Chapter 2

Invariants in real space

It is apparent from the discussions of the previous chapter that determination of invariants for dynamical systems is quite important as these mathematical structures have multitude of applications in different branches of science [4, 7, 73]. In past, many concerted efforts had been made to devise different methods to isolate invariants and find their possible utilities and interpretations for both TD and TID systems [4, 7, 73, 74]. Recently Struckmeier and Riedel (SR) [28] proposed a novel technique for the construction of quadratic invariants for non-autonomous systems. In fact, quadratic invariants are widely studied because of their resemblance with system's Hamiltonian whose kinetic energy part is also quadratic in momenta. Naturally one can be curious to look for more applications of a newly developed technique by employing it to more physical problems and extending the study to search higher order invariants in higher dimensions. Such extension studies would prove a testing ground to assess the strength and weakness of a given method. In this direction, many authors have also paid their attention to investigate cubic and quartic invariants and their applications [4, 7, 73, 75]. The higher order invariants are particularly interesting for establishing super integrability of dynamical systems [89]. With a motivation to expand the catalogue of applications of SR method, two studies have been carried out in the present chapter. In the first one, we investigate exact invariants for some physically interesting one and two dimensional TD systems. In the second one, the same methodology has been extended to search quartic invariants for a couple of TD systems [77].

2.1 Construction of quadratic invariants

Here, in the first study, we find quadratic invariants of five TD dynamical systems viz one dimensional general quartic polynomial potential, Morse potential, Hulthen's potential, a two-dimensional coupled quartic potential and Toda potential within the framework of SR method. For this purpose, in the following subsections, we first give a brief account of the underlying method and subsequently apply it to derive invariants of the above mentioned systems.

2.1.1 Formalism

Consider a two dimensional TD system described by a Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y, t). \tag{2.1}$$

The concomitant equations of motion of the above system are written as

$$\begin{aligned} \dot{x} &= p_x, & \dot{y} &= p_y, \\ \dot{p}_x &= -\frac{\partial V(x,y,t)}{\partial x}, & \dot{p}_y &= -\frac{\partial V(x,y,t)}{\partial y}. \end{aligned} \tag{2.2}$$

The solutions of the above equations, $x(t), y(t)$, $p_x(t)$ and $p_y(t)$, completely describe the system's time evolution within the four-dimensional phase space.

A quantity $I = I(x(t), y(t), p_x(t), p_y(t), t)$ is regarded as an invariant of the particle motion of a system if it conforms the invariance condition

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0, \tag{2.3}$$

where PB stands for Poisson's bracket. Note that the above equation can easily be derived by taking total time derivative of the function I and invoking the equations of motion eq.(2.2). Now we examine the existence of a conserved quantity for a system described by eq.(2.1) with a special ansatz for I being at most quadratic in velocities

$$I = f_2(t)(p_x^2 + p_y^2) + f_1(x, t)p_x + g_1(y, t)p_y + f_0(x, y, t). \tag{2.4}$$

The set of functions $f_2(t)$, $f_1(x, t)$, $g_1(y, t)$ and $f_0(x, y, t)$ that render I invariant are to be determined. With the single particle equations of motion eq.(2.2), a vanishing total time derivative of eq.(2.4) means explicitely

$$\begin{aligned} &(\dot{x}^2 + \dot{y}^2)\frac{df_2}{dt} + \dot{x}\frac{\partial f_1}{\partial t} + \dot{y}\frac{\partial g_1}{\partial t} + \dot{x}^2\frac{\partial f_1}{\partial x} + \dot{y}^2\frac{\partial g_1}{\partial y} + \dot{x}\frac{\partial f_0}{\partial x} \\ &+\dot{y}\frac{\partial f_0}{\partial y} - (2f_2\dot{x} + f_1)\frac{\partial V}{\partial x} - (2f_2\dot{y} + g_1)\frac{\partial V}{\partial y} + \frac{\partial f_0}{\partial t} = 0. \end{aligned} \tag{2.5}$$

Now arrange the terms of above equation with regard to their powers in the velocities $\dot{x}$ and $\dot{y}$. Therefore, the coefficients pertaining to the velocity powers must vanish separately and this gives us following set of equations

$$\frac{df_2}{dt} + \frac{\partial f_1}{\partial x} = 0, \tag{2.6a}$$
$$\frac{df_2}{dt} + \frac{\partial g_1}{\partial y} = 0, \tag{2.6b}$$
$$\frac{\partial f_0}{\partial x} + \frac{\partial f_1}{\partial t} - 2f_2\frac{\partial V}{\partial x} = 0, \tag{2.6c}$$
$$\frac{\partial f_0}{\partial x} + \frac{\partial g_1}{\partial t} - 2f_2\frac{\partial V}{\partial y} = 0, \tag{2.6d}$$
$$f_1\frac{\partial V}{\partial x} + g_1\frac{\partial V}{\partial y} - \frac{\partial f_0}{\partial t} = 0. \tag{2.6e}$$

