Chapter 1

Introduction

In their endeavor to understand the behavior of dynamical systems, such as weather changes, earth quakes, cardiac arrhythmias, growth of populations of various species, propagation of light in optical fibres, fluid and plasma turbulences, neuron's initiation and propagation, energy landscapes in glassy systems, medical imaging etc., usually formulate physical problems through suitable mathematical model equations called dynamical equations or equations of motion. The dynamical equations can have a variety of incarnations like ordinary differential equations, partial differential equations, difference equations, integro-difference equations, algebraic equations and so on, depending upon the nature and range of the underlying forces and parameters involved of a system. Further the equations of motion are said to be linear or nonlinear depending upon whether the governing force is linear or non-linear. Recent studies have clearly shown that it does make a profound difference depending on whether the system is linear or nonlinear [1, 2]. These studies have led to revolutionary new concepts and opened a new interdisciplinary branch called nonlinear dynamics. The next logical step towards problem solving is to find an appropriate method, implement it and interpret obtained results to mimic the dynamics of the concerned system.

Thus, based upon the solvability of the equations of motion, dynamical systems can broadly be classified in two categories (i) integrable systems and (ii) non-integrable systems. An integrable system is that whose equation of motion is solvable in terms of some known functions. The solutions of such systems are often well behaved in the sense that a small change in initial conditions brings about only a small change in the motion and are referred to regular or normal solutions. In actual practice there are only a few integrable

systems like the undamped and unforced Duffing oscillator, the pendulum under small oscillations, the Kepler particle, a rigid symmetrical top, hydrogen atom etc.

It is interesting to note that existence of integrable dynamical systems can be utilized to develop suitable approximation schemes to deal with non-integrable systems by treating the integrable cases as basic zeroth order exact results. In the case of finite dimensional Hamiltonian systems this ultimately lead to the KAM theorem concerning the preservation of phase space tori under perturbation. The infinite degrees of freedom integrable systems can exhibit soliton solutions [1, 2].

On the other hand the equations of motion of nonintegrable [1, 2, 3, 4, 5] systems are not explicitly solvable and often exhibit chaotic [1, 2] behavior in a certain parametric range and greatly hamper the predictability. In fact, it is one of the important problems in nonlinear dynamics to identify whether a given system is integrable or nonintegrable. Therefor, to classify whether a given system is integrable or not, a mathematical criterion is essential and the concept of integrability can serve this purpose. Now let us take a detailed view of this vital concept.

1.1 Notion of integrability

While dealing dynamical a system, be it Hamiltonian or non-Hamiltonian, in nonlinear dynamics a natural question arises that what is the meaning of integrability and under what conditions it occurs. The answer of the former question is somewhat vague as the concept of integrability is itself, in a sense, not well defined and there seems no unique definition for it yet. The latter is even more difficult to answer, as no well defined criteria seem to exist to isolate integrable cases.

A literature survey, however reveals that the integrability nature of a dynamical system can systematically be investigated using the following two broad notions [2]. The first one uses essentially the literal meaning: integrable - integrated with the required number of integration constants; non-integrable - proven not to be integrable. This particular definition of integrability can be related to the existence of single valued, analytic solutions, for nonlinear ordinary differential equations leading to the notion of integrability in the complex plane. This concept, generally called Painleve property, was greatly pursued by many great mathematicians and physicist like Fuchs, Kovalevskaya, Painleve and others [6]. The second notion, particularly applicable to Hamiltonian systems, is to search for

a sufficient number of single valued, analytic, involutive integrals of motion for a Hamiltonian system with n-degrees of freedom, so that the associated Hamilton's equations of motion, in principle, can be integrated by quadratures in the sense of Liouville.

From a qualitative point of view, integrability can be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative information to understand the dynamics of the system locally as well as globally [2, 7].

Integrable models form a beautiful class of physical models in the sense that they are exactly solvable. For this reason they have always provided a treasured class of systems where the underlying structures of the model can be laid bare and understood. To study the cardinal and preeminent properties of a dynamical system, it is important to know that whether the system is integrable, and if it is, one wants to know as many quantities as possible whose values are conserved during the time evolution of the system. The global quantity mentioned above is a function from the space of dependent variables (phase space) to the real (or may be complex) numbers. This function, in literature, variously called as an integral of motion, a conserved quantity, a constant of motion, a second invariant etc. There can be several, even an infinity of constants of motion depending on the number of degrees of freedom of the system. Since invariants are analytic functions, and analytic results are much easier to use, to interpret and to generalize, therefore, these can act as an important link to understand the properties of nonlinear dynamical systems.

