Chapter 4

Invariants in extended complex phase space

It is an established fact that the Hamiltonian formulation of a physical system in real phase space proves very effective to solve equations of motion and paves a way to understand the underlying dynamics. But in some cases, the formulation of the concerned problem in complex phase space can be a better route to get them solved. One can track the utility of complex Hamiltonians in the study of nuclear models, atomic, molecular and nuclear scattering phenomena, chemical reactions, population biology, delocalized transitions in type-II superconductors and laser physics [75, 87, 88, 89]. The complex Hamiltonians now a days become more potent with the advent of $\mathcal{PT}$-Symmetric quantum mechanics [90].

To find some signatures of complex systems in classical mechanics, recently Kaushal and co-workers [81, 83] studied complex invariants for both TD and TID systems within the framework of an extended complex phase space (ECPS) characterized by

$$x = x_1 + ip_2; \quad p = p_1 + ix_2, \tag{4.1}$$

which have been used by Xavier and Aguiar to develop an algorithm for the computation of the semi-classical coherent-state propagator [89] in laser physics. Transformations similar to the above one have also been used in the study of nonlinear evolution equations in the context of amplitude-modulated nonlinear Langmuir waves in plasma [91]. In this transformation both x and p are separately made complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each. From physics point of view p_2 and x_2 can be regarded as fictitious/spurious components of momentum and coordinate respectively and their presence in the above transformation equations as such allow the introduction of some sort of coordinate-momentum coupling of the the dynamical system. For such an interpretation of the imaginary parts of x and p, one needs to modify the transformation equations in the form

$$x = x_1 + idp_2; \quad p = p_1 + id^{-1}x_2,$$

for the dimensional consideration. In the present work, we choose $d = 1$. Since in the ECPS the degrees of freedom of a system get doubled, therefore this complexifying scheme is better suited to study one dimensional systems. A $\mathcal{PT}$-Symmetric [90] form of a complex Hamiltonian in the ECPS can be found by invoking $\mathcal{PT}$ invariance condition

$$\mathcal{PT}(x_1, p_1, x_2, p_2; i) \rightarrow (-x_1, p_1, -x_2, p_2; -i). \tag{4.2}$$

Recently the ECPS approach is further applied to find higher order complex invariants for a number of systems [84, 85]. Some workers have also solved Schrodinger equation for a variety of one and two dimensional complex Hamiltonian systems within the framework of ECPS [92, 93, 94, 95]. Note that a complex invariant of the form $I = ln(p + im\omega x) - i\omega t$ for a real TD harmonic oscillator do exit long back [61].

Thus keeping in view the importance of complex Hamiltonian systems in the description of many phenomena, here in the present chapter, we carried out two studies on the construction of invariants of various orders within the framework of the ECPS [96]. In first one, we derive TID cubic and quartic invariants and TD quadratic invariants of a number of complex systems, including $\mathcal{PT}$-Symmetric cases, by utilizing the rationalization method developed by Kaushal et al. Thereafter, in the second one, we extend the SR approach into the ECPS and obtain quadratic invariants of a general nonlinear TD quartic oscillator.

4.1 Invariants using rationalization method

Recently the rationalization method was developed to find invariants of TID classical dynamical systems within the framework of the ECPS. Here, in the first study, the same method has been used to work out invariants for a number of TID and TD systems in the ECPS. To this end, we first outline the methodology and subsequently take some specific problems to obtain their invariants.

4.1.1 Formalism

The Hamiltonian $H(x,p,t)$ of a one-dimensional dynamical system in complex space can be expressed, in the light of eq.(4.1), as

$$H = H_1(x_1,p_2,p_1,x_2,t) + iH_2(x_1,p_2,p_1,x_2,t). \tag{4.3}$$

The equations of motion for the above complex system are written as

$$\begin{aligned}\dot{x}_1 &= \frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}; \qquad \dot{p}_2 = \frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2},\\ \dot{p}_1 &= -(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}); \quad \dot{x}_2 = -(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}).\end{aligned} \tag{4.4}$$

Now consider a complex phase space function $I(x,p,t)$ of the form

$$I = I_1(x_1,p_2,p_1,x_2,t) + iI_2(x_1,p_2,p_1,x_2,t). \tag{4.5}$$

The function I is said to be a dynamical invariant of the system in complex phase space provided that it conforms the invariance condition

$$\begin{aligned}\frac{dI}{dt} &= \frac{\partial I}{\partial t} + [I,H]_{(x_1,p_1)} - i[I,H]_{(x_1,x_2)} - i[I,H]_{(p_2,p_1)} - [I,H]_{(p_2,x_2)}\\ &= 0,\end{aligned} \tag{4.6}$$

in the present complex space.

The invariance eq.(4.6), after using eqs.(4.3) and (4.5) and thereafter equating real and imaginary parts separately to zero, reduces to the following pair of equations

$$\begin{aligned}&\frac{\partial I_1}{\partial t} + (\frac{\partial I_1}{\partial x_1} + \frac{\partial I_2}{\partial p_2})(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}) - (\frac{\partial I_2}{\partial x_1} - \frac{\partial I_1}{\partial p_2})(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2})\\ &-(\frac{\partial I_1}{\partial p_1} + \frac{\partial I_2}{\partial x_2})(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}) + (\frac{\partial I_2}{\partial p_1} - \frac{\partial I_1}{\partial x_2})(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}) = 0,\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\frac{\partial I_2}{\partial t} + (\frac{\partial I_2}{\partial x_1} - \frac{\partial I_1}{\partial p_2})(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}) + (\frac{\partial I_1}{\partial x_1} + \frac{\partial I_2}{\partial p_2})(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2})\\ &-(\frac{\partial I_2}{\partial p_1} - \frac{\partial I_1}{\partial x_2})(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}) - (\frac{\partial I_1}{\partial p_1} + \frac{\partial I_2}{\partial x_2})(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}) = 0.\end{aligned} \tag{4.8}$$

So in order to find an invariant $I = I_1 + iI_2$ for a given Hamiltonian $H = H_1 + iH_2$ in ECPS, the above set of partial differential equations must simultaneously be rationalized which, in turn, produce a set of partial differential equations for some unknown coefficients which appear in an ansatz for I. The solutions of such resultant equations will finally provide a desired form of an invariant for a given physical system. Now, in what follows, we employ the above procedure to a couple of systems and obtain cubic and quartic invariants [78] for such cases.

4.1.2 Quartic invariants

1. A shifted harmonic oscillator

Consider a case of a TID shifted harmonic oscillator in one dimension, for which the Hamiltonian is written as

$$H = p^2 + ax + bx^2, \tag{4.9}$$

which, within the framework of eq.(4.1), is expressed as

$$H = H_1(x_1, x_2, p_1, p_2) + iH_2(x_1, x_2, p_1, p_2), \tag{4.10}$$

where the real and imaginary parts are given as

$$H_1 = p_1^2 - x_2^2 + a_1x_1 - a_2p_2 + b_1(x_1^2 - p_2^2) - 2b_2x_1p_2, \tag{4.11a}$$

$$H_2 = 2p_1x_2 + a_1p_2 + a_2x_1 + 2b_1x_1p_2 + b_2(x_1^2 - p_2^2). \tag{4.11b}$$

Here we consider $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Now assume the above system admits a complex invariant I, a fourth order polynomial in momentum, as

$$I = a_0(x) + a_2(x)p^2 + a_4(x)p^4. \tag{4.12}$$

Now using eq.(4.1) the above equation in its complex version is written as

$$I = I_1 + iI_2, \tag{4.13}$$

where the real and imaginary parts are given by

$$\begin{aligned} I_1 = {} & a_{0r} + a_{2r}(p_1^2 - x_2^2) - 2a_{2i}p_1x_2 \\ & + a_{4r}(p_1^4 + x_2^4 - 6p_1^2x_2^2) - 4a_{4i}(p_1^2 - x_2^2)p_1x_2 \end{aligned} \tag{4.14a}$$

$$\begin{aligned} I_2 = {} & a_{0i} + a_{2i}(p_1^2 - x_2^2) + 2a_{2r}p_1x_2 \\ & + a_{4i}(p_1^4 + x_2^4 - 6p_1^2x_2^2) + 4a_{4i}(p_1^2 - x_2^2)p_1x_2, \end{aligned} \tag{4.14b}$$

and the complex coefficient functions $a_0(x)$, $a_2(x)$ and $a_4(x)$ in eq.(4.12) are written in the form $a_0(x) = a_{0r}(x_1, p_2) + ia_{0i}(x_1, p_2)$, $a_2(x) = a_{2r}(x_1, p_2)$
$+ ia_{2i}(x_1, p_2)$ and $a_4(x) = a_{4r}(x_1, p_2) + ia_{4i}(x_1, p_2)$ with a_{0r}, a_{0i}, a_{2r}, a_{2i}, a_{4r} and a_{4i} as the real functions of their real arguments. On Substituting eqs.(4.11a), (4.11b), (4.14a) and (4.14b) in eqs.(4.7) and (4.8) and rationalizing the resultant expressions with respect

to the powers of p_1, x_2 and their combinations, the following set of partial differential equations are obtained

$$\frac{\partial a_{0r}}{\partial x_1} + \frac{\partial a_{0i}}{\partial p_2} = 2a_{2r}(a_1 - 2b_2p_2 + 2b_1x_1) \\ - 2a_{2i}(a_2 + 2b_1p_2 + 2b_2x_1), \tag{4.15}$$

$$\frac{\partial a_{0r}}{\partial p_2} - \frac{\partial a_{0i}}{\partial x_1} = -2a_{2r}(a_2 + 2b_1p_2 + 2b_2x_1) \\ - 2a_{2i}(a_1 + 2b_1x_1 + 2b_2p_2), \tag{4.16}$$

$$\frac{\partial a_{2r}}{\partial x_1} + \frac{\partial a_{2i}}{\partial p_2} = 4a_{4r}(a_1 - 2b_2p_2 + 2b_1x_1) \\ - 4a_{4i}(a_2 + 2b_1p_2 + 2b_2x_1), \tag{4.17}$$

$$\frac{\partial a_{2r}}{\partial p_2} - \frac{\partial a_{2i}}{\partial x_1} = -4a_{4r}(a_2 + 2b_1p_2 + 2b_2x_1) \\ - 4a_{2i}(a_1 + 2b_1x_1 - 2b_2p_2), \tag{4.18}$$

$$\frac{\partial a_{4r}}{\partial x_1} + \frac{\partial a_{4i}}{\partial p_2} = 0, \tag{4.19}$$

$$\frac{\partial a_{4r}}{\partial p_2} - \frac{\partial a_{4i}}{\partial x_1} = 0. \tag{4.20}$$

So in order to obtain the complex invariant for the shifted harmonic oscillator, we now find the solutions of the above set of equations.

