Chapter 3

Invariants in complex $z\bar{z}$ space

The theory of invariants is an important branch in the realm of nonlinear dynamics. Since its inception, it is a very active research field which witnessed large activity to develop different methods to isolate conserved quantities, their interpretation and utility to understand the underlying dynamics of a variety of physical systems [4, 7, 73]. Although the real Hamiltonians and their associated exact invariants have been studied extensively [4, 7], the same efforts have not been made to study invariants of complex Hamiltonian systems [28, 29, 74, 75, 80, 81, 82]. Complexification of space can be achieved by employing a number of transformation routes. A brief survey of such transformations can be found in [83]. The construction of invariants in complexified space leads to some mathematical simplification and transparency for both TD and TID systems [4, 7].

A number of interesting studies of the past decade on the quantum mechanics of $\mathcal{PT}$-symmetric Hamiltonians have now opened a new challenging field to study complex potentials which were rejected earlier for not conforming the stringent hermiticity condition. In view of the importance of $\mathcal{PT}$-symmetric Hamiltonian systems, recently many studies have been reported on the construction of invariants in an extended complex phase space [74, 75, 81, 82, 84, 85].

With the motivation to enhance the catalogue of exact invariants in complex space, here in the present study, we use the transformations $z = x + iy$ and $\bar{z} = x - iy$ to complexify the space [86]. These transformations, particularly, proved very useful to establish the integrability of a variety of central potentials rather easily as compared to that in the cartesian case [4, 7]. Here we employ a new robust method recently presented by SR [28] by casting it in $z\bar{z}$ coordinates system with a hope that it will produce TD invariants. To our best knowledge, this is first attempt of using SR approach in $z\bar{z}$ coordinates for deriving invariants for two dimensional non autonomous systems. To achieve this goal, we first develop the SR formalism in $z\bar{z}$-complex space and then demonstrate the efficacy of the proposed method by determining exact quadratic invariants of four physical systems.

3.1 Formalism

Consider a two dimensional TD dynamical system whose Hamiltonian is given as

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y, t). \tag{3.1}$$

The above Hamiltonian is transformed in $z\bar{z}$ complex space as

$$H = \frac{1}{2}|\dot{z}\dot{\bar{z}}| + V(z, \bar{z}, t), \tag{3.2}$$

with concomitant equations of motion

$$\ddot{z} + 2\frac{\partial V}{\partial z} = 0, \qquad \ddot{\bar{z}} + 2\frac{\partial V}{\partial \bar{z}} = 0. \tag{3.3}$$

The solution functions z, $\bar{z}$, $\dot{z}$ and $\dot{\bar{z}}$ define a path within the 4-dimensional phase space that completely describes the system's time evolution. A quantity defined as $I = I(z, \bar{z}, \dot{z}, \dot{\bar{z}}, t)$ constitutes an invariant of the particle motion if it conforms the following condition:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{z\dot{z}} + [I, H]_{\bar{z}\dot{\bar{z}}} = 0, \tag{3.4}$$

where $[I, H]$ is the Poisson Bracket. Now let us explore the possibility of existence of an polynomial invariant, quadratic in velocities, for the system described by eq.(3.2) as

$$I = f_2(t)(\dot{z}^2 + \dot{\bar{z}}^2) + f_1(z, \bar{z}, t)\dot{z} + g_1(z, \bar{z}, t)\dot{\bar{z}} + f_0(z, \bar{z}, t). \tag{3.5}$$

The set of unknown functions $f_2(t)$, $f_1(z, \bar{z}, t)$, $g_1(z, \bar{z}, t)$ and $f_0(z, \bar{z}, t)$ that render I invariant are to be determined for a given physical system. The quantity described in eq.(3.5) is said to be invariant of a system if its total time derivative vanishes, i.e.