In classical mechanics, the Liouville integrability is the most common definition of integrability. A system, with $2n$ degrees of freedom, i.e. n coordinates q_i and n conjugate momenta p_j, with Poisson bracket $[q_i, p_j] = \delta_{ij}$, is said to be Liouville integrable if there are $2n$ independent, well defined, global functions $I_k(p, q)$ such that

$$[I_i, I_j] = 0, \quad \forall \ i, j,$$

i.e. all I_n's are involutive. In case of time-independent (TID) systems, one of the I_n's is H itself. In principle, there can be more than n functionally independent invariants, but they may not be in involution.

There is also a distinction between complete integrability, in the Liouville sense, and partial integrability, as well as a notion of superintegrability and maximal superintegrability. Essentially, these distinctions correspond to the dimensions of the leaves of the foliation. When the number of independent Poisson commuting invariants is less than maximal (but, in the case of autonomous systems, more than one), we say the system is partially inte-

grable. When there exist further functionally independent invariants, beyond the maximal number that can be Poisson commuting, and hence the dimension of the leaves of the invariant foliation are less than n, we say the system is superintegrable. If there is a regular foliation with one-dimensional leaves (curves), this is called maximally superintegrable. The best known cases of this particular class of systems are the n-dimensional harmonic oscillator and time dependent (TD) one dimensional harmonic oscillator.

Some well-known classical integrable systems are worth to mention as the central force motion, two center Newtonian gravitational motion, geodesic motion on ellipsoids, Calogero-Moser-Sutherland models, Neumann oscillator, swinging Atwood's machine with certain choices of parameters, integrable Clebsch and Steklov systems in fluids, Lagrange, Euler and Kovalevskaya tops (motion of a rigid body about a fixed point). Integrable lattice models, such as Toda lattice, Ablowitz-Ladik lattice, Volterra lattice, PDEs in $1+1$-dimensions like Korteweg-de Vries equation $(u_t = u_{xxx} + uu_x)$, sine-Gordon equation $(u_{tt} + u_{xx} = sinu)$, nonlinear Schrödinger equation $(i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + \kappa|\psi|^2\psi)$, Dym equation $(u_t = u^3u_{xxx})$, PDEs in $2+1$-dimensions, such as Kadomtsev-Petviashvili equation, Davey-Stewartson equation and Ishimori equation and PDEs in higher dimensions like Self-dual Yang-Mills equations [2].

Similarly the notion of integrability in quantum mechanics comes naturally as a generalization of a similar notion in classical mechanics [4]. A quantum mechanical system described by the stationary Schrödinger equation

$$H\psi = E\psi, \qquad H = -\frac{\hbar^2}{2m}\nabla^2 + V(x_1, ..., x_n),$$

is said to be completely integrable if there exists a set of $(2n-1)$ algebraically independent linear operators $X_a,\ a = 1, 2, ..., 2n-1$ commuting with the Hamiltonian and among each other i.e.,

$$[H, X_a] = 0, \quad [X_a, X_b] = 0.$$

The operators X_a, X_b etc. are usually assumed to be polynomial in momenta with coordinate dependent coefficients. The existence of such commuting operators lead to separation of variables in the Schrödinger equation. In the quantum setting, functions on phase space must be replaced by self-adjoint operators on a Hilbert space [8, 9], and the notion of Poisson commuting functions replaced by commuting operators.

Since there is no clear definition of independence of operators, except for special classes, the definition of integrable systems, in the quantum sense, is not yet agreed upon. The

working definition that is mostly used is that there is a maximal set of commuting operators, including the Hamiltonian, and a semiclassical limit in which these operators have symbols that are independent Poisson commuting functions on the phase space.