Determination of a_{4r} and a_{4i}: Note that eqs.(4.19) and (4.20) can be reduced to similar second order pde forms for the functions a_{4r} and a_{4i} respectively as

$$\frac{\partial^2 a_{4r}}{\partial {x_1}^2} + \frac{\partial^2 a_{4r}}{\partial {p_2}^2} = 0, \tag{4.21a}$$
$$\frac{\partial^2 a_{4i}}{\partial {p_2}^2} + \frac{\partial^2 a_{4i}}{\partial {x_1}^2} = 0. \tag{4.21b}$$

Assuming separability of a_{4r} and, a_{4i} under addition i.e. $a_{4r}(x_1, p_2) = X_{4r}(x_1) + P_{4r}(p_2)$ and $a_{4i}(x_1, p_2) = X_{4i}(x_1) + P_{4i}(p_2)$, solutions of eqs.(4.21a) and (4.21b) are written in the form

$$a_{4r} = \frac{1}{2}\alpha(x_1^2 - p_2^2) + \alpha_1 x_1 + \alpha_2 p_2 + \delta_1, \tag{4.22a}$$
$$a_{4i} = \frac{1}{2}\beta(x_1^2 - p_2^2) + \beta_1 x_1 + \beta_2 p_2 + \delta_2, \tag{4.22b}$$

which satisfy eqs.(4.19) and (4.20) after having

$$\alpha = \beta = 0, \qquad \alpha_1 = -\beta_2 \quad \alpha_2 = \beta_1. \tag{4.23}$$

Here α's, β's and δ's are arbitrary constants of integration to be determined later. With these choices eqs.(4.22a) and (4.22b) for a_{4r} and a_{4i} now become

$$a_{4r} = \alpha_1 x_1 + \alpha_2 p_2 + \delta_1, \tag{4.24a}$$

$$a_{4i} = \alpha_2 x_1 - \alpha_1 p_2 + \delta_2. \tag{4.24b}$$

Determination of a_{2r} and a_{2i}: To this effect, first reduce eqs.(4.17) and (4.18) for functions a_{2r} and a_{2i}, after substituting a_{4r} and a_{4i}, to similar second order forms respectively and then integration of the resultant expressions provide the following solutions

$$a_{2r} = \psi_1(\frac{1}{3}x_1^3 + p_2^2 x_1) - \psi_2(\frac{1}{3}p_2^3 + p_2 x_1^2) + \psi_3(x_1^2 + p_2^2), \tag{4.25a}$$

$$a_{2i} = \psi_2(\frac{1}{3}x_1^3 + p_2^2 x_1) + \psi_1(\frac{1}{3}p_2^3 + p_2 x_1^2) + \psi_4(x_1^2 + p_2^2), \tag{4.25b}$$

with $\psi_1 = 4(\alpha_1 b_1 - \alpha_2 b_2)$, $\psi_2 = 4(\alpha_1 b_2 + \alpha_2 b_1)$, $\psi_3 = 2(\alpha_1 a_1 - \alpha_2 a_2)$ and $\psi_4 = 2(\alpha_1 a_2 + \alpha_2 a_1)$.

Note that when eqs.(4.25a) and (4.25b) put back into their parent eqs.(4.17) and (4.18) give $\delta_1 = \delta_2 = 0$. Hence the solutions of a_{4r} and a_{4i}, eqs.(4.24a) and (4.24b) are further simplified.

Determination of a_{0r} and a_{0i}: For the solutions of these coefficients, by putting the values of a_{4r}, a_{4i}, a_{2r} and a_{2i} in eqs.(4.15) and (4.16), one can easily obtain the solution functions as

$$\begin{aligned} a_{0r} = {} & \frac{8}{15}(c_1 x_1^5 + c_2 p_2^5) + \frac{4}{3}d_1(-x_1^4 + p_2^4) + \frac{2}{3}(e_1 x_1^3 - e_2 p_2^3) \\ & - \frac{8}{3}(c_2 p_2 x_1^4 + c_1 p_2^4 x_1) - \frac{16}{3}(c_1 p_2^2 x_1^3 + c_2 p_2^3 x_1^2) \\ & + \frac{16}{3}d_2(p_2 x_1^3 + p_2^3 x_1) + 2(e_1 p_2^2 x_1 - e_2 p_2 x_1^2), \end{aligned} \tag{4.26a}$$

$$\begin{aligned} a_{0i} = {} & \frac{8}{15}(c_2 x_1^5 - c_1 p_2^5) + \frac{4}{3}d_2(-x_1^4 + p_2^4) + \frac{2}{3}(e_2 x_1^3 + e_1 p_2^3) \\ & + \frac{8}{3}(c_1 p_2 x_1^4 - c_2 p_2^4 x_1) + \frac{16}{3}(-c_2 p_2^2 x_1^3 + c_1 p_2^3 x_1^2) \\ & - \frac{16}{3}d_1(p_2 x_1^3 + p_2^3 x_1) + 2(e_2 p_2^2 x_1 + e_1 p_2 x_1^2), \end{aligned} \tag{4.26b}$$

where $c_1 = \alpha_1(b_1^2 - b_2^2) - 2\alpha_2 b_1 b_2$, $c_2 = \alpha_2(b_1^2 - b_2^2) + 2\alpha_1 b_1 b_2$, $e_1 = \alpha_1(a_1^2 - a_2^2) - 2\alpha_2 a_1 a_2$, $e_2 = \alpha_2(a_1^2 - a_2^2) + 2\alpha_1 a_1 a_2$, $d_1 = \alpha_2 a_1 b_2 + \alpha_1 a_2 b_2 + \alpha_2 a_2 b_1 - \alpha_1 a_1 b_1$ and $d_2 =$

$-\alpha_1 b_1 a_2 + \alpha_2 a_2 b_2 - \alpha_1 a_1 b_2 - \alpha_2 a_1 b_1.$

Finally, the complex invariant of the shifted harmonic oscillator is obtained from eq.(4.13), after using the solutions for a_{0r}, a_{0i}, a_{2r}, a_{2i}, a_{4r} and a_{4i} in eqs.(4.14a) and (4.14b), as

$$\begin{aligned} I = {} & (p_1 + ix_2)^2\{\frac{1}{3}(\psi_1 + i\psi_2)(x_1^3 + ip_2^3 + 3ix_1p_2(x_1 - ip_2)) \\ & + (\psi_3 + i\psi_4)(x_1^2 + p_2^2)\} + \frac{8}{15}(c_1 + ic_2)\{(x_1 - ip_2)^5 - 10x_1p_2^4 \\ & + 10ip_2x_1^4\} - \frac{4}{3}(d_1 + id_2)\{x_1^4 - p_2^4 + 4i(p_2x_1^3 + p_2^3x_1)\} \\ & + \frac{2}{3}(e_1 + ie_2)\{(x_1 - ip_2)^3 + 6ix_1p_2(x_1 - ip_2)\} \\ & + (\alpha_1 + i\alpha_2)(x_1 - ip_2)(p_1 + ix_2)^4. \end{aligned} \tag{4.27}$$

Note that the above function conforms the invariance condition (4.6).

2. A $\mathcal{PT}$-symmetric shifted harmonic oscillator

In view of the $\mathcal{PT}$-symmetric condition, eq.(4.2), the given Hamiltonian will be $\mathcal{PT}$-symmetric only when $a_1 = b_2 = 0$. Thus the real and imaginary parts of the Hamiltonian eq.(4.9) become

$$H_1 = p_1^2 - x_2^2 - a_2p_2 + b_1(x_1^2 - p_2^2), \tag{4.28a}$$

$$H_2 = 2p_1x_2 + a_2x_1 + 2b_1x_1p_2. \tag{4.28b}$$

The complex invariant for this case turns to be

$$\begin{aligned} I = {} & (\alpha_1 + i\alpha_2)[\{\frac{4b_1}{3}(x_1^3 + ip_2^3 + 3ix_1^2p_2 + 3x_1p_2^2) + 2ia_2(x_1^2 + p_2^2) \\ & + (x_1 - ip_2)(p_1 + ix_2)^2\}(p_1 + ix_2)^2 + \frac{8b_1^2}{15}\{(x_1 - ip_2)^5 - 10x_1p_2^4 \\ & + 10ip_2x_1^4\} + \frac{4ib_1a_2}{3}(x_1^4 - p_2^4 + 4ip_2x_1^3 + 4ip_2^3x_1) \\ & - \frac{2a_2^2}{3}\{(x_1 - ip_2)^3 + 6ix_1^2p_2 + 6x_1p_2^2\}]. \end{aligned} \tag{4.29}$$

3. A simple harmonic oscillator

The Hamiltonian of a simple harmonic oscillator in one dimension can easily be obtained by adjusting $a_1 = a_2 = b_2 = 0$ and $b_1 = \omega^2/2$ in eqs.(4.11a) and (4.11b) as

$$H_1 = p_1^2 - x_2^2 + \frac{1}{2}\omega^2(x_1^2 - p_2^2), \tag{4.30a}$$

$$H_2 = 2p_1x_2 + \omega^2x_1p_2. \tag{4.30b}$$

The solutions of various coefficients a_{4r}, a_{4i}, a_{2r}, a_{2i}, a_{0r} and a_{0i}, in view of the above mentioned conditions, are now written as

$$a_{4r} = \alpha_1 x_1 + \alpha_2 p_2, \tag{4.31}$$

$$a_{4i} = \alpha_2 x_1 - \alpha_1 p_2, \tag{4.32}$$

$$a_{2r} = \frac{2}{3}\alpha_1\omega^2 x_1^3 - 2\alpha_2\omega^2 p_2 x_1^2 + 2\alpha_1\omega^2 p_2^2 x_1 - \frac{2}{3}\alpha_2\omega^2 p_2^3, \tag{4.33}$$

$$a_{2r} = \frac{2}{3}\alpha_2\omega^2 x_1^3 + 2\alpha_1\omega^2 p_2 x_1^2 + 2\alpha_2\omega^2 p_2^2 x_1 + \frac{2}{3}\alpha_1\omega^2 p_2^3, \tag{4.34}$$

$$\begin{aligned} a_{0r} = &\frac{2}{15}\alpha_1\omega^4 x_1^5 + \frac{2}{15}\alpha_2\omega^4 p_2^5 - \frac{2}{3}\alpha_2\omega^4 p_2 x_1^4 \\ &- \frac{2}{3}\alpha_1\omega^4 p_2^4 x_1 - \frac{4}{3}\alpha_1\omega^4 p_2^2 x_1^3 - \frac{4}{3}\alpha_2\omega^4 p_2^3 x_1^2, \end{aligned} \tag{4.35}$$

$$\begin{aligned} a_{0i} = &\frac{2}{15}\alpha_2\omega^4 x_1^5 - \frac{2}{15}\alpha_1\omega^4 p_2^5 + \frac{2}{3}\alpha_1\omega^4 p_2 x_1^4 \\ &- \frac{2}{3}\alpha_2\omega^4 p_2^4 x_1 - \frac{4}{3}\alpha_2\omega^4 p_2^2 x_1^3 + \frac{4}{3}\alpha_1\omega^4 p_2^3 x_1^2. \end{aligned} \tag{4.36}$$

Finally the invariant for the simple harmonic oscillator system becomes

$$\begin{aligned} I = &(\alpha_1 + i\alpha_2)[\frac{2}{3}\omega^2(x_1^3 + ip_2^3 + 3ix_1^2 p_2 + 3x_1 p_2^2)(p_1 + ix_2)^2 \\ &+ \frac{8}{15}\omega^4\{(x_1 - ip_2)^5 - 10x_1 p_2^4 + 10ip_2 x_1^4\} \\ &+ (x_1 - ip_2)(p_1 + ix_2)^4]. \end{aligned} \tag{4.37}$$