$$\begin{aligned}&\frac{df_2}{dt}(\dot{z}^2 + \dot{\bar{z}}^2) + \frac{\partial f_1}{\partial t}\dot{z} + \frac{\partial g_1}{\partial t}\dot{\bar{z}} + \frac{\partial f_0}{\partial t} + \dot{z}^2\frac{\partial f_1}{\partial z} + \dot{z}\dot{\bar{z}}\frac{\partial g_1}{\partial z} + \dot{z}\frac{\partial f_0}{\partial z} + \dot{\bar{z}}\frac{\partial f_0}{\partial \bar{z}} \\ &-4f_2\dot{z}\frac{\partial V}{\partial z} - 2f_1\frac{\partial V}{\partial z} + \dot{\bar{z}}^2\frac{\partial g_1}{\partial \bar{z}} + \dot{z}\dot{\bar{z}}\frac{\partial f_1}{\partial \bar{z}} - 4f_2\dot{\bar{z}}\frac{\partial V}{\partial \bar{z}} - 2g_1\frac{\partial V}{\partial \bar{z}} = 0.\end{aligned} \tag{3.6}$$

The above equation must hold independently on the specific complex phase-space location of the particle which lead to vanishing of coefficients pertaining to the velocity powers separately. Thus we obtain the following set of equations

$$\frac{df_2}{dt} + \frac{\partial f_1}{\partial z} = 0, \tag{3.7a}$$

$$\frac{df_2}{dt} + \frac{\partial g_1}{\partial \bar{z}} = 0. \tag{3.7b}$$

$$\frac{\partial f_0}{\partial z} + \frac{\partial f_1}{\partial t} - 4f_2\frac{\partial V}{\partial z} = 0 \tag{3.7c}$$

$$\frac{\partial f_0}{\partial \bar{z}} + \frac{\partial g_1}{\partial t} - 4f_2\frac{\partial V}{\partial \bar{z}} = 0 \tag{3.7d}$$

$$\frac{\partial g_1}{\partial z} + \frac{\partial f_1}{\partial \bar{z}} = 0, \tag{3.7e}$$

$$2f_1\frac{\partial V}{\partial z} + 2g_1\frac{\partial V}{\partial \bar{z}} - \frac{\partial f_0}{\partial t} = 0. \tag{3.7f}$$

In order to derive an invariant for a given system, one has to solve the above set of equations for various unknown coefficients successively. To this effect, from eqs.(3.7a) and (3.7b), the solutions for f_1 and g_1 can easily be obtained as

$$f_1 = -\dot{f}_2 z + b_1(t), \tag{3.8a}$$

$$g_1 = -\dot{f}_2 \bar{z} + b_2(t). \tag{3.8b}$$

Next with the help of above equations, a consistent solution of eqs.(3.7c) and (3.7d) comes out the following

$$f_0 = 4f_2V + \frac{1}{2}\ddot{f}_2(z^2 + \bar{z}^2) - \dot{b}_1 z - \dot{b}_2\bar{z} + b_3. \tag{3.9}$$

Note that various b's appearing in the above solutions are TD integration constants. The form of solutions, eqs.(3.8a) and (3.8b), for f_1 and g_1 are consistent with eq.(3.7e). Finally eq.(3.7f) is converted into a linear third-order differential equation for f_2 and b's as

$$\begin{aligned}&4\dot{f}_2V + 4f_2\frac{\partial V}{\partial t} + \frac{1}{2}\dddot{f}_2(z^2 + \bar{z}^2) + 2(\dot{f}_2 z - b_1)\frac{\partial V}{\partial z} + 2(\dot{f}_2\bar{z} - b_2)\frac{\partial V}{\partial \bar{z}}\\ &-\ddot{b}_1 z - \ddot{b}_2\bar{z} + \dot{b}_3 = 0,\end{aligned} \tag{3.10}$$

which acts as a constraint on the choices of constants (b's) in the solutions of f_0, f_1 and g_1 for a given potential V. The solutions of the third order differential eq.(3.10) for f_2 , after using eqs.(3.5), (3.8a), (3.8b) and (3.9), will lead us to the following form of the invariant

$$\begin{aligned}I = {}& f_2(\dot{z}^2 + \dot{\bar{z}}^2) + 4f_2V - \dot{f}_2 z\dot{z} - \dot{f}_2\bar{z}\dot{\bar{z}} + \frac{1}{2}\ddot{f}_2(z^2 + \bar{z}^2)\\ &+ b_1\dot{z} + b_2\dot{\bar{z}} - \dot{b}_1 z - \dot{b}_2\bar{z} + b_3.\end{aligned} \tag{3.11}$$

