xy} = -\frac{c_1 E z}{1+\nu} \left[ \frac{(5x^4 + a^4 - 6a^2 x^2) (5y^2 + b^4 - 6b^2 y^2)}{+ v(5x^4 + a^4 - 6a^2 x^2) (5y^2 + b^4 - 6b^2 y^2)} \right], \quad \text{...(4.2.8c)}
\]

**For a square plate:**

The case of a square plate can be obtained by putting \( b = a \). The component \( w_1 \) is given by

\[
w_1 = c_2 xy (x^2 - a^2)^2 (y^2 - a^2)^2, \quad \text{...(4.2.9)}
\]

where \( c_2 \) is obtained from the expression for \( c_1 \) after putting \( b = a \), in equation (4.2.5) and we get:

\[
c_2 = 0.0444 \left(\frac{q_o}{Da^6}\right). \quad \text{...(4.2.10)}
\]
The maximum deflection at the centre \((a/2, a/2)\) of the square plate is given by

\[ w_1(a/2,a/2) = 0.0034804 \left( \frac{q_o a^4}{D} \right) \]

which agrees with the known solution for a square plate. Stress components for a square plate can easily be found out from equation (4.2.8) on putting \(b = a\). The variation of normal stress \(\tau_{xx}\) against \(x/a\) for various values of \(\nu\) has been shown in Fig. [6].

**Solution by Galerkin method:**

This method involves direct use of the differential equation; it does not require the existence of a functional. For this reason the method has a broader range of application than does the Ritz method. It can be observed that the two methods are closely related in the area of solid mechanics. However, there is some advantage in using the Galerkin method over the Ritz method since the equations for the unknown coefficients in the assumed approximate solution are reached more directly than in the Ritz method which deals with the functional.

Let us consider the linear equation

\[ Lu = f, \quad ...(4.2.11) \]

where \(L\) is a function of differential operator and the boundary conditions are homogeneous. The right side may be thought of as a forcing function of some sort and we may formulate a "virtual work" expression for this function as follows:
Fig 6 The variation of normal stress $\tau_{xx}$ with $x$
where $\delta u$ is a "virtual displacement" consistent with the constraints. It must also be true from equation (4.2.11) that
\[
\iint_V (Lu) \delta u \, dv = \iint_V f \delta u \, dv. \tag{4.2.13}
\]

We take the approximate solution as
\[
\bar{u} = \sum_{i=1}^{n} a_i \phi_i, \tag{4.2.14}
\]
where the functions $\phi_i$, called the coordinate functions, satisfy all the boundary conditions of the problem. The function $\bar{u}$ will not satisfy equation (4.2.13), exactly, however to find the coefficient $a_i$ we can choose for the $n$ virtual displacements also the $n$ coordinate functions $\phi_i$ so that
\[
\iint_V (L\bar{u} - f) \phi_i \, dv = 0, \quad i = 1, 2, \ldots, n. \tag{4.2.15}
\]

Using equation (4.2.15) to the plate equation viz.
\[
\nabla^4 w = \left( \frac{q_e}{D} \right) \left( \frac{x}{a} \right), \tag{4.2.16}
\]
we get
\[
\int_0^b \int_0^b \left\{ \nabla^4 \bar{w} - \left( \frac{q_e}{D} \right) \left( \frac{x}{a} \right) \phi_i \right\} dx \, dy = 0. \tag{4.2.17}
\]

The boundary conditions for the problem are given as follows:
\[
w = \frac{\partial \omega}{\partial x} = 0, \quad \text{at} \ x = 0, a. \tag{4.2.18a}
\]
\[ w = \frac{\partial w}{\partial y} = 0, \text{ at } y = 0, 1. \] ...(4.2.18b)

Let us introduce the non-dimensional variables \( \zeta \) and \( \eta \) defined as follows:

\[ \zeta = \frac{x}{a}, \] ...(4.2.19a)

\[ \eta = \frac{y}{b}. \] ...(4.2.19b)

The boundary conditions (4.2.18) can be written as

\[ w = \frac{\partial w}{\partial \zeta} = 0, \text{ on } \zeta = 0, 1, \] ...(4.2.20a)

\[ w = \frac{\partial w}{\partial \eta} = 0, \text{ on } \eta = 0, 1. \] ...(4.2.20b)