The solutions of the first two equations may be expressed as

$$f_1(x,t) = -\dot{f}_2(t)x + b_x(t), \tag{2.7a}$$
$$g_1(y,t) = -\dot{f}_2(t)y + b_y(t), \tag{2.7b}$$

with $b_x(t)$ and $b_y(t)$ as arbitrary functions of time only. Next, using eqs.(2.7a) and (2.7b) in eqs.(2.6c) and (2.6d) and integrating the resultant expressions, the solution for f_0 is finally written as

$$f_0(x,y,t) = 2f_2V + \frac{1}{2}\ddot{f}_2(x^2+y^2) - \dot{b}_x x - \dot{b}_y y. \tag{2.8}$$

The remaining eq.(2.6e) can be expressed as a linear third-order differential equation for f_2, b_x and b_y as

$$\begin{aligned} &2\dot{f}_2V + 2f_2\frac{\partial V}{\partial t} + \frac{1}{2}\dddot{f}_2(x^2+y^2) + \dot{f}_2(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}) \\ &+b_x\ddot{x} - \ddot{b}_x x + b_y\ddot{y} - \ddot{b}_y y = 0. \end{aligned} \tag{2.9}$$

which imposes a condition for I to embody an invariant of the particle motion. With f_2 representing a solution of eq.(2.9), the invariant I, eq.(2.4), follows from eqs.(2.7a), (2.7b) and (2.8) together with the Hamiltonian as

$$\begin{aligned} I = {} & 2f_2H - \dot{f}_2(xp_x + yp_y) + \frac{1}{2}\ddot{f}_2(x^2+y^2) \\ & + b_xp_x - \dot{b}_x x + b_yp_y - \dot{b}_y y. \end{aligned} \tag{2.10}$$

It is easy to show that the invariant (2.10) embodies a time integral of eq.(2.9) by calculating its total time derivative and by inserting the single particle equations of motion (2.2).

Hence, eq.(2.10) provides a time integral of eq.(2.9) if and only if the system's evolution is governed by the equations of motion (2.2).

Since b_x and b_y are arbitrary functions of time only and do not depend on f_2, therefore the terms in eq.(2.9) must vanish separately i.e.

$$\dot{f}_2(2V + x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}) + 2f_2\frac{\partial V}{\partial t} + \frac{1}{2}\dddot{f}_2(x^2 + y^2) = 0, \tag{2.11a}$$

$$b_x\ddot{x} - \ddot{b}_x x = 0, \qquad b_y\ddot{y} - \ddot{b}_y y = 0. \tag{2.11b}$$

We thus obtain the following distinct invariants

$$I_{f_2} = 2f_2H - \dot{f}_2(xp_x + yp_y) + \frac{1}{2}\ddot{f}_2(t)(x^2 + y^2), \tag{2.12a}$$

$$I_{b_x} = b_xp_x - \dot{b}_x x, \qquad I_{b_y} = b_yp_y - \dot{b}_y y. \tag{2.12b}$$

Note that for autonomous systems ($\frac{\partial V}{\partial t} = 0$), $f_2 = constant$ is always a solution of eq.(2.11a) and for such a case $I_{f_2} \propto H$, thus providing the system's total energy, which is a known invariant for Hamiltonian systems with no explicit time dependence. Nevertheless, eq.(2.12a) also allows for solutions $f_2 \neq constant$ for these systems. We thereby obtain other nontrivial invariants for autonomous systems that exist in addition to the total energy.

After developing the methodology to find out invariants for TD systems, in what follows, we construct invariants of a few one and two dimensional TD systems.

2.1.2 Examples

1. A two-dimensional coupled quartic system

Firstly, we investigate a two dimensional nonlinear Hamiltonian system of a TD coupled quartic system defined by

$$\begin{aligned} H = &\frac{1}{2}(p_x^2 + p_y^2) + a_1(t)(x^2 + y^2) + a_2(t)xy + a_3(t)(x^4 + y^4) \\ &+ a_4(t)x^2y^2 + a_5(t)x^3y + a_6(t)xy^3. \end{aligned} \tag{2.13}$$

The related equations of motion are written as

$$\ddot{x} + 2a_1x + a_2y + 4a_3x^3 + 2a_4xy^2 + 3a_5x^2y + a_6y^3 = 0, \tag{2.14a}$$

$$\ddot{y} + 2\dot{a}_1y + \dot{a}_2x + 4\dot{a}_3y^3 + 2\dot{a}_4x^2y + \dot{a}_5x^3y + 3\dot{a}_6xy^2 = 0. \tag{2.14b}$$