Quantum integrable systems can be explicitly solved by Bethe ansatz or quantum inverse scattering method. It has become widely known that there is a one to one correspondence between operators $\hat{A}$ in Hilbert space and functions $A(q,p)$ on phase space i.e., $\hat{A} \rightleftharpoons A(q,p)$ and this relationship is known as Wigner-Weyl transformation, which maps the quantum commutator $[\hat{A},\hat{B}]$ onto the Moyal bracket [10]

$$\begin{aligned}\{A,B\}_{MB} &= \frac{2}{\hbar}A\sin((1/2)\hbar \overset{\leftrightarrow}{\wedge})B \\ &= A \overset{\leftrightarrow}{\wedge} B - \frac{1}{24}\hbar^2 A \overset{\leftrightarrow 3}{\wedge} B + \frac{1}{1920}\hbar^4 A \overset{\leftrightarrow 5}{\wedge} B + ..., \end{aligned} \tag{1.1}$$

here $\{A,B\}_{MB}$ denotes the Moyal bracket and $\overset{\leftrightarrow}{\wedge}$ for a n dimensional system is given by

$$\overset{\leftrightarrow}{\wedge} = \sum_{i=1}^{n}\left[\frac{\overleftarrow{\partial}}{\partial q_i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q_i}\right]. \tag{1.2}$$

Therefore, in order to obtain quantum invariants of a system from the corresponding classical one, the quantum corrections [8, 9, 11] arising from the terms involving $\hbar$ in the expansion of the sine function in eq.(1.1) need to be incorporated. In the classical limit $\hbar \to 0$, the Moyal bracket approaches the Poisson bracket

$$\{\hat{A},\hat{B}\}_{MB} \to A \overset{\leftrightarrow}{\wedge} B = \{A,B\}_{PB}.$$

It is to be noted that if the invariant is at the most second order in momenta, then the Moyal bracket simply reduces to the Poisson bracket and the classical and quantum invariants turn out to be the same. For cubic and higher order invariants, the need of quantum corrections arises and only after incorporation of such corrections, a classical invariant becomes quantum invariant in the spirit of Moyal bracket. The Heisenberg quantum model, the Lieb-Liniger model and the Hubbard model are some well-known quantum integrable systems.

From the above discussion, it is apparent that the invariants play a vital role in establishing integrability of dynamical systems and thus need more elaboration. Next, we present a brief account of invariants and their various forms.

1.2 Types of invariants

Since the main emphasis of the present thesis is to investigate invariants for dynamical systems, therefor it is pertinent to know the meaning and various forms of "invariants" of a dynamical system. To this effect, consider a phase space function $I(x_i, p_i, t)$ for a Hamiltonian system. At the initial time t_0 the function I will get a value, $I(t_0)$, which is determined uniquely through the initial values $x_i(t_0)$ and $p_i(t_0),\ i = 1, 2, ..., N$. As time evolves the coordinates evolve according to the Hamilton equations of motion and as a result, the value of I may also change. If, however, the value of I remains constant under the motion, then I is called a constant of motion or integral of motion or invariant. Note that the constancy of I depends on the choice of the Hamiltonian H, and in particular H itself is a constant of motion for autonomous systems.

As an invariant is basically a phase space function which is in involution with the Hamiltonian as well as other invariants, if they exist for the system. For the invariant to be nontrivial, it is required that I must be functionally independent of H. Functional independency of two functions K and L is easily tested by considering $2N \times 2N$ Jacobian $\partial(K, L)/\partial(x_i, p_i)$ and if its rank is two then K and L are functionally independent.

Although, various functional forms of invariants are investigated and discussed in literature [4], but here we describe only a few widely used forms of invariants.

1.2.1 Polynomial invariants

The total energy of a dynamical system usually is the sum of kinetic energy and potential energy and there is a special status of the kinetic energy term, which is quadratic in momenta, in the Hamiltonian. Therefore, an invariant is often found to possess a polynomial form [4, 7] in momenta and the degree of polynomial gives the order of the invariant.

In general, for a two dimensional system, an ansatz for an n^{th} order invariant can be made as

$$I = a_0 + a_i\xi_i + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{3!}a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l + ..., \tag{1.3}$$

where $\xi_i = \dot{x}_i$ and $i, j, k, l, ... = 1, 2$ and the coefficients a_0, a_i, a_{ij}, a_{ijk}, $a_{ijkl}, ...$ etc. are functions of coordinates only and these coefficients are symmetric with respect to any interchange of their indices i.e. $a_{ij} = a_{ji}$ etc. Note that, for an autonomous Hamiltonian which is even powered in momenta (i.e. it is invariant under the time reflection symmetry),

the invariant will contain either even or odd power terms in momenta. However, if the potential term in H involves momentum-dependence, then all the terms upto desired order in the invariant should be considered. In some cases the invariant can have fractional powers in momenta provided it fulfills mathematical conditions such as single-valuedness, differentiability etc. In past, a number of researchers employed polynomial invariants to study classical and quantum dynamical systems and we shall also consider this form of invariants in the chapters 2-4.

