Chapter 5

Summary and conclusions

For a better understanding of the underlying properties of dynamical systems, suitable mathematical model equations are undertaken which can have a variety of incarnations like ordinary differential equations, partial differential equations etc. The dynamics governing equations are said to be linear or nonlinear depending upon whether the governing force is linear or nonlinear and it does make a profound difference depending on whether the system is linear or nonlinear and form the subject matter of a new interdisciplinary branch called nonlinear dynamics. Nonlinear evolution equations, in general, are difficult to solve analytically and in many cases, show a bizarre behavior characterized as chaotic motion.

So, based upon the solvability of the equations of motion, dynamical systems can broadly be classified as integrable systems and non-integrable systems. Therefor, for such a classification a mathematical criterion is essential and the concept of integrability can serve this purpose. The integrability for a Hamiltonian system can easily be established by searching for a sufficient number of single valued, analytic, involutive constants of motion which, in turn, lead integration of associated Hamilton's equations of motion by quadratures in the sense of Liouville. So, the programme of solving of equations of motion is, therefore, based upon the requirement of isolating sufficient number of invariants. To this effect, various methods have been devised to derive a variety of invariants.

5.1 Conclusions

Keeping in view multitude applications of invariants in various fields of physics, here we extended the approaches described in [28, 83] to determine TD and TID invariants of different orders in momenta for a number of real and complex classical systems. The work of the present thesis is concluded as

- The introductory first chapter of the present thesis contains the details of a number of important methods for obtaining invariants of different forms along with some important applications.

- In chapter two, to expand the catalogue of applications of the SR method, two studies have been carried out. In the first endeavor, we found quadratic invariants of five systems namely a two-dimensional coupled quartic Hamiltonian system, Toda potential, one dimensional general quartic polynomial potential, Morse potential and the Hulthen's potential. In the second work, we generalized the SR method to obtain fourth order invariants of a couple of systems i.e. one dimensional harmonic oscillator and a one dimensional general time dependent potential.

- It is well known that a complex Hamiltonian description of many physical problems is better route to obtain some additional features which were not possible otherwise. One can find innumerable number of such problems in literature. The importance of complex Hamiltonians becomes much more after the development of $\mathcal{PT}$- symmetric quantum mechanics. With a view to find utility of complex Hamiltonians in classical realms and to expand the domain of applicability of the SR method, in the third chapter, we extended the SR method in complex $z\bar{z}$-space with an aim to isolate dynamical invariants of complex classical systems. This type of scheme of coordinate transformation for deriving invariants has also been utilized in many past studies. Here we successfully obtained invariants of four physical systems, namely linearly confining system, a coupled nonlinear oscillator system, a shifted Harmonic oscillator system and a general inverse potential.

- In continuity to our endeavor of dealing of complex Hamiltonian systems, in the fourth chapter, we presented two studies on invariants using two different approaches by scaling the concerned Hamiltonians on an extended complex phase space characterized by $x = x_1 + ip_2$, $p = p_1 + ix_2$.. In the first one, keeping in view the significance of higher order invariants, quartic, cubic and quadratic complex invariants have been determined utilizing the rationalization method for one dimensional TID and TD classical systems. We found quartic invariants of a shifted harmonic oscillator and its $\mathcal{PT}$-symmetric variant and quartic and cubic invariants of a simple harmonic oscillator. We also obtained quadratic invariants of a TD harmonic oscillator and a general TD nonlinear quartic oscillator. In our last work, the SR method has again been developed within the framework of the ECPS to derive quadratic invariants of a TD nonlinear quartic oscillator. These studies can have some interesting bearings in the realms of newly developing field of $\mathcal{PT}$-symmetric classical and quantum mechanics.

5.2 Future outlook

The present thesis work can further be extended on the following fronts:

- The SR method can further be used to find second and higher order invariants of other TD and TID real and complex Hamiltonian systems.
- In the present study we transformed the SR method in two different coordinate spaces, therefore, in the same way the SR method can also be developed in other coordinate spaces like spherical polar coordinate etc. to find invariants of some central potentials.
- It will be interesting to look for physical interpretations and applications of invariants investigated in the present study.