The invariant I_{f_2} is immediately found by writing down the general invariant (2.12a) with the Hamiltonian H given by eq.(2.13)

$$\begin{aligned} I_{f_2} = & f_2[p_x^2 + p_y^2 + 2a_1(x^2+y^2) + 2a_2xy + 2a_3(x^4+y^4) \\ & + 2a_4x^2y^2 + 2a_5x^3y + 2a_6xy^3] - \dot{f}_2(xp_x + yp_y) \\ & + \frac{1}{2}\ddot{f}_2(x^2+y^2). \end{aligned} \tag{2.15}$$

The function f_2 for this particular case is given as a solution of the third order differential equation

$$\dddot{f}_2 + 4f_2\dot{a}_1 + 8\dot{f}_2a_1 + \frac{1}{x^2+y^2}[4f_2A + 4\dot{f}_2B] = 0, \tag{2.16}$$

where $A = (\dot{a}_2xy + \dot{a}_3(x^4+y^4) + \dot{a}_4x^2y^2 + \dot{a}_5x^3y + \dot{a}_6xy^3)$ and $B = (2a_2xy + 3a_3(x^4+y^4) + 3a_4x^2y^2 + 3a_5x^3y + 3a_6xy^3)$, which follows from eq.(2.11a). The particle trajectories $x(t)$ and $y(t)$ are explicitly contained in eqs.(2.14a) and (2.14b) respectively, and must be known prior to integrating eq.(2.16). The third order eq.(2.16) may be reduced to second order differential equation

$$\ddot{f}_2 - \frac{\dot{f}_2^2}{2f_2} + 4f_2a_1 - \frac{g_{x,y}(t)}{f_2} = 0, \tag{2.17}$$

provided that the time derivative of the function $g_{x,y}$, introduced in eq.(2.17), is given by

$$\dot{g}_{x,y} = -\frac{(4f_2^2A + 4f_2\dot{f}_2B)}{x^2+y^2}. \tag{2.18}$$

Finally, by using the auxiliary eq.(2.17), the invariant (2.15) is turned to be

$$\begin{aligned} I_{f_2} = & \frac{g_{x,y}}{2f_2}(x^2+y^2) + \frac{1}{f_2}[(f_2p_x - \frac{1}{2}\dot{f}_2x)^2 + (f_2p_y - \frac{1}{2}\dot{f}_2y)^2 \\ & + 2f_2^2xy(a_2 + a_4xy + a_5x^2 + a_6y^2) + 2f_2^2a_3(x^4+y^4)]. \end{aligned} \tag{2.19}$$

2. The Toda potential

Secondly, from the challenging series of exponential potential systems, we investigate the two dimensional nonlinear TD Toda potential

$$V(x,y,t) = e^{\alpha(t)x-y} + e^{-\alpha(t)x-y} \tag{2.20}$$

whose Hamiltonian is given as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + e^{\alpha x - y} + e^{-\alpha x - y}. \tag{2.21}$$

The related equations of motion are

$$\ddot{x} + \alpha(t)[e^{\alpha x - y} - e^{-\alpha x - y}] = 0, \tag{2.22a}$$

$$\ddot{y} - [e^{\alpha x - y} + e^{-\alpha x - y}] = 0. \tag{2.22b}$$

Again, following the same sequence of steps, the third order differential equation

$$\dddot{f}_2 + 4x(f_2\dot{\alpha} + \dot{f}_2\alpha)[\frac{e^{\alpha x - y} - e^{-\alpha x - y}}{x^2 + y^2}] + 2\dot{f}_2[e^{\alpha x - y} + e^{-\alpha x - y}]\frac{2 - y}{x^2 + y^2} = 0, \tag{2.23}$$

may be converted into a coupled set of first and second order equations. The second order equation is written as

$$\ddot{f}_2 - \frac{\dot{f}_2}{2f_2} - \frac{g_{x,y}}{f_2} = 0, \tag{2.24}$$

where

$$\dot{g}_{x,y} = -[4x(f_2^2\dot{\alpha} + f_2\dot{f}_2\alpha)(e^{\alpha x - y} - e^{-\alpha x - y}) + 2f_2\dot{f}_2(e^{\alpha x - y} + e^{-\alpha x - y})(2 - x)](\frac{1}{x^2 + y^2}). \tag{2.25}$$

The modified form of the invariant (2.12a) for the system described by (2.21) is obtained, by making use of eq.(2.24), as

$$I_{f_2} = \frac{1}{f_2}[(f_2\dot{x} - \frac{1}{2}\dot{f}_2 x)^2 + (f_2\dot{x} - \dot{f}_2 x)^2 + \frac{g_{x,y}}{2}(x^2 + y^2) + 2f_2^2(e^{\alpha x - y} + e^{-\alpha x - y})]. \tag{2.26}$$