4.1.3 Cubic Invariants

1. Simple harmonic oscillator

Here, we demonstrate the rationalization method to determine a cubic complex invariant by taking an example of the TID simple harmonic oscillator with Hamiltonian

$$H = p^2 + \frac{1}{2}\omega^2 x^2, \tag{4.38}$$

and the corresponding real and imaginary parts of H are given in eqs.(4.30a) and (4.30b). Let the system admits a complex invariant I of third order in momentum of the form

$$I = a_1(x)p + a_3(x)p^3, \tag{4.39}$$

which in its complex version, after using eq.(4.1), is written as

$$I = I_1 + iI_2, \tag{4.40}$$

where the real and imaginary parts are given as

$$I_1 = a_{1r}p_1 - a_{1i}x_2 + a_{3r}p_1^3 - 3a_{3r}p_1x_2^2 + a_{3i}x_2^3 - 3a_{3i}p_1^2x_2 \tag{4.41a}$$
$$I_2 = a_{1r}x_2 + a_{1i}p_1 + a_{3i}p_1^3 - 3a_{3i}p_1x_2^2 - a_{3r}x_2^3 + 3a_{3r}p_1^2x_2. \tag{4.41b}$$

The complex coefficient functions $a_1(x)$ and $a_3(x)$ in eq.(4.39) are written in the forms $a_1(x) = a_{1r}(x_1, p_2) + ia_{1i}(x_1, p_2)$, $a_3(x) = a_{3r}(x_1, p_2) + ia_{3i}(x_1, p_2)$, with a_{1r}, a_{1i}, a_{3r} and a_{3i} as the real functions of their real arguments. Now inserting eqs.(4.30a), (4.30b), (4.41a) and (4.41b) in eqs.(4.7) and (4.8) and rationalizing the resultant expressions with respect to the powers of p_1, x_2 and their combinations, we get the following set of equations

$$\frac{\partial a_{1r}}{\partial x_1} + \frac{\partial a_{1i}}{\partial p_2} = 6\omega^2 a_{3r}x_1 + 6\omega^2 a_{3i}p_2, \tag{4.42a}$$
$$\frac{\partial a_{1i}}{\partial x_1} - \frac{\partial a_{1r}}{\partial p_2} = 6\omega^2 a_{3i}x_1 + 6\omega^2 a_{3r}p_2, \tag{4.42b}$$
$$\frac{\partial a_{3r}}{\partial x_1} + \frac{\partial a_{3i}}{\partial p_2} = 0, \tag{4.42c}$$
$$\frac{\partial a_{3i}}{\partial p_2} - \frac{\partial a_{3i}}{\partial x_1} = 0, \tag{4.42d}$$
$$4\omega^2 a_{1r}x_1 - 4\omega^2 a_{1i}p_2 = 0, \tag{4.42e}$$
$$4\omega^2 a_{1i}x_1 + 4\omega^2 a_{1r}p_2 = 0. \tag{4.42f}$$

Next we solve the above set of equations in order to find cubic invariant for the simple harmonic oscillator.

Solutions of the coefficients a_{3r} and a_{3i}: Note that eqs.(4.42c) and (4.42d) can be reduced to similar second order partial differential forms for the functions a_{3r} and a_{3i}, respectively as

$$\frac{\partial^2 a_{3r}}{\partial {x_1}^2} + \frac{\partial^2 a_{3r}}{\partial {p_2}^2} = 0, \tag{4.43a}$$
$$\frac{\partial^2 a_{3i}}{\partial {p_2}^2} + \frac{\partial^2 a_{3i}}{\partial {x_1}^2} = 0. \tag{4.43b}$$

Assuming the separability of a_{3r} and a_{3i} under addition as $a_{3r}(x_1, p_2) = X_{3r}(x_1)+P_{3r}(p_2)$ and $a_{3i}(x_1, p_2) = X_{3i}(x_1) + P_{3i}(p_2)$, we obtain the solutions of eqs.(4.43a) and (4.43b) as

$$a_{3r} = \frac{1}{2}\alpha(x_1^2 - p_2^2) + \alpha_1 x_1 + \alpha_2 p_2 + \delta_1, \quad (4.44a)$$
$$a_{3i} = \frac{1}{2}\beta(x_1^2 - p_2^2) + \beta_1 x_1 + \beta_2 p_2 + \delta_2, \quad (4.44b)$$

which satisfy eqs.(4.42c) and (4.42d) only when

$$\alpha = \beta = 0, \ \ \alpha_1 = -\beta_2 \quad and \quad \alpha_2 = \beta_1. \quad (4.45)$$

Here α's, β's and δ's are arbitrary constants of integration to be determined later. With the choices given in eq.(4.45) for the arbitrary constants, eqs.(4.44a) and (4.44b) for a_{3r} and a_{3i} now take the form

$$a_{3r} = \alpha_1 x_1 + \alpha_2 p_2 + \delta_1, \quad (4.46a)$$
$$a_{3i} = \alpha_2 x_1 - \alpha_1 p_2 + \delta_2. \quad (4.46b)$$

Determination of the coefficients a_{1r} and a_{1i}: Now substitution of a_{3r} and a_{3i} in eqs.(4.42a) and (4.42b) lead us to the following pair of equations for the functions a_{1r} and a_{1i} respectively

$$\frac{\partial^2 a_{1r}}{\partial {x_1}^2} + \frac{\partial^2 a_{1r}}{\partial {p_2}^2} = 12\omega^2\alpha_1 x_1 - 12\omega^2\alpha_2 p_2, \quad (4.47a)$$
$$\frac{\partial^2 a_{1i}}{\partial {p_2}^2} + \frac{\partial^2 a_{1i}}{\partial {x_1}^2} = 12\omega^2\alpha_1 p_2 + 12\omega^2\alpha_2 x_1. \quad (4.47b)$$

The above set of equations have following solutions

$$a_{1r} = 2\omega^2(\alpha_1 x_1^3 - \alpha_2 p_2^3) + 6\omega^2(\alpha_1 x_1 p_2^2 - \alpha_2 p_2 x_1^2), \quad (4.48a)$$
$$a_{1i} = 2\omega^2(\alpha_2 x_1^3 + \alpha_1 p_2^3) + 6\omega^2(\alpha_2 x_1 p_2^2 + \alpha_1 p_2 x_1^2). \quad (4.48b)$$

And putting back these results into eqs.(4.42a) and (4.42b), we get

$$\delta_1 = \alpha_1 x_1 + \alpha_2 p_2; \quad \delta_2 = -\alpha_1 p_2 + \alpha_2 x_1. \quad (4.49)$$

Hence the forms for a_{3r} and a_{3i} are further simplified as

$$a_{3r} = 2\alpha_1 x_1 + 2\alpha_2 p_2; \quad a_{3i} = 2\alpha_2 x_1 - 2\alpha_1 p_2. \quad (4.50)$$

Now by substituting solutions of a_{1r} and a_{1i} in eqs.(4.42e) and (4.42f), we get following restrictions

$$
\begin{aligned}
&-8\omega^4 x_1(\alpha_1 x_1^3 - \alpha_2 p_2^3) - 24\omega^4 x_1(\alpha_1 x_1 p_2^2 - \alpha_2 p_2 x_1^2) \\
&+8\omega^4 p_2(\alpha_2 x_1^3 + \alpha_1 p_2^3) + 24\omega^4 p_2(\alpha_2 x_1 p_2^2 + \alpha_1 p_2 x_1^2) = 0, \qquad (4.51a) \\
&8\omega^4 x_1(\alpha_2 x_1^3 + \alpha_1 p_2^3) + 24\omega^4 x_1(\alpha_2 x_1 p_2^2 + \alpha_1 p_2 x_1^2) \\
&+8\omega^4 p_2(\alpha_1 x_1^3 - \alpha_2 p_2^3) + 24\omega^4 p_2(\alpha_1 x_1 p_2^2 - \alpha_2 p_2 x_1^2) = 0, \qquad (4.51b)
\end{aligned}
$$

and these equations can be unified by adding i times eq.(4.51b) to eq.(4.51a) as

$$
8\omega^4(\alpha_1 - i\alpha_2)(x_1^4 - p_2^4 - 4ix_1p_2^3 - 4ip_2x_1^3) = 0. \qquad (4.52)
$$

The above equation further put a restriction on the choice of coefficients α_1 and α_2 i.e. $\alpha_1 = i\alpha_2$.

Now finally plugging eqs.(4.48a), (4.48b) and (4.50) in eqs.(4.41a) and (4.41b), we get

$$
\begin{aligned}
I_1 &= \{2\omega^2(\alpha_1 x_1^3 - \alpha_2 p_2^3) + 6\omega^2(\alpha_1 x_1 p_2^2 - \alpha_2 p_2 x_1^2)\}p_1 \\
&\quad - \{2\omega^2(\alpha_2 x_1^3 + \alpha_1 p_2^3) + 6\omega^2(\alpha_2 x_1 p_2^2 + \alpha_1 p_2 x_1^2)\}x_2 \\
&\quad + (2\alpha_1 x_1 + 2\alpha_2 p_2)(p_1^3 - 3p_1 x_2^2) \\
&\quad + (2\alpha_2 x_1 - 2\alpha_1 p_2)(x_2^3 - 3p_1^2 x_2), \qquad (4.53a) \\
I_2 &= \{2\omega^2(\alpha_1 x_1^3 - \alpha_2 p_2^3) + 6\omega^2(\alpha_1 x_1 p_2^2 - \alpha_2 p_2 x_1^2)\}x_2 \\
&\quad + \{2\omega^2(\alpha_2 x_1^3 + \alpha_1 p_2^3) + 6\omega^2(\alpha_2 x_1 p_2^2 + \alpha_1 p_2 x_1^2)\}p_1 \\
&\quad + (2\alpha_2 x_1 - 2\alpha_1 p_2)(p_1^3 - 3p_1 x_2^2) \\
&\quad + (2\alpha_1 x_1 + 2\alpha_2 p_2)(x_2^3 - 3p_1^2 x_2). \qquad (4.53b)
\end{aligned}
$$

Using the results of eqs.(4.53a) and (4.53b) in eq.(4.40) one can easily obtain the complex invariant I, after invoking the condition $\alpha_1 = i\alpha_2$, as

$$
\begin{aligned}
I = 4\omega^2\alpha_1(p_1 + ix_2)\big(&x_1^3 - ip_2^3 + 3x_1p_2^2 + 3ip_2x_1^2 \\
&+ (x_1 - ip_2)(p_1 + ix_2)^2/\omega^2\big). \qquad (4.54)
\end{aligned}
$$

It is to be noted that eq.(4.54) satisfies the invariance condition (4.6).