1.2.2 Rational invariants

Invariants can also be considered as rational functions of the momenta [4]. In particular, an invariant I for a system is considered of the form

$$I = \frac{R}{S},$$

where R and S are polynomials in momenta and $S \neq 0$, so that not only S^{-1} exists but also $RS^{-1} = S^{-1}R$ holds. Clearly, the vanishing of the Poisson bracket of this form of I and H requires

$$S[H, R]_{PB} = R[H, S]_{PB},$$

which is equivalent to the pair

$$[H, R]_{PB} = GR; \quad [H, S]_{PB} = GS,$$

with G as some rational function of momenta. It is worth to mention that if $I = R/S$ is an invariant of the system then

$$I = (aR + bS)/(cR + dS),$$

is also an invariant with a, b, c and d arbitrary constants for which, $ad - bc \neq 0$. For more discussion on rational invariants see ref.[7].

1.2.3 Transcendental invariants

Transcendental invariants [4, 12] are a generalization of rational invariants. Or in other way, we can say that the rational invariants provide the transcendental invariants. By definition, a transcendental invariant I is an arbitrary function K of two different polynomials R and S in momenta i.e.

$$I = K(R, S),$$

where the Poisson bracket of H and I yields

$$K_R[H, R] + K_S[H, S] = 0,$$

and the function K can be solved from the above equation only if

$$[H, R] = GP_1(R, S),$$

$$[S, H] = GP_2(R, S),$$

where P_i's are some polynomials in R and S. In this case, not only the functional form of K but also the degrees of polynomials R and S will suggest further classification of transcendental invariants.

In literature, depending upon the demand of a particular physical situation, there also exist some other forms of invariants such as exponential type. However, out of various available forms of invariants, the polynomial form is widely prevailed and has been a part of different studies. In present study we also isolate polynomial invariants in different frameworks.

1.3 Methods for construction of invariants

A. E. Noether first discovered that constants of motion correspond to some symmetries and thus these can be found through symmetries of the system [13]. M. Lutzky applied Lie's theory to differential equations of motion for mechanical systems, and studied Lie symmetries and constants of motion of dynamical systems [14]. Hojman gave a new conservation theorem, and a conserved quantity was constructed in terms of a symmetry transformation vector of the equations of motion only [15]. In past, many methods [4, 7, 13, 16, 17, 18, 19] have been developed to obtain constants of motion, that span from elementary algebraic methods to symmetry considerations evaluated through symplectic group transformations or Noether's theorem. Recently various researchers have applied some new methods for construction of invariants [20, 21, 22, 23]. But none of such methods have a universal character, and in most of the cases one or more adhoc assumptions are to be made for obtaining concrete results.

There are, of course, many approaches, which one can try to search invariants if system seems to be integrable. However, while dealing practical problems the prospects of integrability should be tested by other means, e.g. numerically and by singularity analysis

[19, 24]. If the system does not fail either test, the next step would be to search for the invariant(s).

Now we describe some important and widely used methods for the investigation of invariants.

1.3.1 Rationalization method

Whittaker [3] introduced the rationalization method for the construction of invariants, second order in momenta, of TID systems. Subsequently, this method has been used by many researchers for finding invariants of both TID and TD systems in one and two dimensions [20, 21, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. A brief description of this method is as follows.

Consider a three-dimensional TD dynamical system whose Hamiltonian is given as

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z, t). \tag{1.4}$$

Considering the existence of a constant of motion I (say, fourth order in momenta) for the system, eq.(1.4), of the form

$$I = a_0 + a_i\xi_i + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{3!}a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l, \tag{1.5}$$

where $i, j, k, l = 1, 2, 3, 4$, $\xi_1 = \dot{x}_1$, $\xi_2 = \dot{x}_2$, $\xi_3 = \dot{x}_3$ and a_0, a_i, a_{ij}, a_{ijk}, a_{ijkl}, are functions of coordinates x_1, x_2 and x_3 only.