3. A one-dimensional quartic potential

Next, we consider the case of a general one-dimensional nonlinear TD quartic polynomial potential whose Hamiltonian takes the form

$$H = \frac{1}{2}p^2 + a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3 + a_4(t)x^4. \tag{2.27}$$

The related equation of motion follows as

$$\ddot{x} + a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 = 0. \tag{2.28}$$

In the same manner as is done above, the third order differential equation

$$\dddot{f}_2 + 8\dot{f}_2 a_2 + 4f_2\dot{a}_2 + x(10\dot{f}_2 a_3 + 4f_2\dot{a}_3) + x^2(12\dot{f}_2 a_4 + 4f_2\dot{a}_4) + \frac{1}{x}(6\dot{f}_2 a_1 + 4f_2\dot{a}_1) + \frac{1}{x^2}(4\dot{f}_2 a_0 + 4f_2\dot{a}_0) = 0, \tag{2.29}$$

may be converted to the second order differential equation which is given as

$$\ddot{f}_2 - \frac{\dot{f}_2^2}{2f_2} + 4f_2 a_2 - \frac{g_x}{f_2} = 0, \tag{2.30}$$

where

$$\dot{g}_x = -f_2[x(10\dot{f}_2 a_3 + 4f_2\dot{a}_3) + x^2(12\dot{f}_2 a_4 + 4f_2\dot{a}_4) + \frac{1}{x}(6\dot{f}_2 a_1 + 4f_2\dot{a}_1) + \frac{1}{x^2}(4\dot{f}_2 a_0 + 4f_2\dot{a}_0)]. \tag{2.31}$$

Making use of eq.(2.27) along with the help of the auxiliary eq.(2.30) the invariant (2.12a) takes the form

$$I_{f_2} = \frac{1}{f_2}[(f_2 p - \frac{1}{2}\dot{f}_2 x)^2 + \frac{g_x}{2}x^2 + 2f_2^2(a_0 + a_1 x + a_3 x^3 + a_4 x^4)]. \tag{2.32}$$

4. The Morse potential

Fourthly, we consider the case of the Morse potential$V(x,t) = V_0(t)(e^{-2ax} - 2e^{-ax})$ where V_0 and a are TD coupling parameters and, the Hamiltonian of the system is given by

$$H = \frac{1}{2}p^2 + V_0(t)(e^{-2ax} - 2e^{-ax}.) \tag{2.33}$$

The related equation of motion follows as

$$\ddot{x} - 2V_0 a(e^{-2ax} - e^{-ax}) = 0. \tag{2.34}$$

Again following the same method the third order differential equation for this particular case is obtained as

$$\frac{1}{2}\dddot{f}_2 x^2 + 2\dot{f}_2[(e^{-2ax} - 2e^{-ax}) - ax(e^{-2ax} - e^{-ax})]V_0 + 2f_2[(e^{-2ax} - 2e^{-ax})\frac{\partial V_0}{\partial t} - 2\dot{a}xV_0(e^{-2ax} - e^{-ax})] = 0, \tag{2.35}$$

which follows from (2.11a). Since the particle trajectory is explicitly contained in (2.34), it must be known prior to integrating eq.(2.35). The third order equation (2.35) may be transformed to second order linear differential equation

$$\ddot{f}_2 - \frac{\dot{f}^2}{2f_2} - \frac{g_x}{f_2} = 0, \tag{2.36}$$

provided the time derivative of the function g_x, introduced in (2.36) is given by

$$\dot{g}_x = \frac{4}{x}V_0(t)(e^{-2ax} - e^{-ax})(f_2\dot{f}_2 a + 2f_2^2\dot{a}) - \frac{4}{x^2}(e^{-2ax} - 2e^{-ax})(f_2\dot{f}_2 V_0 + f_2^2\dot{V}_0). \tag{2.37}$$

The invariant for this particular exponential system, represented by (2.33), is found to be

$$I_{f_2} = \frac{1}{f_2}[(f_2 p - \frac{1}{2}\dot{f}_2 x)^2 + 2f_2^2 V(x,t) + \frac{1}{2}g_x x^2]. \tag{2.38}$$

5. The Hulthen's potential

Finally, we investigate an interesting potential due to Hulthen i.e.