4.1.4 Quadratic invariants

1. A TD harmonic oscillator

A harmonic oscillator potential is generally used as a toy model for most of the newly developed methods in physics and describes the dynamics of many idealized physical systems. Therefore, we start with a TD harmonic oscillator and construct a complex invariant for it.
So consider a TD harmonic oscillator described by the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}w^2(t)x^2. \tag{4.55}$$

The above system is $\mathcal{PT}$-symmetric in view of the condition (4.2). The complex version of the above system is obtained by using eq.(4.1) in eq.(4.55) as $H = H_1 + iH_2$ with

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2 + \omega^2 x_1^2 - \omega^2 p_2^2);\ H_2 = p_1 x_2 + \omega^2 x_1 p_2. \tag{4.56}$$

Now let us assume that the above systems possesses a TD invariant I of the form

$$I = f_0(x,t) + f_1(x,t)p + f_2(x,t)p^2, \tag{4.57}$$

which can be translated in the form $I = I_1 + iI_2$ by using eq.(4.1) with

$$\begin{aligned} I_1 &= f_{2r}(p_1^2 - x_2^2) - 2f_{2i}p_1x_2 + f_{1r}p_1 - f_{1i}x_2 + f_{0r}; \\ I_2 &= f_{2i}(p_1^2 - x_2^2) + 2f_{2r}p_1x_2 + f_{1r}x_2 + f_{1i}p_1 + f_{0i}. \end{aligned} \tag{4.58}$$

The coefficient functions $f_0(x,t)$, $f_1(x,t)$ and $f_2(x,t)$ are complex and defined as $f_0(x,t) = f_{0r}(x_1,p_2,t) + if_{0i}(x_1,p_2,t)$, $f_1(x,t) = f_{1r}(x_1,p_2,t) + if_{1i}(x_1,p_2,t)$ and $f_2(x,t) = f_{2r}(x_1,p_2,t) + if_{2i}(x_1,p_2,t)$ where f_{0r}, f_{0i}, f_{1r}, f_{1i}, f_{2r} and f_{2i} are real functions of their real arguments.
Now using the particle equations of motion (4.4) and substituting eqs.(4.56) and (4.58) in

eqs.(4.7) and (4.8), we obtain the following expressions

$$
\begin{aligned}
&(p_1^2-x_2^2)\frac{\partial f_{2r}}{\partial t}+4f_{2r}\omega^2(-p_1x_1+x_2p_2)-2p_1x_2\frac{\partial f_{2i}}{\partial t}\\
&+4f_{2i}\omega^2(x_2x_1+p_1p_2)-(4p_1f_{2r}-4x_2f_{2i}+2f_{1r})(2\omega^2x_1)\\
&+(4p_1f_{2i}+4x_2f_{2r}+2f_{1i})(2\omega^2p_2)+\Bigg[(p_1^2-x_2^2)(\frac{\partial f_{2r}}{\partial x_1}+\frac{\partial f_{2i}}{\partial p_2})\\
&+2p_1x_2(\frac{\partial f_{2r}}{\partial p_2}-\frac{\partial f_{2i}}{\partial x_1})+p_1(\frac{\partial f_{1r}}{\partial x_1}+\frac{\partial f_{1i}}{\partial p_2})+x_2(\frac{\partial f_{1r}}{\partial p_2}-\frac{\partial f_{1i}}{\partial x_1})\\
&+(\frac{\partial f_{0r}}{\partial x_1}+\frac{\partial f_{0i}}{\partial p_2})\Bigg](2p_1)-\Bigg[(p_1^2-x_2^2)(\frac{\partial f_{2i}}{\partial x_1}-\frac{\partial f_{2r}}{\partial p_2})+2p_1x_2\\
&(\frac{\partial f_{2r}}{\partial x_1}+\frac{\partial f_{2i}}{\partial p_2})+p_1(\frac{\partial f_{1i}}{\partial x_1}-\frac{\partial f_{1r}}{\partial p_2})+x_2(\frac{\partial f_{1r}}{\partial x_1}+\frac{\partial f_{1i}}{\partial p_2})\\
&+(\frac{\partial f_{0i}}{\partial x_1}-\frac{\partial f_{0r}}{\partial p_2})\Bigg](2x_2)+p_1\frac{\partial f_{1r}}{\partial t}-2f_{1r}\omega^2x_1\\
&+2f_{1i}\omega^2p_2-x_2\frac{\partial f_{1i}}{\partial t}+\frac{\partial f_{0r}}{\partial t}=0,
\end{aligned}
\tag{4.59}
$$

$$
\begin{aligned}
&(p_1^2-x_2^2)\frac{\partial f_{2i}}{\partial t}+4f_{2i}\omega^2(-p_1x_1+x_2p_2)-2p_1x_2\frac{\partial f_{2r}}{\partial t}\\
&-4f_{2r}\omega^2(x_2x_1+p_1p_2)-(4p_1f_{2i}+4x_2f_{2r}+2f_{1i})(2\omega^2x_1)\\
&-(4p_1f_{2r}-4x_2f_{2i}+2f_{1r})(2\omega^2p_2)+\Bigg[(p_1^2-x_2^2)(\frac{\partial f_{2i}}{\partial x_1}-\frac{\partial f_{2r}}{\partial p_2})\\
&+2p_1x_2(\frac{\partial f_{2r}}{\partial x_1}+\frac{\partial f_{2i}}{\partial p_2})+x_2(\frac{\partial f_{1r}}{\partial x_1}+\frac{\partial f_{1i}}{\partial p_2})+p_1(\frac{\partial f_{1i}}{\partial x_1}-\frac{\partial f_{1r}}{\partial p_2})\\
&+(\frac{\partial f_{0i}}{\partial x_1}-\frac{\partial f_{0r}}{\partial p_2})\Bigg](2p_1)-\Bigg[(p_1^2-x_2^2)(\frac{\partial f_{2r}}{\partial x_1}+\frac{\partial f_{2i}}{\partial p_2})+2p_1x_2\\
&(\frac{\partial f_{2r}}{\partial p_2}-\frac{\partial f_{2i}}{\partial x_1})+p_1(\frac{\partial f_{1r}}{\partial x_1}+\frac{\partial f_{1i}}{\partial p_2})+x_2(\frac{\partial f_{1r}}{\partial p_2}-\frac{\partial f_{1i}}{\partial x_1})\\
&+(\frac{\partial f_{0r}}{\partial x_1}+\frac{\partial f_{0i}}{\partial p_2})\Bigg](2x_2)+x_2\frac{\partial f_{1r}}{\partial t}-2f_{1r}\omega^2p_2\\
&-2f_{1i}\omega^2x_1+x_2\frac{\partial f_{1r}}{\partial t}+\frac{\partial f_{0i}}{\partial t}=0.
\end{aligned}
\tag{4.60}
$$

The above two relations can now be rationalized with respect to the powers of p_1, x_2 and their combinations which, in turn, give the following set of coupled partial differential equations amongst the unknown coefficients of I as

$$\frac{\partial f_{1r}}{\partial t}+2(\frac{\partial f_{0r}}{\partial x_1}+\frac{\partial f_{0i}}{\partial p_2})-12f_{2r}\omega^2x_1+12f_{2i}\omega^2p_2=0, \tag{4.61a}$$

$$\frac{\partial f_{1i}}{\partial t}+2(\frac{\partial f_{0i}}{\partial x_1}-\frac{\partial f_{0r}}{\partial p_2})-12f_{2i}\omega^2x_1-12f_{2r}\omega^2p_2=0, \tag{4.61b}$$

$$\frac{\partial f_{2r}}{\partial t}+2(\frac{\partial f_{1r}}{\partial x_1}+\frac{\partial f_{1i}}{\partial p_2})=0, \tag{4.61c}$$

$$\frac{\partial f_{2i}}{\partial t}-2(\frac{\partial f_{1r}}{\partial p_2}-\frac{\partial f_{1i}}{\partial x_1})=0, \tag{4.61d}$$

$$\frac{\partial f_{2r}}{\partial x_1}+\frac{\partial f_{2i}}{\partial p_2}=0, \tag{4.61e}$$

$$\frac{\partial f_{2i}}{\partial x_1}-\frac{\partial f_{2r}}{\partial p_2}=0, \tag{4.61f}$$

$$\frac{\partial f_{0r}}{\partial t}-6f_{1r}\omega^2x_1+6f_{1i}\omega^2p_2=0, \tag{4.61g}$$

$$\frac{\partial f_{0i}}{\partial t}-6f_{1i}\omega^2x_1-6f_{1r}\omega^2p_2=0. \tag{4.61h}$$

So for construction of complex invariants of a given system, one has to find out solutions for unknown parameters f_{0r}, f_{0i}, f_{1r}, f_{1i}, f_{2r} and f_{2i} which are functions of (x_1, p_2, t).

Solutions for f_{2r} and f_{2i} : For determination of f_{2r} and f_{2i}, eqs.(4.61e) and (4.61f) can be reduced to similar second-order forms for the functions f_{2r}, f_{2i}, respectively, as

$$\frac{\partial^2 f_{2r}}{\partial x_1^2}+\frac{\partial^2 f_{2r}}{\partial p_2^2}=0; \quad \frac{\partial^2 f_{2i}}{\partial x_1^2}+\frac{\partial^2 f_{2i}}{\partial p_2^2}=0. \tag{4.62}$$

Assuming separability of f_{2r} and f_{2i} under addition as $f_{2r}=X_{2r}(x_1,t)+P_{2r}(p_2,t)$, $f_{2i}=X_{2i}(x_1,t)+P_{2i}(p_2,t)$, one can obtain the solutions of eq.(4.62) in the form

$$f_{2r}=\frac{\alpha(t)}{2}(x_1^2-p_2^2)+\alpha_1(t)x_1+\alpha_2(t)p_2+\delta_1(x_1,p_2,t);$$
$$f_{2i}=\frac{\beta(t)}{2}(x_1^2-p_2^2)+\beta_1(t)x_1+\beta_2(t)p_2+\delta_2(x_1,p_2,t). \tag{4.63}$$

where α and β are separations constants and α_1, α_2, β_1, β_2, δ_1 and δ_2 are arbitrary constants of integration to be determined later with the condition that δ_1 and δ_2 are linear in x_1 and p_2 simultaneously and the same linearity has been established in eq.(4.69). The following conditions are obtained so that the solution functions f_{2r} and f_{2i} satisfy their parent eqs.(4.61e) and (4.61f)

$$\alpha=\beta=0, \quad \beta_2=-\alpha_1, \quad \alpha_2=\beta_1. \tag{4.64}$$

In view of the above restrictions, the solution functions f_{2r} and f_{2i} turn out to be

$$f_{2r}=\alpha_1x_1+\alpha_2p_2+\delta_1,$$
$$f_{2i}=\alpha_2x_1-\alpha_1p_2+\delta_2. \tag{4.65}$$

Solutions for f_{1r} and f_{1i}: In order to find f_{1r} and f_{1i}, take partial time derivative of eq.(4.65) and using eqs.(4.61c) and (4.61d), we get

$$\frac{\partial f_{1r}}{\partial x_1} + \frac{\partial f_{1i}}{\partial p_2} + \frac{\dot{\alpha}_1}{2} x_1 + \frac{\dot{\alpha}_2}{2} p_2 + \frac{\dot{\delta}_1}{2} = 0; \tag{4.66a}$$

$$\frac{\partial f_{1i}}{\partial x_1} - \frac{\partial f_{1r}}{\partial p_2} + \frac{\dot{\alpha}_2}{2} x_1 - \frac{\dot{\alpha}_1}{2} p_2 + \frac{\dot{\delta}_2}{2} = 0. \tag{4.66b}$$