With the boundary conditions (4.2.20) in mind we set the following functions:

\[ g_1 = \zeta (\zeta^2 - 1)^2, \] ...(4.2.21a)

\[ g_2 = \zeta^3 (\zeta^2 - 1)^2, \] ...(4.2.21b)

\[ h_1 = \eta (\eta^2 - 1)^2, \] ...(4.2.21c)

\[ h_2 = \eta^3 (\eta^2 - 1)^2. \] ...(4.2.21d)

We can then formulate a set of four coordinate functions \( \phi_i \) that satisfy the given boundary conditions:

\[ \phi_1 = g_1 h_1 = \zeta \eta (\zeta^2 - 1)^2 (\eta^2 - 1)^2, \] ...(4.2.22a)

\[ \phi_2 = g_1 h_2 = \zeta \eta^3 (\zeta^2 - 1)^2 (\eta^2 - 1)^2, \] ...(4.2.22b)

\[ \phi_3 = g_2 h_1 = \zeta^3 \eta (\zeta^2 - 1)^2 (\eta^2 - 1)^2, \] ...(4.2.22c)

\[ \phi_4 = g_2 h_2 = \zeta^3 \eta^3 (\zeta^2 - 1)^2 (\eta^2 - 1)^2. \] ...(4.2.22d)
The trial function \( w \) is given as
\[
\bar{w} = c_{11}g_{1}h_{1} + c_{12}g_{1}h_{2} + c_{21}g_{2}h_{1} + c_{22}g_{2}h_{2}, \quad \text{(4.2.23)}
\]
where the c's are to be determined by the Galerkin method.

Now equation (4.2.17) need only be integrated over one quadrant as a result of symmetry. Also it is noted that
\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{a^2 \partial \zeta^2} + \frac{\partial^2}{b^2 \partial \eta^2}. \quad \text{(4.2.24)}
\]

We thus have the following four equations by equations (4.2.22) on multiplying through by \( b^4 \):
\[
\int_{0}^{1} \int_{0}^{1} \left[ \left( \frac{b^2}{a^2} \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \eta^2} \right) \left( c_{11}g_{1}h_{1} + c_{12}g_{1}h_{2} + + c_{21}g_{2}h_{1} + c_{22}g_{2}h_{2} \right) - b^4 \left( \frac{q_0}{D} \right) \left( \frac{\zeta}{a} \right) \right] \times (g_{1}h_{1})d\zeta d\eta = 0 \quad \text{(4.2.25)}
\]
\[
\int_{0}^{1} \int_{0}^{1} \left[ \left( \frac{b^2}{a^2} \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \eta^2} \right) \left( c_{11}g_{1}h_{1} + c_{12}g_{1}h_{2} + + c_{21}g_{2}h_{1} + c_{22}g_{2}h_{2} \right) - b^4 \left( \frac{q_0}{D} \right) \left( \frac{\zeta}{a} \right) \right] \times (g_{1}h_{2})d\zeta d\eta = 0 \quad \text{(4.2.26)}
\]
\[
\int_{0}^{1} \int_{0}^{1} \left[ \left( \frac{b^2}{a^2} \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \eta^2} \right) \left( c_{11}g_{1}h_{1} + c_{12}g_{1}h_{2} + + c_{21}g_{2}h_{1} + c_{22}g_{2}h_{2} \right) - b^4 \left( \frac{q_0}{D} \right) \left( \frac{\zeta}{a} \right) \right] \times (g_{2}h_{1})d\zeta d\eta = 0 \quad \text{(4.2.27)}
\]
\[
\int_{0}^{1} \int_{0}^{1} \left[ \left( \frac{b^2}{a^2} \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \eta^2} \right) \left( c_{11}g_{1}h_{1} + c_{12}g_{1}h_{2} + + c_{21}g_{2}h_{1} + c_{22}g_{2}h_{2} \right) - b^4 \left( \frac{q_0}{D} \right) \left( \frac{\zeta}{a} \right) \right] \times (g_{2}h_{2})d\zeta d\eta = 0 \quad \text{(4.2.28)}
\]

For a given ratio \( b^2/a^2 \), we may solve the preceding equations for coefficients \( c_\eta \) in terms of \( D, b \) and \( q_0 \). In the case of a square plate \( a = b \) and \( b^2/a^2 = 1 \).
The above equations reduce to

\[
\begin{bmatrix}
0.803082 & 1.3755160 & 1.3755160 & 1.999870 \\
0.197116 & 0.5128103 & 0.4107521 & 0.785643 \\
0.197162 & 0.5451260 & 0.5128090 & 0.309608 \\
0.054060 & 0.1563016 & 0.1563016 & 0.309608
\end{bmatrix}
\begin{bmatrix}
c_{11} \\
c_{12} \\
c_{21} \\
c_{22}
\end{bmatrix} = \frac{a^4q_0}{D}
\begin{bmatrix}
0.044444 \\
0.011111 \\
0.004230 \\
0.001050
\end{bmatrix}.
\]