$$V(x,t) = \frac{e^{-a(t)x}}{1 - e^{-a(t)x}}, \tag{2.39}$$

and the corresponding Hamiltonian is written as

$$H = \frac{1}{2}p^2 + \frac{e^{-a(t)x}}{1 - e^{-a(t)x}}, \tag{2.40}$$

with following equation of motion

$$\ddot{x} - aV(1+V) = 0. \tag{2.41}$$

Now by following the same procedure outlined in the above cases, the third order differential equation

$$\dddot{f}_2 - \frac{1}{x}V(1-V)(4f_2\dot{a} + 2\dot{f}_2 a) + 4V\frac{\dot{f}_2}{x^2} = 0, \tag{2.42}$$

may be reduced to second order differential equation as

$$\ddot{f}_2 - \frac{\dot{f}_2^2}{2f_2} - \frac{g_x}{f_2} = 0, \tag{2.43}$$

where

$$\dot{g}_x = \frac{f_2}{x}[V(1-V)(4f_2\dot{a} + 2\dot{f}_2 a) - 4V\frac{\dot{f}_2}{x}]. \tag{2.44}$$

The particle trajectory must be obtained by integrating (2.41) before going for the integration of (2.42). Finally using eq.(2.40) along with the auxiliary eq.(2.43) the invariant (2.12a) takes the form

$$I_{f_2} = \frac{1}{f_2}[(f_2 p - \frac{1}{2}\dot{f}_2 x)^2 + 2f_2^2 V + \frac{g_x}{2}x^2]. \tag{2.45}$$

After deriving quadratic invariants for some dynamical systems, now we take up the task of finding quartic invariants for a couple of physical systems.

2.2 Construction of quartic invariants

The existence of higher order invariants is found useful in many physical problems and many studies have been reported on the utilities of cubic and quartic invariants [78]. This section presents the construction of exact quartic invariants for one-dimensional TD dynamical systems using the SR approach [79]. The invariants are found to contain a function of time $f_4(t)$ which is a solution of a linear fourth order differential equation.

2.2.1 Formalism

We start with a one-dimensional single particle system described by a Hamiltonian

$$H = \frac{1}{2}p^2 + V(x,t), \tag{2.46}$$

with following concomitant equations of motion

$$\dot{x} = p, \quad \dot{p} = -\frac{\partial V}{\partial x}. \tag{2.47}$$

A phase space quantity $I = I(x(t), p(t), t)$ will be an invariant of the particle motion if its total time derivative vanishes. Now the main concern in the invariant theory is to design an appropriate form of I. In past, researcher's investigated a variety of functional forms of invariants [7], however, a polynomial in momenta form was most frequently studied. So here, we examine the existence of a conserved quantity for the system described by eq.(2.46) with a special ansatz for I being quartic polynomial in momentum as

$$I = f_4(t)p^4 + f_3(x,t)p^3 + f_2(x,t)p^2 + f_1(x,t)p + f_0(x,t). \tag{2.48}$$

The set of functions $f_4(t)$, $f_3(x,t)$, $f_2(x,t)$, $f_1(x,t)$ and $f_0(x,t)$ that render I invariant are to be determined. With the single particle equations of motion (2.47), a vanishing total time derivative of eq.(2.48) means explicitly

$$\begin{aligned} &\frac{df_4}{dt}p^4 + 4f_4p^3\dot{p} + \frac{\partial f_3}{\partial t}p^3 + 3f_3p^2\dot{p} + p^3\frac{\partial f_3}{\partial x}\dot{x} + \frac{\partial f_2}{\partial t}p^2 + 2f_2p\dot{p} \\ &\quad + p^2\frac{\partial f_2}{\partial x}\dot{x} + \frac{\partial f_1}{\partial t}p + f_1\dot{p} + p\frac{\partial f_1}{\partial x}\dot{x} + \frac{\partial f_0}{\partial t} + \frac{\partial f_0}{\partial x}\dot{x} = 0. \end{aligned} \tag{2.49}$$

Next arrange the terms of this equation with regard to their powers in momentum and equating the coefficients pertaining to various momentum powers of the resultant equation, after using equations of motion, separately to zero gives the following set of first order coupled differential equations

$$\frac{df_4}{dt} + \frac{\partial f_3}{\partial x} = 0, \tag{2.50a}$$
$$4f_4\frac{\partial V}{\partial x} - \frac{df_3}{dt} - \frac{\partial f_2}{\partial x} = 0, \tag{2.50b}$$
$$3f_3\frac{\partial V}{\partial x} - \frac{df_2}{dt} - \frac{\partial f_1}{\partial x} = 0, \tag{2.50c}$$
$$2f_2\frac{\partial V}{\partial x} - \frac{df_1}{dt} - \frac{\partial f_0}{\partial x} = 0, \tag{2.50d}$$
$$\frac{\partial f_0}{\partial t} - f_1\frac{\partial V}{\partial x} = 0. \tag{2.50e}$$