Now partially differentiating eqs.(4.66a) and (4.66b) with respect to x_1 and p_2 respectively and then subtracting the resultant later equation from the former one and again partially differentiating eqs.(4.66a) and (4.66b) with respect to p_2 and x_1 respectively and then adding the resultant later equation to the former one, produce the following set of differential equations

$$\frac{\partial^2 f_{1r}}{\partial x_1^2} + \frac{\partial^2 f_{1r}}{\partial p_2^2} + \dot{\alpha}_1 = 0, \tag{4.67a}$$

$$\frac{\partial^2 f_{1i}}{\partial p_2^2} + \frac{\partial^2 f_{1i}}{\partial x_1^2} + \dot{\alpha}_2 = 0. \tag{4.67b}$$

The solutions of the above equations, for functions f_{1r} and f_{1i}, are as follow

$$f_{1r} = -\frac{\dot{\alpha}_1}{2}(x_1^2 + p_2^2); \quad f_{1i} = -\frac{\dot{\alpha}_2}{2}(x_1^2 + p_2^2). \tag{4.68}$$

Again by putting the above solution functions in their original eqs.(4.66a) and (4.66b) give the following constraint conditions

$$\delta_1 = \alpha_1 x_1 + \alpha_2 p_2, \quad \delta_2 = -\alpha_1 p_2 + \alpha_2 x_1. \tag{4.69}$$

Hence the forms for f_{2r} and f_{2i} finally become

$$f_{2r} = 2\alpha_1 x_1 + 2\alpha_2 p_2, \quad f_{2i} = 2\alpha_2 x_1 - 2\alpha_1 p_2. \tag{4.70}$$

Solutions for f_{0r} and f_{0i}: Now using eqs.(4.65), (4.68) and (4.69) in eqs.(4.61a) and (4.61b) we get

$$\frac{\ddot{\alpha}_1}{2}(x_1^2 + p_2^2) + 2\frac{\partial f_{0r}}{\partial x_1} + 2\frac{\partial f_{0i}}{\partial p_2} - 24\alpha_1\omega^2(x_1^2 + p_2^2) = 0, \tag{4.71a}$$

$$\frac{\ddot{\alpha}_2}{2}(x_1^2 + p_2^2) + 2\frac{\partial f_{0i}}{\partial x_1} - 2\frac{\partial f_{0r}}{\partial p_2} - 24\alpha_2\omega^2(x_1^2 + p_2^2) = 0. \tag{4.71b}$$

The above set of equations are conveniently be solved as

$$\begin{aligned} f_{0r} &= \frac{16}{3}\omega^2(\alpha_1 x_1^3 - \alpha_2 p_2^3) - 16\omega^2 x_1 p_2(\alpha_2 x_1 - \alpha_1 p_2) \\ &\quad + \frac{1}{12}(\ddot{\alpha}_1 x_1^3 - \ddot{\alpha}_2 p_2^3) - \frac{x_1 p_2}{4}(\ddot{\alpha}_2 x_1 - \ddot{\alpha}_1 p_2), \qquad (4.72a) \\ f_{0i} &= \frac{16}{3}\omega^2(\alpha_2 x_1^3 + \alpha_1 p_2^3) + 16\omega^2 x_1 p_2(\alpha_1 x_1 + \alpha_2 p_2) \\ &\quad + \frac{1}{12}(\ddot{\alpha}_2 x_1^3 + \ddot{\alpha}_1 p_2^3) + \frac{x_1 p_2}{4}(\ddot{\alpha}_1 x_1 + \ddot{\alpha}_2 p_2). \qquad (4.72b) \end{aligned}$$

The remaining two eqs. (4.61g) and (4.61h) impose the following restrictions

$$\begin{aligned} &\frac{x_1^2 + p_2^2}{12}(\dddot{\alpha}_1 x_1 - 3\dddot{\alpha}_2 p_2) + 3\omega^2(\dot{\alpha}_1 x_1 - \dot{\alpha}_2 p_2)(x_1^2 + p_2^2) \\ &-16 x_1 p_2 \frac{\partial}{\partial t}\Omega^2(\alpha_2 x_1 - \alpha_1 p_2) + \frac{16}{3}\frac{\partial}{\partial t}\Omega^2(\alpha_1 x_1^3 - \alpha_2 p_2^3) = 0, \end{aligned} \qquad (4.73a)$$

$$\begin{aligned} &\frac{x_1^2 + p_2^2}{12}(\dddot{\alpha}_2 x_1 + 3\dddot{\alpha}_1 p_2) + 3\omega^2(\dot{\alpha}_2 x_1 + \dot{\alpha}_1 p_2)(x_1^2 + p_2^2) \\ &+16 x_1 p_2 \frac{\partial}{\partial t}\Omega^2(\alpha_1 x_1 + \alpha_2 p_2) + \frac{16}{3}\frac{\partial}{\partial t}\Omega^2(\alpha_2 x_1^3 + \alpha_1 p_2^3) = 0. \end{aligned} \qquad (4.73b)$$

After finding the solutions of unknown coefficients, the explicit forms of the real and imaginary parts of the complex invariant can now be obtained by using the results of eqs.(4.68), (4.70), (4.72a) and (4.72b) for f_{2r}, f_{2i}, f_{1r}, f_{1i}, f_{0r}, and f_{0i} in eq.(4.58) as

$$\begin{aligned} I_1 &= 2(\alpha_1 x_1 + \alpha_2 p_2)(p_1^2 - x_2^2) - 4(\alpha_2 x_1 - \alpha_1 p_2)p_1 x_2 \\ &\quad - \frac{1}{2}(\dot{\alpha}_1 p_1 - \dot{\alpha}_2 x_2)(x_1^2 + p_2^2) + \frac{1}{12}(\ddot{\alpha}_1 x_1^3 - \ddot{\alpha}_2 p_2^3) \\ &\quad - \frac{x_1 p_2}{4}(\ddot{\alpha}_2 x_1 - \ddot{\alpha}_1 p_2) + \frac{16}{3}\omega^2(\alpha_1 x_1^3 - \alpha_2 p_2^3) \\ &\quad - 16\omega^2 x_1 p_2(\alpha_2 x_1 - \alpha_1 p_2); \qquad (4.74a) \\ I_2 &= 2(\alpha_2 x_1 - \alpha_1 p_2)(p_1^2 - x_2^2) + 4(\alpha_1 x_1 + \alpha_2 p_2)p_1 x_2 \\ &\quad - \frac{1}{2}(\dot{\alpha}_1 x_2 + \dot{\alpha}_2 p_1)(x_1^2 + p_2^2) + \frac{1}{12}(\ddot{\alpha}_2 x_1^3 + \ddot{\alpha}_1 p_2^3) \\ &\quad + \frac{x_1 p_2}{4}(\ddot{\alpha}_1 x_1 + \ddot{\alpha}_2 p_2) + \frac{16}{3}\omega^2(\alpha_2 x_1^3 + \alpha_1 p_2^3) \\ &\quad + 16\omega^2 x_1 p_2(\alpha_1 x_1 + \alpha_2 p_2). \qquad (4.74b) \end{aligned}$$

and the complex invariant $I = I_1 + iI_2$ finally be written as

$$\begin{aligned} I &= 2\Lambda x^* p^2 - \frac{\dot{\Lambda}}{2}(x_1^2 + p_2^2)p + \frac{\ddot{\Lambda}}{12}(x_1^3 + ip_2^3) + \frac{\ddot{\Lambda}}{4}ix^* x_1 p_2 \\ &\quad + 16i\Lambda\omega^2 x^* x_1 p_2 + \frac{\Lambda\omega^2}{3}(x_1^3 + ip_2^3), \qquad (4.75) \end{aligned}$$

where $x^* = x_1 - ip_2$, $\Lambda = \alpha_1 + i\alpha_2$ and $\Lambda^* = \alpha_1 - i\alpha_2$. Also the constraint eqs.(4.73a) and (4.73b) can be unified into a unique third order differential equation of the form

$$\frac{\dddot{\Lambda}}{12}(x_1^3 + ip_2^3) + \frac{\ddot{\Lambda}}{4} ix^* x_1 p_2 + \frac{16}{3}(x_1^3 + ip_2^3)\frac{\partial \Lambda\omega^2}{\partial t}$$
$$+16i\dot{\Lambda}\omega^2 x^* \frac{\partial \Lambda\omega^2}{\partial t} + 16ix^* x_1 p_2 + 3\dot{\Lambda}\omega^2 x(x_1^2 + p_2^2) = 0. \qquad (4.76)$$

So the complex invariant, eq.(4.75), conforms the invariance condition eq.(4.6) subject to the condition (4.76).

2. A general nonlinear quartic potential

Now consider the case of a one dimensional general nonlinear quartic oscillator, for which the Hamiltonian is written as

$$H = \frac{1}{2}p^2 + a_0 + ax + bx^2 + cx^3 + dx^4. \qquad (4.77)$$

Using the complexification eq.(4.1), the above Hamiltonian is expressed as

$$\begin{aligned} H_1 = {} & \frac{1}{2}(p_1^2 - x_2^2) + a_{0r} + a_1 x_1 - a_2 p_2 + b_1(x_1^2 - p_2^2) - 2b_2 x_1 p_2 \\ & + c_1 x_1^3 - 3c_1 x_1 p_2^2 + c_2 p_2^3 - 3c_2 x_1^2 p_2 + d_1(x_1^2 - p_2^2)^2 \\ & - 4d_1 x_1^2 p_2^2 - 4d_2 x_1 p_2 (x_1^2 - p_2^2), \qquad (4.78a) \\ H_2 = {} & p_1 x_2 + a_{0i} + a_1 p_2 + a_2 x_1 + 2b_1 x_1 p_2 + b_2(x_1^2 - p_2^2) \\ & - c_1 p_2^3 + 3c_1 x_1^2 p_2 + c_2 x_1^3 - 3c_2 x_1 p_2^2 + 4d_1 x_1 p_2 (x_1^2 - p_2^2) \\ & + d_2(x_1^2 - p_2^2)^2 - 4d_2 x_1^2 p_2^2, \qquad (4.78b) \end{aligned}$$

where the complex coupling coefficients $a_0 = a_{0r} + ia_{0i}$, $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$ and $d = d_1 + id_2$ are assumed to be time dependent.
Next assume that for this system also there exists a complex invariant of the form expressed in eq.(4.57). Again rationalization of eqs.(4.7) and (4.8) for the present system produces a set of differential equations amongst the unknown coefficient functions f_{2r}, f_{2i}, f_{1r}, f_{1i}, f_{0r} and f_{0i}. Now following the procedure as outlined in the previous section,

the solutions for these coefficients are derived in the form

$$f_{2r} = 2\alpha_1 x_1 + 2\alpha_2 p_2; \quad f_{2i} = 2\alpha_2 x_1 - 2\alpha_1 p_2. \tag{4.79a}$$