It can easily be shown that the invariant (3.11) contains a time integral of eq.(3.10) by determining the total time derivative of (3.11) and then inserting the equations of motion (3.3). Thus, eq.(3.11) provides a time integral of eq.(3.10) if and only if the system's evolution is governed by the equations of motion (3.3). As b_1, b_2 and b_3 are arbitrary functions of time and independent of f_2, therefore, the terms in eq.(3.10) must vanish separately as

$$4\dot{f}_2 V + 4f_2\frac{\partial V}{\partial t} + \frac{1}{2}\dddot{f}_2(z^2+\bar{z}^2) + 2\dot{f}_2 z\frac{\partial V}{\partial z} + 2\dot{f}_2\bar{z}\frac{\partial V}{\partial \bar{z}} = 0, \tag{3.12a}$$

$$-2b_1\frac{\partial V}{\partial z} - 2b_2\frac{\partial V}{\partial \bar{z}} - \ddot{b}_1 z - \ddot{b}_2\bar{z} + \dot{b}_3 = 0, \tag{3.12b}$$

which provide us the following two distinct forms of invariants

$$I_{f_2} = f_2(\dot{z}^2+\dot{\bar{z}}^2) + 4f_2V - \dot{f}_2 z\dot{z} - \dot{f}_2\bar{z}\dot{\bar{z}} + \frac{1}{2}\ddot{f}_2(z^2+\bar{z}^2), \tag{3.13a}$$

$$I_b = b_1\dot{z} + b_2\dot{\bar{z}} - \dot{b}_1 z - \dot{b}_2\bar{z} + b_3. \tag{3.13b}$$

Note that eq.(3.12a) will have a trivial solution $f_2(t) = constant$ for autonomous systems ($\frac{\partial V}{\partial t} = 0$) and the corresponding invariant will be total energy. Nevertheless, eq.(3.12a) also allows nontrivial solutions for such systems and thus one can derive other nontrivial invariants for autonomous systems other than the total energy.

3.1.1 Examples

After developing the mathematical structure for the construction of invariants for TD systems in complex phase space, now in what follows, we present some illustrative examples to check the strength of the proposed method.

1. A linearly confining potential

As a simple example, firstly we investigate an invariant of a linearly confining system whose Hamiltonian is given as

$$H = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \omega(t)(x^2+y^2)^{1/2} - \beta(t)(x^2+y^2)^{-1/2}, \tag{3.14}$$

which after complexification becomes

$$H = \frac{1}{2}|\dot{z}\dot{\bar{z}}| + \omega(t)(z\bar{z})^{1/2} - \beta(t)(z\bar{z})^{-1/2}. \tag{3.15}$$

The related equations of motions are written as

$$\ddot{z} + \bar{z}[\omega(z\bar{z})^{-1/2} + \beta(z\bar{z})^{-3/2}] = 0, \tag{3.16a}$$

$$\ddot{\bar{z}} + z[\omega(z\bar{z})^{-1/2} + \beta(z\bar{z})^{-3/2}] = 0. \tag{3.16b}$$

The third-order differential equation (3.12a) for the present system, whose solutions provide f_2, turns out as

$$\dddot{f}_2 + \frac{2}{(z^2+\bar{z}^2)}[6\dot{f}_2(\omega(z\bar{z})^{1/2} - \beta(z\bar{z})^{-1/2}) + 4f_2(\dot{\omega}(z\bar{z})^{1/2} - \dot{\beta}(z\bar{z})^{-1/2})] = 0. \tag{3.17}$$

The above equation may be converted into a second-order differential equation

$$\ddot{f}_2 - \frac{\dot{f}_2^2}{2f_2} - \frac{g_{z,\bar{z}}}{f_2} = 0, \tag{3.18}$$

where the function $g_{z,\bar{z}}(t)$ is the solution of a first order differential equation given as

$$\dot{g}_{z,\bar{z}} = -\frac{2f_2}{(z^2+\bar{z}^2)}[6\dot{f}_2(\omega(z\bar{z})^{1/2} - \beta(z\bar{z})^{-1/2}) + 4f_2(\dot{\omega}(z\bar{z})^{1/2} - \dot{\beta}(z\bar{z})^{-1/2})]. \tag{3.19}$$

Finally using the eq.(3.18), the invariant (3.13a) for the given system may be expressed in an alternative form as

$$I_{f_2} = \frac{1}{f_2}[(f_2\dot{z} - \frac{1}{2}\dot{f}_2 z)^2 + (f_2\dot{\bar{z}} - \frac{1}{2}\dot{f}_2\bar{z})^2] + 4f_2[\omega(z\bar{z})^{1/2} - \beta(z\bar{z})^{-1/2}] + \frac{g}{2f_2}(z^2+\bar{z}^2), \tag{3.20}$$

which conforms the invariance condition (3.4).