On solving the above set of equation successively, one can obtain the following forms for f_3, f_2, f_1 and f_0 in terms of f_4 and unknown integration constants a, b, c and d as

$$f_3 = -\dot{f}_4 x + a(t), \tag{2.51}$$

$$f_2 = 4f_4V + \frac{1}{2}\ddot{f}_4x^2 - \dot{a}x + b(t), \tag{2.52}$$

$$\begin{aligned} f_1 = &-3\dot{f}_4\int x\frac{\partial V}{\partial x}dx - 4\dot{f}_4\int V dx - 4f_4\int \frac{\partial V}{\partial t}dx \\ &-\frac{1}{6}\dddot{f}_4x^3 + 3aV + \frac{1}{2}\ddot{a}x^2 - \dot{b}x + c(t), \end{aligned} \tag{2.53}$$

$$\begin{aligned} f_0 = 8f_4\int V\frac{\partial V}{\partial x}dx &+ 4f_4\int dx\int\frac{\partial^2 V}{\partial t^2}dx + 3\dot{f}_4\int dx\int\frac{\partial^2 V}{\partial x\partial t}dx \\ &+ 8\dot{f}_4\int dx\int\frac{\partial V}{\partial t}dx + \ddot{f}_4\int x^2\frac{\partial V}{\partial x}dx + 3\dot{f}_4\int dx\int x\frac{\partial V}{\partial x}dx \\ &+ 4\ddot{f}_4\int dx\int V dx + \frac{\ddddot{f}_4x^4}{24} - 3a\int\frac{\partial V}{\partial t}dx - 2\dot{a}\int x\frac{\partial V}{\partial x}dx \\ &- 3\dot{a}\int V dx - \frac{1}{6}\dddot{a}x^3 + 2bV + \frac{1}{2}\ddot{b}x^2 - \dot{c}x + d(t). \end{aligned} \tag{2.54}$$

The remaining eq.(2.50e) may be expressed as a set of third order and fourth-order differential equations with respect to its dependence and non dependence on f_4 respectively as

$$\ddddot{f}_4 - \frac{1}{2}\frac{\dddot{f}_4^{\,2}}{\ddot{f}_4} - \frac{g(t)}{\ddot{f}_4} = 0, \tag{2.55}$$

$$\begin{aligned} 3aV\frac{\partial V}{\partial x} &+ \frac{1}{2}\ddot{a}x^2\frac{\partial V}{\partial x} - \dot{b}x\frac{\partial V}{\partial x} + c\frac{\partial V}{\partial x} + 2\ddot{a}\int x\frac{\partial V}{\partial t}dx \\ +\dot{a}\int x\frac{\partial^2 V}{\partial t^2}dx &+ 3\ddot{a}\int V dx + 6\dot{a}\int\frac{\partial V}{\partial t}dx + 3a\int\frac{\partial^2 V}{\partial t^2}dx \\ &+\frac{1}{6}\ddddot{a}x^3 + 2\dot{b}V - 2b\frac{\partial V}{\partial t} - \frac{1}{2}\dddot{b}x^2 + \ddot{c}x - \dot{d} = 0. \end{aligned} \tag{2.56}$$

The time derivative of the function $g_x(t)$ introduced in eq.(2.55) is given as

$$\begin{aligned}\dot{g} = -\frac{24\ddot{f}_4}{x^4}\Big[\dddot{f}_4\Big\{\frac{1}{6}x^3\frac{\partial V}{\partial x} + \int x^2\frac{\partial V}{\partial x}dx + 3\int dx\int x\frac{\partial V}{\partial x}dx \\ + 4\int dx\int Vdx\Big\} + \ddot{f}_4\Big\{\int x^2\frac{\partial^2 V}{\partial x\partial t}dx + 6\int dx\int x\frac{\partial^2 V}{\partial x\partial t}dx \\ + 12\int dx\int\frac{\partial V}{\partial t}dx\Big\} + \dot{f}_4\Big\{3\frac{\partial V}{\partial x}\int x\frac{\partial V}{\partial x}dx + 4\frac{\partial V}{\partial x}\int Vdx \\ + 8\int V\frac{\partial V}{\partial x}dx + 3\int dx\int x\frac{\partial^3 V}{\partial x\partial t^2}dx + 12\int dx\int\frac{\partial^2 V}{\partial t^2}dx\Big\} \\ + f_4\Big\{4\frac{\partial V}{\partial x}\int\frac{\partial V}{\partial t}dx + 8\frac{\partial}{\partial t}\int V\frac{\partial V}{\partial x}dx + 4\int dx\int\frac{\partial^3 V}{\partial t^3}dx\Big\}\Big]\end{aligned} \tag{2.57}$$