$$f_{1r} = -\frac{\dot{\alpha}_1}{2}(x_1^2 + p_2^2); \quad f_{1i} = -\frac{\dot{\alpha}_2}{2}(x_1^2 + p_2^2). \tag{4.79b}$$

$$\begin{aligned} f_{0r} = {} & \frac{1}{12}(\ddot{\alpha}_1 x_1^3 - \ddot{\alpha}_2 p_2^3) - \frac{x_1 p_2}{4}(\ddot{\alpha}_2 x_1 - \ddot{\alpha}_1 p_2) + 16(a_1\alpha_1 - a_2\alpha_2) \\ & (x_1^2 + p_2^2) + \frac{32}{3}\alpha_1(b_1 x_1^3 - b_2 p_2^3) - \frac{32}{3}\alpha_2(b_2 x_1^3 + b_1 p_2^3) + 16 \\ & (c_1\alpha_1 - c_2\alpha_2)(x_1^4 - p_2^4) + \frac{64}{5}\alpha_1(d_1 x_1^5 + d_2 p_2^5) - \frac{64}{5}\alpha_2(d_2 x_1^5 \\ & - d_1 p_2^5) + 32\alpha_1(b_1 x_1 p_2^2 - b_2 p_2 x_1^2) - 32\alpha_2(b_2 x_1 p_2^2 + b_1 p_2 x_1^2) \\ & - 32(c_2\alpha_1 + c_1\alpha_2)(x_1 p_2^3 + p_2 x_1^3) - 64\alpha_1(d_1 x_1 p_2^4 + d_2 p_2 x_1^4) \\ & + 64\alpha_2(d_2 x_1 p_2^4 - d_1 p_2 x_1^4), \end{aligned} \tag{4.79c}$$

$$\begin{aligned} f_{0i} = {} & \frac{1}{12}(\ddot{\alpha}_2 x_1^3 + \ddot{\alpha}_1 p_2^3) + \frac{x_1 p_2}{4}(\ddot{\alpha}_1 x_1 + \ddot{\alpha}_2 p_2) + 16(a_1\alpha_2 + a_2\alpha_1) \\ & (x_1^2 + p_2^2) + \frac{32}{3}\alpha_2(b_1 x_1^3 - b_2 p_2^3) + \frac{32}{3}\alpha_1(b_2 x_1^3 + b_1 p_2^3) + 16 \\ & (c_1\alpha_2 + c_2\alpha_1)(x_1^4 - p_2^4) + \frac{64}{5}\alpha_2(d_1 x_1^5 + d_2 p_2^5) + \frac{64}{5}\alpha_1(d_2 x_1^5 \\ & - d_1 p_2^5) + 32\alpha_2(b_1 x_1 p_2^2 - b_2 p_2 x_1^2) + 32\alpha_1(b_2 x_1 p_2^2 + b_1 p_2 x_1^2) \\ & + 32(c_1\alpha_1 - c_2\alpha_2)(x_1 p_2^3 + p_2 x_1^3) - 64\alpha_2(d_1 x_1 p_2^4 + d_2 p_2 x_1^4) \\ & - 64\alpha_1(d_2 x_1 p_2^4 - d_1 p_2 x_1^4). \end{aligned} \tag{4.79d}$$

The invariant for this case is written as

$$\begin{aligned} I = {} & 2\Lambda x^* p^2 - \frac{\dot{\Lambda}}{2}(x_1^2 + p_2^2)p + \frac{\ddot{\Lambda}}{12}x_1^3 - \frac{i\ddot{\Lambda}^*}{12}p_2^3 + \frac{i x_1 p_2 \ddot{\Lambda}}{4}x^* \\ & + 16\Lambda a(x_1^2 + p_2^2) + \frac{32\Lambda}{3}b(x_1^3 + ip_2^3) + 16\Lambda c(x_1^4 - p_2^4) \\ & + \frac{64\Lambda}{5}d(x_1^5 - ip_2^5) - 32ix_1p_2\Lambda bx - 32ix_1p_2\Lambda^* c^*(x_1^2 + p_2^2) \\ & + 64ix_1p_2\Lambda d(x_1^3 + p_2^3). \end{aligned} \tag{4.80}$$

And the constraint equation for the above invariant comes out to be

$$\begin{aligned} & \frac{\dddot{\Lambda}}{12}x_1^3 - \frac{i\dddot{\Lambda}^*}{12}p_2^3 + \frac{i\dddot{\Lambda}}{4}x_1p_2x^* + 16(x_1^2 + p_2^2)\frac{\partial}{\partial t}\Lambda a \\ & + \frac{32}{3}(x_1^3 + ip_2^3)\frac{\partial}{\partial t}\Lambda b + 16(x_1^4 - p_2^4)\frac{\partial}{\partial t}\Lambda c + \frac{64}{5}(x_1^5 - ip_2^5)\frac{\partial}{\partial t}\Lambda d \\ & -32x_1p_2x\frac{\partial}{\partial t}\Lambda b - 4\dot{\Lambda}(a + 2bx + 3cx^2 + 4dx^3)(x_1^2 + p_2^2) \\ & +64ix_1p_2(x_1^3 + ip_2^3)\frac{\partial}{\partial t}\Lambda d - 32ix_1p_2(x_1^2 + p_2^2)\frac{\partial}{\partial t}\Lambda^* c^* = 0, \end{aligned} \tag{4.81}$$

where $c^* = c_1 - ic_2$.

3. The $\mathcal{PT}$-symmetric case of the general nonlinear quartic potential

Now consider the $\mathcal{PT}$-symmetric version of the general quartic potential (4.77) which, in view of the condition (4.2), is obtained by adjusting $a_{0i} = a_1 = b_2 = c_1 = d_2 = 0$. With these substitutions in eq.(4.77) one easily obtains

$$\begin{aligned} H_1 &= \frac{1}{2}(p_1^2 - x_2^2) + a_{0r} - a_2 p_2 + b_1(x_1^2 - p_2^2) + c_2 p_2^3 \\ &\quad - 3c_2 x_1^2 p_2 + d_1(x_1^2 - p_2^2)^2 - 4d_1 x_1^2 p_2^2, \qquad (4.82a) \\ H_2 &= p_1 x_2 + a_{0i} + a_2 x_1 + 2b_1 x_1 p_2 + c_2 x_1^3 \\ &\quad - 3c_2 x_1 p_2^2 + 4d_1 x_1 p_2(x_1^2 - p_2^2), \qquad (4.82b) \end{aligned}$$

and the invariant for this particular case becomes

$$\begin{aligned} &2\Lambda x^* p^2 - \frac{\dot{\Lambda}}{2}(x_1^2 + p_2^2)p + \frac{\ddot{\Lambda}}{12}x_1^3 - \frac{i\ddot{\Lambda}^*}{12}p_2^3 + 16i\Lambda a_2(x_1^2 + p_2^2) \\ &+\frac{ix_1 p_2 \ddot{\Lambda}}{4}x^* + \frac{32\Lambda}{3}b_1(x_1^3 + ip_2^3) + 16i\Lambda c_2(x_1^4 - p_2^4) \\ &+\frac{64\Lambda}{5}d_1(x_1^5 - ip_2^5)32ix_1 p_2 \Lambda b_1 x - 32x_1 p_2 \Lambda^* c_2(x_1^2 + p_2^2) \\ &+64ix_1 p_2 \Lambda d_1(x_1^3 + p_2^3), \qquad (4.83) \end{aligned}$$

with following constraint condition

$$\begin{aligned} &\frac{\dddot{\Lambda}}{12}x_1^3 - \frac{i\dddot{\Lambda}^*}{12}p_2^3 + \frac{i\dddot{\Lambda}}{4}x_1 p_2 x^* + 16i(x_1^2 + p_2^2)\frac{\partial}{\partial t}\Lambda a_2 \\ &+\frac{32}{3}(x_1^3 + ip_2^3)\frac{\partial}{\partial t}\Lambda b_1 + 16i(x_1^4 - p_2^4)\frac{\partial}{\partial t}\Lambda c_2 + \frac{64}{5}(x_1^5 - ip_2^5)\frac{\partial}{\partial t}\Lambda d_1 \\ &-32x_1 p_2 x\frac{\partial}{\partial t}\Lambda b_1 - 4\dot{\Lambda}(ia_2 + 2b_1 x + 3ic_2 x^2 + 4d_1 x^3)(x_1^2 + p_2^2) \\ &+64ix_1 p_2(x_1^3 + ip_2^3)\frac{\partial}{\partial t}\Lambda d_1 - 32x_1 p_2(x_1^2 + p_2^2)\frac{\partial}{\partial t}\Lambda^* c_2 = 0. \qquad (4.84) \end{aligned}$$

4.2 Invariants using Struckmeier and Riedel approach

Here, in our last study, we develop the SR approach in the ECPS characterized by eq.(4.1) to obtain invariants for TD systems. As an illustration, we take a TD quartic oscillator and find a quadratic invariant for it.

4.2.1 Formalism

Consider a system of a single particle which is moving in an explicitly TD and velocity-independent potential and described by a Hamiltonian

$$H = \frac{1}{2}p^2 + V(x,t). \tag{4.85}$$

The Hamiltonian $H(x,p,t)$ of a one-dimensional system in complex space can be expressed, using eq.(4.1), as

$$H = H_1(x_1,p_2,p_1,x_2,t) + iH_2(x_1,p_2,p_1,x_2,t). \tag{4.86}$$

Note that (x_1,p_1), (x_2,p_2) constitute canonical pairs. The Hamiltonian, eq.(4.85), in ECPS can be written as

$$H = \frac{1}{2}(p_1^2 - x_2^2 + 2ip_1x_2) + (V_r(x_1,p_2,t) + iV_i(x_1,p_2,t)), \tag{4.87}$$

which, after separating real and imaginary parts, can further be written as

$$H_1 = \frac{1}{2}(p_1^2 - x_2^2) + V_r, \tag{4.88a}$$
$$H_2 = p_1x_2 + V_i, \tag{4.88b}$$

where V_r and V_i are real and imaginary parts of the potential. In ECPS, one can easily develop

$$\frac{\partial}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_2}); \quad \frac{\partial}{\partial p} = \frac{1}{2}(\frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_2}). \tag{4.89}$$

Therefore, in view of the above equation, the Hamilton's equations of motion for complex H can now be given as

$$\begin{aligned} \dot{x}_1 &= \frac{1}{2}(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}); \quad \dot{p}_2 = \frac{1}{2}(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2}), \\ \dot{p}_1 &= -\frac{1}{2}(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}); \quad \dot{x}_2 = -\frac{1}{2}(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}). \end{aligned} \tag{4.90}$$

Now consider a complex phase space function $I(x,p,t)$ as

$$I = I_1(x_1,p_2,p_1,x_2,t) + iI_2(x_1,p_2,p_1,x_2,t). \tag{4.91}$$

The function I is said to be a TD invariant of the system in complex phase space if it conforms the invariance condition eq.(4.6). Now we examine the existence of a conserved quantity eq.(4.91) for a system described by eq.(4.85) with a special ansatz for I being at most quadratic in the velocities

$$I = f_2(t)(p_1^2 - x_2^2 + 2ip_1x_2) + f_1(x,t)(p_1 + ix_2) + f_0(x,t), \tag{4.92}$$

where $f_1 = f_{1r}(x_1,t) + if_{1i}(p_2,t)$ and $f_0 = f_{0r}(x_1,p_2,t) + if_{0i}(x_1,p_2,t)$ that render I invariant are to be determined as a function of f_2. The above equation is written as

$$I_1 = f_2(p_1^2 - x_2^2) + f_{1r}p_1 - f_{1i}x_2 + f_{0r}, \tag{4.93a}$$

$$I_2 = 2f_2p_1x_2 + f_{1r}x_2 + f_{1i}p_1 + f_{0i}, \tag{4.93b}$$

after separating real and imaginary parts.