The invariance condition of the function I implies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0, \tag{1.6}$$

where $[..]_{PB}$ is Poisson bracket. On rationalizing the expression, obtained after using eqs.(1.4) and (1.5) in eq.(1.6), with respect to the powers of ξ_i, ξ_j, ξ_k and their all possible products, we get a system of over-determined coupled first order differential equations for unknown coefficient functions $a_0, a_i, a_{ij}, a_{ijk}$ and a_{ijkl}. The mutually consistent solutions of these partial differential equations (PDEs) for potential V give the desired invariant. As this method gives exact invariants for a system, one can utilize it to find higher order invariants for both real and complex Hamiltonian systems in two or higher dimensions. We shall further elaborate this method in the chapters 2, 3 and 4 for construction of higher order real and complex classical invariants of a number of dynamical systems.

1.3.2 Lie-algebraic approach

This method is accounted for obtaining invariants of a variety of TD systems [28, 35, 36, 37] using the closure property of the dynamical Lie-algebra generated by phase space functions. In this approach the Hamiltonian $H(x, y, p_x, p_y, t)$ of the system is expressed as

$$H = \sum_n h_n(t)\Gamma_n(p_x, p_y, x, y), \tag{1.7}$$

and the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ do not bear explicit time dependence whereas the coefficients $h_n(t)$ are functions of time. The Γ_n's in eq.(1.7) generate a closed dynamical algebra, implies

$$[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l, \tag{1.8}$$

where C^l_{nm} are the structure constants of the algebra. If the Γ_n's in eq.(1.8) are not sufficient to close the algebra then, the set of Γ_n must be extended by adding new Γ_l's, such that $\Gamma_l = [\Gamma_n, \Gamma_m]$, until the closure is obtained along with additional $h_l(t)$'s which are taken to be zero. It is to be noted that the algebra contains the important structural information for the dynamical behavior of the system besides its straightforward extension to the quantum regime. Since the dynamical invariant I is also a part of Lie-algebra, then one can express it as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(p_x, p_y, x, y), \tag{1.9}$$

where $\lambda_k(t)$'s are TD coefficients. The invariance of $I(t)$ for a TD system requires

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0. \tag{1.10}$$

Thus, inserting eqs.(1.7) and (1.9) in eq.(1.10), we get a system of linear, first order differential equations

$$\dot{\lambda}_r + \sum_n \left[\sum_m C^r_{nm}h_m(t)\right]\lambda_n = 0, \tag{1.11}$$

in λ_n's whose solutions, in turn, provide classical invariant of a given system. It is worth to note that the Lie-algebraic method gives exact TD invariants for classical systems and can easily be extended in quantum domains. Lie-algebra based studies may be found in [28, 35, 36, 37].

1.3.3 Transformation group method

Transformation group method [38] was introduced and developed, based upon the transformation group techniques, by J. Ray [39] and Burgan *et al* [40] which deals with the transformation of both dependent and independent variables. The arrangement of the coefficients, which are unknown functions of the transformation, is such that the equation of motion remains invariant under the transformation and the energy-integral in the new coordinates gives the desired invariant of the system.

For the sake of demonstration of the method, consider a one dimensional TD harmonic oscillator whose equation of motion is given as

$$\ddot{x} + \omega^2(t)x = g(t). \tag{1.12}$$

Now assume transformations for the system described by eq.(1.12) as

$$x' = \frac{x}{C(t)} + A(t); \quad t' = D(t), \tag{1.13}$$

where A, C and D are arbitrary time functions. With the use of above transformations, eq.(1.12) turns out to be

$$\begin{aligned} C\dot{D}^2\frac{d^2x'}{dt'^2} + (2\dot{C}\dot{D} + C\ddot{D})\frac{dx'}{dt'} + [\ddot{C} + \omega^2 C]x' \\ -[\ddot{C}A + 2\dot{C}A + \omega^2 CA + C\ddot{A} + g] = 0. \end{aligned} \tag{1.14}$$

On demanding the invariance under eq.(1.13), the coefficients of (dx'/dt') in eq.(1.14) must vanish, yielding $\dot{D} = dt'/dt = 1/C^2$ and, hence eq.(1.14) takes the form

$$\frac{d^2x'}{dt'^2} + C^3[\ddot{C}x' + \omega^2 Cx' - \ddot{C}A - 2\dot{C}A - \omega^2 CA - C\ddot{A} - g] = 0. \tag{1.15}$$