Since a, b, c and d are arbitrary functions of time and don't depend on $f_4(t)$, thus we obtain the following distinct invariants using eqs.(2.51-2.55) into eq.(2.48) as

$$\begin{aligned}I_{f_4} = f_4p^4 - \dot{f}_4xp^3 + 4f_4Vp^2 + \frac{1}{2}\ddot{f}_4x^2p^2 + 4f_4\int dx\int\frac{\partial^2 V}{\partial t^2}dx \\ - 4f_4p\int\frac{\partial V}{\partial t}dx - \frac{1}{6}\dddot{f}_4x^3p + 8f_4\int V\frac{\partial V}{\partial x}dx - 3\dot{f}_4p\int x\frac{\partial V}{\partial x}dx \\ + 3\dot{f}_4\int dx\int x\frac{\partial^2 V}{\partial x\partial t}dx + 8\dot{f}_4\int dx\int\frac{\partial V}{\partial t}dx + \ddot{f}_4\int x^2\frac{\partial V}{\partial x}dx \\ + 3\ddot{f}_4\int dx\int x\frac{\partial V}{\partial x}dx + 4\ddot{f}_4\int dx\int Vdx - 4\dot{f}_4p\int Vdx \\ + \frac{x^4}{24}\Big(\frac{\dddot{f}_4^{\,2}}{2\ddot{f}_4} + \frac{g}{\ddot{f}_4}\Big)\end{aligned} \tag{2.58}$$

$$\begin{aligned}I_{abcd} = -3a\int\frac{\partial V}{\partial t}dx - 2\dot{a}\int x\frac{\partial V}{\partial x}dx - 3\dot{a}\int Vdx \\ - \frac{1}{6}\dddot{a}x^3 + 2bV + \frac{1}{2}\ddot{b}x^2 - \dot{c}x + d + ap^3 - \dot{a}xp^2 \\ + 3aVp + bp^2 + \frac{1}{2}\ddot{a}x^2p - \dot{b}xp + cp\end{aligned} \tag{2.59}$$

The invariant I_{f_4} of the particle motion obtained above remains conserved along the phase-space path representing the system's time evolution. After developing the mathematical setup for construction of invariants, in what follows, we take a couple of examples to demonstrate the applicability of the above procedure.

2.2.2 Examples

1. A dynamical harmonic oscillator

As a simple example, firstly we investigate the one-dimensional TD harmonic oscillator defined by the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)x^2. \tag{2.60}$$

The corresponding equation of motion follows as

$$\ddot{x} + \omega^2 x = 0. \tag{2.61}$$

Now using eqs.(2.60) and (2.61) in the general invariant eq.(2.58), one immediately write the invariant I_{f_4} for the one-dimensional TD oscillator as

$$\begin{aligned} I_{f_4} = {} & f_4[\{p^2 + \omega^2 x^2\}^2 - \frac{4}{3}\omega\dot{\omega}x^3 p + \frac{1}{3}\dot{\omega}^2 x^4 + \frac{1}{3}\omega\ddot{\omega}x^4] \\ & + \dot{f}_4\{-xp^3 - \frac{5}{3}\omega^2 x^3 p + \frac{7}{6}\omega\dot{\omega}x^4\} + \ddot{f}_4\{\frac{1}{2}x^2p^2 + \frac{2}{3}\omega^2 x^4\} \\ & - \frac{1}{6}\dddot{f}_4 x^3 p + \frac{x^4}{24}\{\frac{1}{2}\frac{\dddot{f}_4^{\,2}}{\ddot{f}_4} + \frac{g_x}{\ddot{f}_4}\}. \end{aligned} \tag{2.62}$$

The function f_4 is given as a solution of the fourth-order differential equation

$$\ddddot{f}_4 - \frac{1}{2}\frac{\dddot{f}_4^{\,2}}{\ddot{f}_4} - \frac{g_x}{\ddot{f}_4} = 0, \tag{2.63}$$

where the time derivative of the function g_x introduced in (2.63) is given as

$$\begin{aligned} \dot{g}_x = {} & -\frac{24\ddot{f}_4}{x^4}[\frac{5}{6}\dddot{f}_4\omega^2 x^4 + \frac{5}{2}\ddot{f}_4\omega\dot{\omega}x^4 + \dot{f}_4\{\frac{8}{3}\omega^4 x^4 + \frac{3}{2}\dot{\omega}^2 x^4 \\ & + \frac{3}{2}\omega\ddot{\omega}x^4\} + f_4\{\frac{16}{3}\dot{\omega}\omega^3 x^4 + \dot{\omega}\ddot{\omega}x^4 + \frac{1}{3}\omega\dddot{\omega}x^4\}]. \end{aligned} \tag{2.64}$$

Since the particle trajectories $x = x(t)$ are explicitly contained in eq.(2.63), therefore these must be known prior to integrating eq.(2.63) and are obtained by integrating the equation of motion (2.61). Now it can easily be shown that I_{f_4} is essentially a constant of motion by taking the total time derivative of eq.(2.62), inserting the Hamiltonian (2.60) and equation of motion eq.(2.61), we arrive at eq.(2.63), which is fulfilled by definition of f_4 for given trajectories.