Now using eqs. (4.88a), (4.88b), (4.90), (4.93a) and (4.93b) in eqs.(4.7) and (4.8), we obtain the following expressions

$$\begin{aligned}
&\dot{f}_2(p_1^2 - x_2^2) - f_2(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2})p_1 + f_2(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2})x_2 + \frac{\partial f_{1r}}{\partial t}p_1 \\
&+\frac{\partial f_{1r}}{\partial x_1}(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2})\frac{p_1}{2} - \frac{f_{1r}}{2}(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}) + \frac{\partial f_{1i}}{\partial t}x_2 \\
&-\frac{\partial f_{1i}}{\partial p_2}(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2})\frac{x_2}{2} - \frac{f_{1i}}{2}(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}) + \frac{1}{2}\frac{\partial f_{0r}}{\partial x_1} \\
&(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}) + \frac{1}{2}\frac{\partial f_{0r}}{\partial p_2}(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2}) + \frac{\partial f_{0r}}{\partial t} = 0,
\end{aligned} \tag{4.94a}$$

$$\begin{aligned}
&2\dot{f}_2p_1x_2 - f_2(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2})x_2 - f_2(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2})p_1 + \frac{\partial f_{1r}}{\partial t}x_2 \\
&+\frac{\partial f_{1r}}{\partial x_1}(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2})\frac{x_2}{2} - \frac{f_{1r}}{2}(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_2}) + \frac{\partial f_{1i}}{\partial t}p_1 \\
&+\frac{\partial f_{1i}}{\partial p_2}(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2})\frac{p_1}{2} - \frac{f_{1i}}{2}(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_2}) + \frac{1}{2}\frac{\partial f_{0i}}{\partial x_1} \\
&(\frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_2}) + \frac{1}{2}\frac{\partial f_{0i}}{\partial p_2}(\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_2}) + \frac{\partial f_{0i}}{\partial t} = 0.
\end{aligned} \tag{4.94b}$$

The rationalization of the above pair of equations for a given Hamiltonian with respect to powers of p_1, x_2 and their various products will yield a set of coupled partial differential equations for arbitrary unknown complex coefficients appear in the ansatz for invariant and solutions of which, in turn, will yield the final form of invariant I. In the next subsection, we employ the above line of action to determine quadratic invariant for a general one-dimensional nonlinear complex quartic TD potential.

4.2.2 Quadratic invariants

As a general example, we consider a one-dimensional nonlinear complex Hamiltonian system, defined by

$$H = \frac{1}{2}p^2 + a_1(t)x + a_2(t)x^2 + a_3(t)x^3 + a_4(t)x^4. \tag{4.95}$$

The complexified form of the above Hamiltonian is given as

$$\begin{aligned} H = {} & \frac{1}{2}(p_1^2 - x_2^2 + 2ip_1x_2) + (a_{1r} + ia_{1i})(x_1 + ip_2) \\ & + (a_{2r} + ia_{2i})(x_1 + ip_2)^2 + (a_{3r} + ia_{3i})(x_1 + ip_2)^3 \\ & + (a_{4r} + ia_{4i})(x_1 + ip_2)^4, \end{aligned} \tag{4.96}$$

which can further be separated as real and imaginary parts respectively as

$$\begin{aligned} H_1 = {} & \frac{1}{2}(p_1^2 - x_2^2) + a_{1r}x_1 - a_{1i}p_2 + a_{2r}(x_1^2 - p_2^2) - 2a_{2i}x_1p_2 \\ & + a_{3r}(x_1^3 - 3x_1p_2^2) - a_{3i}(3x_1^2p_2 - p_2^3) \\ & + a_{4r}(x_1^4 + p_2^4 - 6x_1^2p_2^2) - 4a_{4i}(x_1^3p_2 - x_1p_2^3), \end{aligned} \tag{4.97a}$$

$$\begin{aligned} H_2 = {} & p_1x_2 + a_{1r}p_2 + a_{1i}x_1 + a_{2i}(x_1^2 - p_2^2) + 2a_{2r}x_1p_2 \\ & + a_{3i}(x_1^3 - 3x_1p_2^2) + a_{3r}(3x_1^2p_2 - p_2^3) \\ & + a_{4i}(x_1^4 + p_2^4 - 6x_1^2p_2^2) + 4a_{4r}(x_1^3p_2 - x_1p_2^3). \end{aligned} \tag{4.97b}$$

The related equations of motion follow as

$$\begin{aligned} & \dot{x}_1 = p_1, \;\; \dot{p}_2 = x_2, \\ & \dot{p}_1 = -a_{1r} - 2a_{2r}x_1 + 2a_{2i}p_2 - 3a_{3r}x_1^2 + 3a_{3r}p_2^2 + 6a_{3i}x_1p_2 \\ & \qquad - 4a_{4r}x_1^3 + 12a_{4r}x_1p_2^2 - 4a_{4i}p_2^3 + 12a_{4i}x_1^2p_2, \\ & \dot{x}_2 = -a_{1i} - 2a_{2r}p_2 - 2a_{2i}x_1 - 3a_{3i}x_1^2 + 3a_{3i}p_2^2 - 6a_{3r}x_1p_2 \\ & \qquad + 4a_{4r}p_2^3 - 12a_{4r}x_1^2p_2 - 4a_{4i}x_1^3 + 12a_{4i}x_1p_2^2. \end{aligned} \tag{4.98}$$

Now using eqs.(4.97a) and (4.97b) in eqs.(4.94a) and (4.94b) and rationalizing the resultant expressions with respect to the powers of momentum p_1, x_2 and their combinations, we get the following set of partial differential equations

$$\dot{f}_2 + \frac{\partial f_{1r}}{\partial x_1} = 0, \tag{4.99}$$

$$\dot{f}_2 + \frac{\partial f_{1i}}{\partial p_2} = 0, \tag{4.100}$$

$$2\dot{f}_2 + \frac{\partial f_{1r}}{\partial x_1} + \frac{\partial f_{1i}}{\partial p_2} = 0, \tag{4.101}$$

$$\frac{\partial f_{0r}}{\partial x_1} + \frac{\partial f_{1r}}{\partial t} - 2f_2(a_{1r} + 2a_{2r}x_1 - 2a_{2i}p_2 + 3a_{3r}x_1^2 - 3a_{3r}p_2^2$$
$$-6a_{3i}x_1p_2 + 4a_{4r}x_1^3 - 12a_{4r}x_1p_2^2 + 4a_{4i}p_2^3 - 12a_{4i}x_1^2p_2) = 0, \tag{4.102}$$

$$\frac{\partial f_{0r}}{\partial p_2} - \frac{\partial f_{1i}}{\partial t} + 2f_2(a_{1i} + 2a_{2r}p_2 + 2a_{2i}x_1 + 3a_{3i}x_1^2 - 3a_{3i}p_2^2$$
$$+6a_{3r}x_1p_2 - 4a_{4r}p_2^3 + 12a_{4r}x_1^2p_2 + 4a_{4i}x_1^3 - 12a_{4i}x_1p_2^2) = 0, \tag{4.103}$$

$$\frac{\partial f_{0i}}{\partial x_1} + \frac{\partial f_{1i}}{\partial t} - 2f_2(a_{1i} + 2a_{2r}p_2 + 2a_{2i}x_1 + 3a_{3i}x_1^2 - 3a_{3i}p_2^2$$
$$+6a_{3r}x_1p_2 - 4a_{4r}p_2^3 + 12a_{4r}x_1^2p_2 + 4a_{4i}x_1^3 - 12a_{4i}x_1p_2^2) = 0, \tag{4.104}$$

$$\frac{\partial f_{0i}}{\partial p_2} + \frac{\partial f_{1r}}{\partial t} - 2f_2(a_{1r} + 2a_{2r}x_1 - 2a_{2i}p_2 + 3a_{3r}x_1^2 - 3a_{3r}p_2^2$$
$$-6a_{3i}x_1p_2 + 4a_{4r}x_1^3 - 12a_{4r}x_1p_2^2 + 4a_{4i}p_2^3 - 12a_{4i}x_1^2p_2) = 0, \tag{4.105}$$

$$\frac{\partial f_{0r}}{\partial t} - f_{1r}(a_{1r} + 2a_{2r}x_1 - 2a_{2i}p_2 + 3a_{3r}x_1^2 - 3a_{3r}p_2^2 - 6a_{3i}x_1p_2$$
$$+4a_{4r}x_1^3 - 12a_{4r}x_1p_2^2 + 4a_{4i}p_2^3 - 12a_{4i}x_1^2p_2) + f_{1i}(a_{1i} + 2a_{2r}p_2$$
$$+2a_{2i}x_1 + 3a_{3i}x_1^2 - 3a_{3i}p_2^2 + 6a_{3r}x_1p_2 - 4a_{4r}p_2^3 + 12a_{4r}x_1^2p_2$$
$$+4a_{4i}x_1^3 - 12a_{4i}x_1p_2^2) = 0, \tag{4.106}$$

$$\frac{\partial f_{0i}}{\partial t} - f_{1r}(a_{1i} + 2a_{2r}p_2 + 2a_{2i}x_1 + 3a_{3i}x_1^2 - 3a_{3i}p_2^2 + 6a_{3r}x_1p_2$$
$$-4a_{4r}p_2^3 + 12a_{4r}x_1^2p_2 + 4a_{4i}x_1^3 - 12a_{4i}x_1p_2^2) - f_{1i}(a_{1r} + 2a_{2r}x_1$$
$$-2a_{2i}p_2 + 3a_{3r}x_1^2 - 3a_{3r}p_2^2 - 6a_{3i}x_1p_2 + 4a_{4r}x_1^3 - 12a_{4r}x_1p_2^2$$
$$+4a_{4i}p_2^3 - 12a_{4i}x_1^2p_2) = 0. \tag{4.107}$$

Next we solve the above set of partial differential equations for different coupling functions appear in the invariant. The solutions of eqs.(4.99) and (4.100) are easily obtained as

$$f_{1r} = -\dot{f}_2x_1 + \alpha_1(t), \qquad f_{1i} = -\dot{f}_2p_2 + \alpha_2(t). \tag{4.108}$$