In order to identify eq.(1.15) with equation of motion of a TID harmonic oscillator in transformed coordinates, one has to choose A and C in eq.(1.15) such that

$$\ddot{C} + \omega^2(t)C = k/C^3, \tag{1.16}$$

$$\ddot{A} + (kA/C^4) + (2\dot{C}\dot{A}/C) + g/C = 0, \tag{1.17}$$

where, C and A respectively are the solutions of eqs.(1.16) and (1.17). The energy integral for eq.(1.15) has the form

$$I = \frac{1}{2}\left(\frac{dx'}{dt'}\right)^2 + kx'^2, \tag{1.18}$$

which after carrying out the inverse transformation becomes

$$I = \frac{1}{2}(C\dot{x} - \dot{C}x + C^2\dot{A})^2 + \frac{1}{2}(x/C + A)^2. \tag{1.19}$$

This method has widely been used to obtain exact solutions of the TD Schrödinger equation for nonlinear potentials [39, 40, 41, 42, 43, 44, 45, 46, 47, 48].

1.3.4 Lutzky's approach

The Lutzky approach [14] is based upon the formulation of Noether's theorem [13] and used for TD one dimensional systems [4, 19, 21]. The group operator $X = \xi(x,t)\frac{\partial}{\partial t} + \eta(x,t)\frac{\partial}{\partial x}$ describes the symmetry transformation for a TD system. If this symmetry transformation leaves the action $A = \int L(x,\dot{x},t)dt$ invariant, then the combination of the terms $\xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L$ provides total time derivative of the function $f(x,t)$ as

$$\xi\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi})\frac{\partial L}{\partial \dot{x}} + \dot{\xi}L = \dot{f}, \tag{1.20}$$

which further implies the constant of motion for the system as

$$I = (\dot{x}\xi - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f, \tag{1.21}$$

where $\dot{\eta}, \dot{\xi}$ and $\dot{f}$ introduced in eq.(1.20) are given as

$$\dot{\xi} = \frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x};\ \ \dot{\eta} = \frac{\partial \eta}{\partial t} + \dot{x}\frac{\partial \eta}{\partial x};\ \ \dot{f} = \frac{\partial f}{\partial t} + \dot{x}. \tag{1.22}$$

This method has successfully applied not only to TD harmonic oscillator but also to several of its generalizations.

1.3.5 Struckmeier and Riedel method

Recently, for construction of exact invariants for TD classical Hamiltonians systems, Struckmeier and Riedel [28] gave a formulation by considering a system of a non-relativistic ensemble of N-particles of the same species moving in an explicitly TD and velocity-independent potential, whose Hamiltonian H takes the form

$$H = \sum \frac{1}{2}[p_x^2 + p_y^2 + p_z^2] + V(x,y,z,t), \tag{1.23}$$

where x, y and z represent the N component vectors of the spatial coordinates of all particles and for each particle i, from the canonical equations, the equations of motion are given as

$$\dot{x} = p_x \; ; \quad \dot{p}_x \; = \; - \frac{\partial V(x,y,z,t)}{\partial x}, \tag{1.24}$$

and likewise for the y and z degrees of freedom. The solution functions $x(t)$, $y(t)$, $z(t)$, $p_x(t)$, $p_y(t)$, and $p_z(t)$ define a path within the $6N$-dimensional phase space that completely describes the system's time evolution. A quantity $I = I[x(t), y(t), z(t), p_x(t), p_y(t), p_z(t), t]$ constitutes an invariant of the particle motion if its total time derivative vanishes i.e.

$$\begin{aligned} \frac{dI}{dt} &= \frac{\partial I}{\partial t} + \sum \left[\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \frac{\partial I}{\partial z}\dot{z} + \frac{\partial I}{\partial p_x}\dot{p}_x + \frac{\partial I}{\partial p_y}\dot{p}_y + \frac{\partial I}{\partial p_z}\dot{p}_z \right] \\ &= 0. \end{aligned} \tag{1.25}$$