Similarly the other invariant I_{abcd} can easily be obtained by using eq.(2.60) in invariant eq.(2.59) along with a third order differential equation derived from eq.(2.56).

2. A general time dependent potential

Secondly, we investigate a general one-dimensional Hamiltonian system defined by

$$H = \frac{1}{2}p^2 + a(t)x^m + b(t)x^n, \tag{2.65}$$

where m and n are arbitrary numbers. The related equation of motion is given as

$$\ddot{x} + ma(t)x^{m-1} + nb(t)x^{n-1} = 0. \tag{2.66}$$

Now using eqs.(2.65) and (2.66) in eq.(2.58), we find the invariant I_{f_4} for the present system as

$$\begin{aligned} I_{f_4} &= f_4[\{p^2 + 2(ax^m + bx^n)\}^2 - 4\{\frac{\dot{a}x^{m+1}}{m+1} + \frac{\dot{b}x^{n+1}}{n+1}\}p \\ &+ 4\{\frac{\ddot{a}x^{m+2}}{(m+1)(m+2)} + \frac{\ddot{b}x^{n+2}}{(n+1)(n+2)}\}] + \dot{f}_4\{-xp^3 \\ &- \frac{3m+4}{m+1}ax^{m+1}p + \frac{3n+4}{n+1}bx^{n+1}p + \frac{3m+8}{(m+1)(m+2)}\dot{a}x^{m+2} \\ &+ \frac{3n+8}{(n+1)(n+2)}\dot{b}x^{n+2}\} + \ddot{f}_4\{\frac{1}{2}x^2p^2 + \frac{m+2}{m+1}ax^{m+2} \\ &+ \frac{n+2}{n+1}bx^{n+2}\} - \frac{\dddot{f}_4}{6}x^3p + \frac{x^4}{24}\{\frac{\dddot{f}_4^2}{2\ddot{f}_4} + \frac{g_x}{\ddot{f}_4}\}, \end{aligned} \tag{2.67}$$

where the function $f_4(t)$ is given as a solution of the following fourth-order differential equation

$$\ddddot{f}_4 - \frac{1}{2}\frac{\dddot{f}_4^2}{\ddot{f}_4} - \frac{g_x}{\ddot{f}_4} = 0. \tag{2.68}$$

The time derivative of the function g_x associated with eq.(2.68) is given as

$$\begin{aligned}
\dot{g}_x = -\frac{24\ddot{f}_4}{x^4}[\frac{1}{6}\dddot{f}_4\{\frac{(m+3)(m+4)}{m+1}ax^{m+2} + \frac{(n+3)(n+4)}{n+1}bx^{n+2}\} \\
+ \ddot{f}_4\{\frac{(m+3)(m+4)}{(m+1)(m+2)}\dot{a}x^{m+2} + \frac{(n+3)(n+4)}{(n+1)(n+2)}\dot{b}x^{n+2}\} \\
+ \dot{f}_4\{\frac{(3m^2+8m+4)}{(m+1)}a^2x^{2m} + \frac{(3n^2+8n+4)}{(n+1)}b^2x^{2n} \\
+ \frac{(3m+4)}{(m+1)}abnx^{m+n} + \frac{(3n+4)}{(n+1)}abmx^{m+n} \\
\frac{(3m+4)}{(m+1)(m+2)}\ddot{a}x^{m+2} + \frac{(3n+4)}{(n+1)(n+2)}\ddot{b}x^{n+2}\} \\
+ 4f_4\{\frac{3m+2}{m+1}a\dot{a}x^{2m} + \frac{3n+2}{n+1}b\dot{b}x^{2n} \\
+ \frac{1}{(m+1)(m+2)}\dddot{a}x^{m+2} + \frac{1}{(n+1)(n+2)}\dddot{b}x^{n+2} \\
+ \frac{2n+m+2}{n+1}a\dot{b}x^{m+n} + \frac{2m+n+2}{m+1}b\dot{a}x^{m+n}\}]. \qquad (2.69)
\end{aligned}$$

It may be noted that the invariant of the one dimensional harmonic oscillator, eq.(2.62), can easily be retrieved from eq.(2.67) by adjusting $b = 0, a = \frac{\omega}{2}$ and $m = 2$.