Here α's are integration constant and to be obtained separately. Now using the results of f_{1r} and f_{1i} from eq.(4.108) in eqs.(4.102), (4.103) and eqs.(4.104), (4.105) separately, we get

$$f_{0r} = \frac{\ddot{f}_2}{2}(x_1^2 - p_2^2) + 2f_2[a_{1r}x_1 - a_{1i}p_2 + a_{2r}(x_1^2 - p_2^2) - 2a_{2i}x_1p_2$$
$$+ a_{3r}(x_1^3 - 3x_1p_2^2) - a_{3i}(3x_1^2p_2 - p_2^3) + a_{4r}(x_1^4 + p_2^4 - 6x_1^2p_2^2)$$
$$+ 4a_{4i}(x_1p_2^3 - x_1^3p_2)] - \dot{\alpha}_1x_1 + \dot{\alpha}_2p_2 + \beta, \tag{4.109a}$$
$$f_{0i} = \ddot{f}_2x_1p_2 + 2f_2[a_{1i}x_1 + a_{1r}p_2 + a_{2i}(x_1^2 - p_2^2) + 2a_{2r}x_1p_2$$
$$+ a_{3i}(x_1^3 - 3x_1p_2^2) + a_{3r}(3x_1^2p_2 - p_2^3) + a_{4i}(x_1^4 + p_2^4 - 6x_1^2p_2^2)$$
$$- 4a_{4r}(x_1p_2^3 - x_1^3p_2)] - \dot{\alpha}_2x_1 - \dot{\alpha}_1p_2 + \gamma. \tag{4.109b}$$

Here γ and β are again integration constants and to be determined separately. Similarly using the results of f_{1r} and f_{1i} from eq.(4.108) and partial time derivatives of eqs.(4.109a) and (4.109b) in eqs.(4.106) and (4.107), we get the following set of third order differential equations

$$\frac{\dddot{f}_2}{2}(x_1^2 - p_2^2) + \dot{f}_2[3a_{1r}x_1 - 3a_{1i}p_2 + 4a_{2r}(x_1^2 - p_2^2) - 8a_{2i}x_1p_2$$
$$+5a_{3r}(x_1^3 - 3x_1p_2^2) - 5a_{3i}(3x_1^2p_2 - p_2^3) + 6a_{4r}(x_1^4 + p_2^4 - 6x_1^2p_2^2)$$
$$+24a_{4i}(x_1p_2^3 - x_1^3p_2)] + 2f_2[\dot{a}_{1r}x_1 - \dot{a}_{1i}p_2 + \dot{a}_{2r}(x_1^2 - p_2^2) - 2\dot{a}_{2i}$$
$$x_1p_2 + \dot{a}_{3r}(x_1^3 - 3x_1p_2^2) - \dot{a}_{3i}(3x_1^2p_2 - p_2^3) + \dot{a}_{4r}(x_1^4 + p_2^4 - 6x_1^2p_2^2)$$
$$+4\dot{a}_{4i}(x_1p_2^3 - x_1^3p_2)] - \alpha_1(a_{1r} + 2a_{2r}x_1 - 2a_{2i}p_2 + 3a_{3r}x_1^2 - 3a_{3r}p_2^2$$
$$-6a_{3i}x_1p_2 + 4a_{4r}x_1^3 - 12a_{4r}x_1p_2^2 + 4a_{4i}p_2^3 - 12a_{4i}x_1^2p_2) + \alpha_2(a_{1i}$$
$$+2a_{2r}p_2 + 2a_{2i}x_1 + 3a_{3i}x_1^2 - 3a_{3i}p_2^2 + 6a_{3r}x_1p_2 - 4a_{4r}p_2^3$$
$$+12a_{4r}x_1^2p_2 + 4a_{4i}x_1^3 - 12a_{4i}x_1p_2^2) - \ddot{\alpha}_1x_1 + \ddot{\alpha}_2p_2 + \dot{\beta} = 0, \tag{4.110a}$$
$$\dddot{f}_2x_1p_2 + \dot{f}_2[3a_{1i}x_1 + 3a_{1r}p_2 + 4a_{2i}(x_1^2 - p_2^2) + 8a_{2r}x_1p_2 + 5a_{3i}(x_1^3$$
$$-3x_1p_2^2) + 5a_{3r}(3x_1^2p_2 - p_2^3) + 6a_{4i}(x_1^4 + p_2^4 - 6x_1^2p_2^2) - 24a_{4r}(x_1p_2^3$$
$$-x_1^3p_2)] + 2f_2[\dot{a}_{1i}x_1 + \dot{a}_{1r}p_2 + \dot{a}_{2i}(x_1^2 - p_2^2) + 2\dot{a}_{2r}x_1p_2 + \dot{a}_{3i}(x_1^3$$
$$-3x_1p_2^2) + \dot{a}_{3r}(3x_1^2p_2 - p_2^3) + \dot{a}_{4i}(x_1^4 + p_2^4 - 6x_1^2p_2^2) - 4\dot{a}_{4r}(x_1p_2^3$$
$$-x_1^3p_2)] - \alpha_2(a_{1r} + 2a_{2r}x_1 - 2a_{2i}p_2 + 3a_{3r}x_1^2 - 3a_{3r}p_2^2 - 6a_{3i}x_1$$
$$p_2 + 4a_{4r}x_1^3 - 12a_{4r}x_1p_2^2 + 4a_{4i}p_2^3 - 12a_{4i}x_1^2p_2) - \alpha_1(a_{1i} + 2a_{2r}$$
$$p_2 + 2a_{2i}x_1 + 3a_{3i}x_1^2 - 3a_{3i}p_2^2 + 6a_{3r}x_1p_2 - 4a_{4r}p_2^3 + 12a_{4r}x_1^2p_2$$
$$+4a_{4i}x_1^3 - 12a_{4i}x_1p_2^2) - \ddot{\alpha}_2x_1 - \ddot{\alpha}_1p_2 + \dot{\gamma} = 0. \tag{4.110b}$$

Now inserting the different solution functions from eqs.(4.108), (4.109a) and (4.109b) in eqs.(4.93a) and (4.93b), the real and imaginary parts of the invariant defined already in eq.(4.91) are obtained as

$$
\begin{aligned}
I_1 = {} & f_2(p_1^2 - x_2^2) - \dot{f}_2(x_1p_1 - p_2x_2) + \frac{\ddot{f}_2}{2}(x_1^2 - p_2^2) + 2f_2[a_{1r}x_1 \\
& - a_{1i}p_2 + a_{2r}(x_1^2 - p_2^2) - 2a_{2i}x_1p_2 + a_{3r}(x_1^3 - 3x_1p_2^2) - a_{3i} \\
& (3x_1^2p_2 - p_2^3) + a_{4r}(x_1^4 + p_2^4 - 6x_1^2p_2^2) + 4a_{4i}(x_1p_2^3 - x_1^3p_2)] \\
& - \dot{\alpha}_1x_1 + \dot{\alpha}_2p_2 + \beta, \qquad (4.111a)
\end{aligned}
$$

$$
\begin{aligned}
I_2 = {} & 2f_2p_1x_2 - \dot{f}_2(x_1x_2 + p_1p_2) + \ddot{f}_2x_1p_2 + 2f_2[a_{1i}x_1 + a_{1r}p_2 \\
& + a_{2i}(x_1^2 - p_2^2) + 2a_{2r}x_1p_2 + a_{3i}(x_1^3 - 3x_1p_2^2) + a_{3r}(3x_1^2p_2 - p_2^3) \\
& + a_{4i}(x_1^4 + p_2^4 - 6x_1^2p_2^2) - 4a_{4r}(x_1p_2^3 - x_1^3p_2)] \\
& - \dot{\alpha}_2x_1 - \dot{\alpha}_1p_2 + \gamma. \qquad (4.111b)
\end{aligned}
$$

Here one can easily verify that the third order differential equations (4.110a) and (4.110b) are merely total time derivatives of eqs.(4.111a) and (4.111b) respectively and provide solution function f_2. Further using the definition given in eq.(4.91), a final form of the invariant is given as

$$
\begin{aligned}
I = {} & f_2(p_1^2 - x_2^2 + 2ip_1x_2) - \dot{f}_2(x_1 + ip_2)(p_1 + ix_2) + 2f_2[(a_{1r} + ia_{1i}) \\
& (x_1 + ip_2) + (a_{2r} + ia_{2i})(x_1 + ip_2)^2 + (a_{3r} + ia_{3i})(x_1 + ip_2)^3 \\
& + (a_{4r} + ia_{4i})(x_1 + ip_2)^4] + \frac{\ddot{f}_2}{2}(x_1 + ip_2)^2 + (\alpha_1 + i\alpha_2)(p_1 + ix_2) \\
& - (\dot{\alpha}_1 + i\dot{\alpha}_2)(x_1 + ip_2) + (\beta + i\gamma). \qquad (4.112)
\end{aligned}
$$

The above equation can further be written in a compact form as

$$
\begin{aligned}
I = {} & 2f_2H - \dot{f}_2(x_1 + ip_2)(p_1 + ix_2) + \frac{\ddot{f}_2}{2}(x_1 + ip_2)^2 + (\alpha_1 + i\alpha_2) \\
& (p_1 + ix_2) - (\dot{\alpha}_1 + i\dot{\alpha}_2)(x_1 + ip_2) + (\beta + i\gamma). \qquad (4.113)
\end{aligned}
$$

Here function f_2 is given as a solution of the following third order equation

$$
\begin{aligned}
& \dot{f}_2[3(a_{1r} + ia_{1i})(x_1 + ip_2) + 4(a_{2r} + ia_{2i})(x_1 + ip_2)^2 + 5(a_{3r} + ia_{3i}) \\
& (x_1 + ip_2)^3 + 6(a_{4r} + ia_{4i})(x_1 + ip_2)^4] + 2f_2[(\dot{a}_{1r} + i\dot{a}_{1i})(x_1 + ip_2) \\
& +(\dot{a}_{2r} + i\dot{a}_{2i})(x_1 + ip_2)^2 + (\dot{a}_{3r} + i\dot{a}_{3i})(x_1 + ip_2)^3 + (\dot{a}_{4r} + i\dot{a}_{4i})(x_1 \\
& +ip_2)^4] + \frac{\dddot{f}_2}{2}(x_1 + ip_2)^2 - (\alpha_1 + i\alpha_2)[(a_{1r} + ia_{1i}) + 2(a_{2r} + ia_{2i}) \\
& (x_1 + ip_2) + 3(a_{3r} + ia_{3i})(x_1 + ip_2)^2 + 4(a_{4r} + ia_{4i})(x_1 + ip_2)^3] \\
& -(\ddot{\alpha}_1 + i\ddot{\alpha}_2)(x_1 + ip_2) + (\dot{\beta} + i\dot{\gamma}) = 0, \qquad (4.114)
\end{aligned}
$$

which is obtained by adding iota times of eq.(4.110b) to eq.(4.110a). At this end let us examine the invariance of the quantity obtained in eq.(4.113). If the function f_2 assumed to be constant and different integration constants set to zero then $I \propto H$ where H, the total energy, itself is a well known invariant. Again it is important to note that the quantity obtained in eq.(4.113), for real case, turns out to be $I = 2f_2H - \dot{f}_2xp + \frac{\ddot{f}_2}{2}x^2 + \alpha p - \dot{\alpha}x + \beta$ which in turn match the invariant obtained earlier by Struckmeier [28] for $\alpha = b_x$ and $\beta = 0$.