\textbf{...(4.2.29)}

Solving with the aid of a computer we get:

\[
\begin{bmatrix}
c_{11} \\
c_{12} \\
c_{21} \\
c_{22}
\end{bmatrix} = \frac{a^4q_0}{D}
\begin{bmatrix}
0.08020 \\
0.02089 \\
-0.15840 \\
0.00726
\end{bmatrix}
\]

The approximate solution to the problem is then given as follows:

\[
\frac{\overline{w}}{\overline{w}} =
\begin{pmatrix}
(0.08020)\left\{\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)\left(\frac{x^2}{a^2} - 1\right)^2\left(\frac{y^2}{b^2} - 1\right)^2\right\} \\
+ (0.02089)\left\{\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)^3\left(\frac{x^2}{a^2} - 1\right)^2\left(\frac{y^2}{b^2} - 1\right)^2\right\} \\
+ (-0.15840)\left\{\left(\frac{x}{a}\right)^3\left(\frac{y}{b}\right)\left(\frac{x^2}{a^2} - 1\right)^2\left(\frac{y^2}{b^2} - 1\right)^2\right\} \\
+ (0.00726)\left\{\left(\frac{x}{a}\right)^3\left(\frac{y}{b}\right)^3\left(\frac{x^2}{a^2} - 1\right)^2\left(\frac{y^2}{b^2} - 1\right)^2\right\}
\end{pmatrix}
\]

The deflection at the centre \((a/2, a/2)\) of the square plate is given as

\[
\overline{w_{\text{max}}} = 0.0034804\left(\frac{q_0 a^4}{D}\right).
\]

62
Solution by Kantorovich method:

A serious deficiency of the Ritz method as well as the Galerkin method is that the obtained results have a strong dependence on the coordinate functions chosen. The method of Kantorovich which we now consider will decrease this dependence of the results on the choice of the coordinate function thereby making the process more effective. However, this gain will not be reached without additional computational efforts. The Kantorovich method can be reformulated to be the solution of a Galerkin integral.

Let us define the coordinate function $\phi_i(y)$ as

$$\phi_i(y) = y \left(y^2 - b^2\right)^2.$$  \hspace{1cm} (4.2.30)

The trial function $W$ is given as follows:

$$W = c(x) \left(y^2 - b^2\right)^2,$$  \hspace{1cm} (4.2.31)

where $c(x)$ is an unknown function of $x$. Thus we have initially

$$\int_0^b \left[ \int_0^1 \left( \nabla^4 W - \frac{q}{D} \right) \phi_i(y) dy \right] dx = 0,$$  \hspace{1cm} (4.2.32)

where $q$ is the triangular load given as $q = q_o x / a$.

The equation (4.2.32) will be satisfied if we take

$$\int_0^b \left( \nabla^4 W - \frac{q}{D} \right) \phi_i(y) dy = 0.$$  \hspace{1cm} (4.2.33)

Substituting equations (4.2.30) and (4.2.31) into equation (4.2.33), we get
\[ \int_0^\gamma \left( \nabla^2 \{ c(x) y(y^2 - b^2) \}^2 - \frac{q_0}{\text{D}} \left( \frac{y}{a} \right) \right) y(y^2 - b^2)^2 \, dy = 0. \quad \ldots(4.2.34) \]

Therefore,
\[ \int_0^\gamma \left( 120 c(x) y + 8y(5y^2 - 3b^2) \frac{d^2 c}{dx^2} + y(y^2 - b^2) \frac{d^4 c}{dx^4} - \frac{q_0}{\text{D}} \frac{x}{a} \right) y(y^2 - b^2)^2 \, dy = 0. \quad \ldots(4.2.35) \]

Integrating equation (4.2.35), we then get on multiplying through by \((384/10395b^7)\).
\[ \frac{b^4}{\text{D}} \frac{d^4 c}{dx^4} - 22b^2 \frac{d^2 c}{dx^2} + \frac{3465}{4} c = \frac{3465}{768} \left( \frac{q_0}{\text{bD/a}} \right). \quad \ldots(4.2.36) \]