2. A shifted harmonic oscillator potential

In the second case, consider a shifted Harmonic oscillator system with a Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + a(x+y) + \frac{\omega^2}{2}(x^2+y^2), \tag{3.21}$$

where a and ω are TD quantities. The corresponding Hamiltonian for the system in complex space becomes

$$H = \frac{1}{2}|\dot{z}\dot{\bar{z}}| + \frac{1}{2}a(t)(z+\bar{z}) - i\frac{1}{2}a(t)(z-\bar{z}) + \frac{1}{2}\omega^2(t)(z\bar{z}), \tag{3.22}$$

with following concomitant equations of motions as

$$\ddot{z} + a - ia + \omega^2 \bar{z} = 0, \tag{3.23a}$$

$$\ddot{\bar{z}} + a + ia + \omega^2 z = 0. \tag{3.23b}$$

The third-order differential eq.(3.12a) for the function $f_2(t)$ becomes

$$\dddot{f}_2 + \frac{1}{(z^2 + \bar{z}^2)}[(8\dot{f}_2 a + 4f_2\dot{a})(z + \bar{z} - i(z - \bar{z})) + (12\dot{f}_2\omega^2 + 8f_2\omega\dot{\omega})z\bar{z}] = 0. \tag{3.24}$$

The third order differential eq.(3.24) can again be converted into a coupled second-order equation of the form

$$\ddot{f}_2 - \frac{\dot{f}_2^2}{2f_2} - \frac{g_{z,\bar{z}}(t)}{f_2} = 0, \tag{3.25}$$

where time derivative of $g_{z,\bar{z}}(t)$ is governed by the following equation

$$\dot{g}_{z,\bar{z}}(t) = -\frac{f_2}{(z^2 + \bar{z}^2)}[(8\dot{f}_2 a + 4f_2\dot{a})(z + \bar{z} - i(z - \bar{z})) + (12\dot{f}_2\omega^2 + 8f_2\omega\dot{\omega})z\bar{z}]. \tag{3.26}$$

The invariant of the present system then expressed in the form

$$I_{f_2} = \frac{1}{f_2}[(f_2\dot{z} - \frac{1}{2}\dot{f}_2 z)^2 + (f_2\dot{\bar{z}} - \frac{1}{2}\dot{f}_2\bar{z})^2] + \frac{g}{2f_2}(z^2 + \bar{z}^2) + 2f_2[\omega^2 z\bar{z} + a(z + \bar{z}) - ia(z - \bar{z})]. \tag{3.27}$$

3. A coupled nonlinear oscillator

Next we investigate an invariant for a coupled nonlinear oscillator expressed by the Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\omega^2}{2}(x^2 + y^2) + \beta x(x^2 + y^2), \tag{3.28}$$

where ω and β are TD parameters. The above Hamiltonian is similar to the well-known Henon-Heiles system which describes the motion of stars about a galactic center and also useful in the description of the dynamics of vibrating triatomic molecules and triatomic solids. The corresponding Hamiltonian for the system in complex space becomes

$$H = \frac{1}{2}|\dot{z}\dot{\bar{z}}| + \frac{1}{2}\omega^2 z\bar{z} + \frac{1}{2}\beta(z\bar{z}^2 + \bar{z}z^2). \tag{3.29}$$