Now a special ansatz for I being at most quadratic in the velocities is given as

$$\begin{aligned} I = f_2(t)(P_x^2 + P_y^2 + P_z^2) + f_1(x,t)p_x + g_1(y,t)p_y \\ + h_1(z,t)p_z + f_0(x,y,z,t), \end{aligned} \tag{1.26}$$

where the set of functions $f_2(t)$, $f_1(x,t)$, $g_1(y,t)$, $h_1(z,t)$ and $f_0(x,y,z,t)$ that render I invariant are to be determined. The set of unknown functions $f_1(x,t)$, $g_1(y,t)$, $h_1(z,t)$ and $f_0(x,y,z,t)$ are obtained in terms of $f_2(t)$ by rationalizing the the total time derivative of eq.(1.26), after inserting equations of motion (1.24), with respect to the powers of velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ and their combinations.

This form of invariants has successfully been used to assess accuracy of numerical simulations of TD systems. Real and complex forms of invariant, eq.(1.26), are studied in the present thesis in the chapters 2-4.

1.3.6 Lax-Pair method

This method was introduced by P. D. Lax [17] for solving the nonlinear differential equations. To understand the basics of the method, consider an autonomous system

$$\frac{d\mathbf{u}}{dt} = V(u), \quad \mathbf{u} \equiv (u_1, u_2, ..., u_m),$$

which can further be written as

$$\frac{dL}{dt} = [A, L](t) \equiv [A(t), L(t)], \tag{1.27}$$

such that A and L are $n \times n$ matrices and $[A, L] \equiv AL - LA$. These matrices are called a Lax pair where Lax representation is denoted by eq.(1.27).

Thus to find out constants of motion, eq.(1.27) is further expressed as $\frac{dL^k}{dt} = [A, L^k](t)$ where $tr(L^k)$ with $k = 1, 2, ...$ are constants of motion and it is also mentioned that if L^{-1} exists, then $tr(L^{-1})$ is also an invariant, where tr(.) denotes the trace. For more elaboration of the method, consider a Hamiltonian function

$$H(x, p) = \frac{1}{2}(p_1^2 + p_2^2) + e^{x_2 - x_1}. \tag{1.28}$$

Now with the definition

$$a = \frac{1}{2}e^{(x_2 - x_1)/2};\ b = \frac{1}{2}p_1;\ c = \frac{1}{2}p_2, \tag{1.29}$$

the Lax matrices for the system are identified as

$$L = \begin{pmatrix} b & a \\ a & c \end{pmatrix};\quad A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \tag{1.30}$$

such that the Lax representation, using eq.(1.30), is written as

$$[A, L] \equiv AL - LA = \begin{pmatrix} 2a^2 & a(c-b) \\ a(c-b) & -2a^2 \end{pmatrix}, \tag{1.31}$$

where the constants of motion found are written as

$$I_1(a, b, c) = tr(L) = b + c,$$

$$I_2(a, b, c) = tr(L^2) = 2a^2 + b^2 + c^2.$$

So following the steps described above the invariants for other systems can be obtained by just identifying Lax pair L and A.

1.3.7 Field method

The Field method algorithm was developed for obtaining conservation laws of linear one-degree of freedom oscillators. Recently I. Kovacic *et al* [23] has obtained some approximate or adiabatic invariants of nonlinear autonomous oscillators using this technique and found a beneficial technique for studying problems of disparate areas of mechanics. So consider a system

$$\ddot{x} + G(x) = 0, \tag{1.32}$$

with

$$x(0) = a; \quad \dot{x}(0) = 0, \tag{1.33}$$

and $G(x)$ is an odd function of coordinate x, which does not necessarily have a linear term and overdot implies differentiation with respect to time t. Now within the framework of field method, the system (1.32) can further be written as

$$\dot{x} = p\,; \quad \dot{p} = -\omega^2 x + F, \tag{1.34}$$

where $F \equiv F(t) + \omega^2(x) + G(x(t))$ and ω is the frequency to be found. The coordinate x can be represented as a field depending on time t and momentum p i.e. $x = U(t,p)$ whose partial differentiation in combination with eq.(1.34) yields

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial p} + [-\omega^2 U + F(t)] - p = 0. \tag{1.35}$$