The characteristic equation for the differential equation (4.2.36) is given as follows:
\[ b^4 p^4 - 22 b^2 p^2 + \frac{3465}{4} = 0, \quad \ldots(4.2.37) \]
whose roots are given by
\[ p_{1,2,3,4} = \frac{1}{b}(\pm \alpha \pm \beta), \quad \ldots(4.2.38) \]
where
\[ \alpha = 4.49, \]
\[ \beta = 3.03. \]

The complementary function (C.F.) is given by
\[ \text{C.F.} = B_1 e^{(\alpha + \beta)x - b} + B_2 e^{(\alpha - \beta)x - b} + B_3 e^{(-\alpha + \beta)x + b} + B_4 e^{(-\alpha - \beta)x + b}. \quad \ldots(4.2.39) \]

Expanding \(e^{\pm \alpha}\) in terms of hyperbolic functions and \(e^{\pm \beta}\) in terms of sine and cosine functions, the C.F. becomes
The particular integral (P.I.) is

\[
P.I. = \frac{1}{192} \left( \frac{q_0}{D} \right) \left( \frac{x}{ab} \right).
\]

Taking \( c(x) \) to be an even function of \( x \), we can set \( A_2 = A_4 = 0 \), so that the general solution for \( c(x) \) is given by

\[
c(x) = A_1 \cosh \alpha x/b \cos \beta x/b + A_3 \sinh \alpha x/b \sin \beta x/b + \frac{1}{192} \left( \frac{q_0}{D} \right) \left( \frac{x}{ab} \right).
\]

On using the condition \( c(a) = c'(a) = 0 \) to satisfy the boundary condition for \( W \), we can find out the remaining constants \( A_1 \) and \( A_3 \) easily.

Therefore,

\[
A_1 \cosh \alpha a/b \cos \beta a/b + A_3 \sinh \alpha a/b \sin \beta a/b + \frac{1}{192} \left( \frac{q_0}{D} \right) \left( \frac{1}{ab} \right) = 0 \quad \ldots(4.2.43)
\]

\[
A_1 \left\{ \frac{\alpha}{b} \cosh \alpha a/b \cos \beta a/b - \frac{\beta}{b} \cosh \alpha a/b \sin \beta a/b \right\} +
+A_3 \left\{ \frac{\alpha}{b} \cosh \alpha a/b \sin \beta a/b + \frac{\beta}{b} \sinh \alpha a/b \cos \beta a/b \right\} + \frac{1}{192} \left( \frac{q_0}{D} \right) \left( \frac{1}{ab} \right) = 0 \quad \ldots(4.2.44)
\]

From equations (4.2.43) and (4.2.44), we get

\[
A_1 = \frac{1}{192} \left( \frac{q_0}{D} \right) \left[ \frac{1}{b} + \gamma_3 \left\{ \frac{1}{b} \left( \frac{\gamma_2}{\gamma_0} \right) - \frac{1}{a} \left( \frac{\gamma_1}{\gamma_0} \right) \right\} \right] \left( \frac{1}{\gamma_1} \right) \quad \ldots(4.2.45)
\]
\[
A_3 = \frac{1}{192} \left( \frac{q_0}{D} \right) \left[ \frac{1}{b} \left( \frac{\gamma_2}{\gamma_0} \right) - \frac{1}{a} \left( \frac{\gamma_1}{\gamma_0} \right) \right], \tag{4.2.46}
\]

where

\[
\begin{align*}
\gamma_0 &= \beta \sinh \alpha \mu \cosh \alpha \mu + \alpha \sin \beta \mu \cos \beta \mu, \\
\gamma_1 &= \cosh \alpha \mu \cos \beta \mu, \\
\gamma_2 &= \alpha \sinh \alpha \mu \cos \beta \mu - \cosh \alpha \mu \sin \beta \mu, \\
\gamma_3 &= \sinh \alpha \mu \sin \beta \mu,
\end{align*}
\]

and \( \mu = a/b \).

The solution is given by

\[
W = \frac{1}{192} \left( \frac{q_0}{D} \right) \left\{ A_1 \cosh x/b + A_3 \sinh x/b + \frac{x}{ab} \right\} y(y^2 - b^2)^2. \tag{4.2.47}
\]

The maximum deflection at the centre of the square plate is obtained by putting \( x = a/2, y = a/2 \) in equation (4.2.47) and is given as:

\[
W_{\text{max}} = 0.0034704 \left( \frac{q_0 a^4}{D} \right).
\]