The equations of motion are obtained as

$$\ddot{z} + \omega^2 \bar{z} + \beta(2z\bar{z} + \bar{z}^2) = 0, \tag{3.30a}$$

$$\ddot{\bar{z}} + \omega^2 z + \beta(2z\bar{z} + z^2) = 0. \tag{3.30b}$$

The function $f_2(t)$ is given as a solution of the third-order differential equation

$$\dddot{f}_2 + \frac{2}{(z^2 + \bar{z}^2)}[(4\dot{f}_2\omega^2 + 4f_2\omega\dot{\omega})z\bar{z} + (5\dot{f}_2\beta + 2f_2\dot{\beta})(z\bar{z}^2 + \bar{z}z^2)] = 0, \tag{3.31}$$

which follows from eq.(3.12a). The second-order differential equation

$$\ddot{f}_2 - \frac{\dot{f_2}^2}{2f_2} - \frac{g_{z,\bar{z}}}{f_2} = 0 \tag{3.32}$$

is equivalent to eq.(3.31) if the time derivative of $g_{z,\bar{z}}(t)$, introduced in eq. (3.32), is given by

$$\dot{g}_{z,\bar{z}} = -\frac{2f_2}{(z^2 + \bar{z}^2)}[(4\dot{f}_2\omega^2 + 4f_2\omega\dot{\omega})z\bar{z} + (5\dot{f}_2\beta + 2f_2\dot{\beta})(z\bar{z}^2 + \bar{z}z^2)]. \tag{3.33}$$

With the help of the auxiliary equation (3.32), the invariant for the present system is expressed as

$$I_{f_2} = \frac{1}{f_2}[(f_2\dot{z} - \frac{1}{2}\dot{f}_2 z)^2 + (f_2\dot{\bar{z}} - \frac{1}{2}\dot{f}_2\bar{z})^2] + 2f_2 z\bar{z}[\omega^2 + \beta(z + \bar{z})z\bar{z}] + \frac{g}{2f_2}(z^2 + \bar{z}^2). \tag{3.34}$$

4. An inverse potential

Finally consider a general two-dimensional nonlinear complex Hamiltonian of the form

$$H = \frac{1}{2}|\dot{z}\dot{\bar{z}}| + \omega\frac{z}{\bar{z}} + \beta\frac{\bar{z}}{z}, \tag{3.35}$$

where parameters ω and β are TD parameters. The equations of motion for this case are as follow

$$\ddot{z} + 2[\frac{\omega}{\bar{z}} - \frac{\beta\bar{z}}{z^2}] = 0, \tag{3.36a}$$

$$\ddot{\bar{z}} - 2[\frac{\omega z}{\bar{z}^2} - \frac{\beta}{z}] = 0. \tag{3.36b}$$

The third-order differential equation for function f_2 is given as

$$\dddot{f}_2 + \frac{2}{(z^2+\bar{z}^2)}[4\dot{f}_2(\omega\frac{z}{\bar{z}} + \beta\frac{\bar{z}}{z}) + 4f_2(\dot{\omega}\frac{z}{\bar{z}} + \dot{\beta}\frac{\bar{z}}{z})] = 0, \tag{3.37}$$

which follows from (3.12a). Again the following second-order equation

$$\ddot{f}_2 - \frac{\dot{f_2}^2}{2f_2} - \frac{g_{z,\bar{z}}}{f_2} = 0 \tag{3.38}$$

is equivalent to (3.37) if the time derivative of $g_{z,\bar{z}}(t)$, introduced in eq. (3.38), is given by

$$\dot{g}_{z,\bar{z}} = -\frac{2f_2}{(z^2+\bar{z}^2)}[4\dot{f}_2(\omega\frac{z}{\bar{z}} + \beta\frac{\bar{z}}{z}) + 4f_2(\dot{\omega}\frac{z}{\bar{z}} + \dot{\beta}\frac{\bar{z}}{z})]. \tag{3.39}$$

With the help of the eq.(3.38), the invariant (3.13a) is formulated as

$$\begin{aligned} I_{f_2} = &\frac{1}{f_2}[(f_2\dot{z} - \frac{1}{2}\dot{f}_2 z)^2 + (f_2\dot{\bar{z}} - \frac{1}{2}\dot{f}_2\bar{z})^2] \\ &+ 4f_2[\omega\frac{z}{\bar{z}} + \beta\frac{\bar{z}}{z}] + \frac{g}{2f_2}(z^2+\bar{z}^2). \end{aligned} \tag{3.40}$$

Thus on the basis of above studies, we can emphasize that the SR method developed here in the $z\bar{z}$ coordinates is capable of producing invariants of TD systems in complex space and can straightforwardly be employed to find more applications.