The solution of this PDE can be assumed in the form $U = Ap + f(p)$ [47], with A being a constant and f is an unknown function of time. Now using $U = Ap + f(p)$ into eq.(1.35) and equating the terms involving p and the independent terms with zero, we obtain

$$A_{\frac{1}{2}} = \sqrt{A} = \pm\frac{i}{\omega}, \;\; f(t) = Ce^{A\omega^2 t} - Ae^{A\omega^2 t}\int F(\tau)e^{A\omega^2\tau}d\tau, \tag{1.36}$$

where C appear as a constant and i represents an imaginary unit. For two values of the constant A algebraic transformations of the assumed form (i.e. $U = Ap + f(p)$) lead to the expressions in which the convolution integrals appear as:

$$x - \frac{ip}{w} + \frac{i}{w}\int [\omega^2(\tau) - G(x(\tau))]e^{-\omega(\tau - t)}d\tau = C_1 e^{i\omega t}, \tag{1.37}$$

$$x - \frac{ip}{w} - \frac{i}{w}\int [\omega^2(\tau) - G(x(\tau))]e^{-\omega(\tau - t)}d\tau = C_2 e^{i\omega t}. \tag{1.38}$$

To solve the integrals (1.37) and (1.38), the solution for the coordinate x inside the square brackets is assumed as $x(\tau) \approx \frac{C_1^{i\omega\tau} + C_2 e^{-i\omega\tau}}{2}$ which in accordance with the initial conditions (1.33) gives the constants $C_1 = C_2 = a$ and eqs.(1.37) and (1.38) are expressed as

$$\begin{aligned} a = \frac{i}{\omega}\int \left[\omega^2 \frac{a + ae^{-i2\omega\tau}}{2} - e^{-i\omega t}G\left(\frac{ae^{i\omega\tau} + ae^{-i\omega\tau}}{2}\right)\right]d\tau \\ + \left[x - \frac{ip}{\omega}\right]e^{-i\omega t}, \end{aligned} \tag{1.39}$$

$$a = -\frac{i}{\omega}\int\left[\omega^2\frac{a+ae^{i2\omega\tau}}{2} - e^{i\omega t}G\left(\frac{ae^{i\omega\tau}+ae^{-i\omega\tau}}{2}\right)\right]d\tau + \left[x+\frac{ip}{\omega}\right]e^{-i\omega t}, \tag{1.40}$$

where the frequency ω will be calculated from the elimination of secular terms among the terms generated by the integrals. Also the elimination of the secular terms and integration of the remaining terms shall provide some function of time. Together with the terms in front of the integrals, they will form two independent linear adiabatic invariants. They provide additional information about the behavior of the system being considered, giving us the combinations of the parameters of the system which remain almost constant during time. Besides, they enable us to find a quadratic approximate invariant as their product which can be written in the following form

$$I = x^2 + \frac{p^2}{\omega^2} + xD_1(t) + pD_2(t) + D_3(t) - a^2, \tag{1.41}$$

where $D_1(t)$, $D_2(t)$ and $D_3(t)$ are some functions of time.

1.3.8 Prelle-Singer method

This method was proposed by Prelle and Singer [18] for solving first order ordinary differential equations (ODEs) in terms of elementary functions. The advantage of this method is that it gives guarantee of finding solution if the given system of first order ODEs has a solution in terms of elementary functions. Recently, using this approach, many authors [49, 50] found first integrals of motion for second order ODEs. For a brief description of this method, we consider a second order ODE as

$$\ddot{x} = \frac{P}{Q}, \qquad P, Q \in\ C[t, x, \dot{x}], \tag{1.42}$$

where overdot shows differentiation with respect to time and P and Q are polynomials in x, $\dot{x}$ and t with complex coefficients. Consider that the ODE, eq.(1.42), admits a first integral $I(x, \dot{x}, t) = C$, with C constant on the solutions, so that the total time derivative of I gives

$$\frac{dI}{dt} = \frac{\partial I}{\partial t}dt + \frac{\partial I}{\partial x}dx + \frac{\partial I}{\partial \dot{x}}d\dot{x} = 0. \tag{1.43}$$

Now on rewriting eq.(1.43) in the form $\frac{P}{Q}dt - d\dot{x} = 0$ and adding a null term $S(x, \dot{x}, t)\dot{x}dt - S(x, \dot{x}, t)dx$ to the latter, we obtain

$$\left(\frac{P}{Q} + S\dot{x}\right)dt - Sdx - d\dot{x} = 0. \tag{1.44}$$

So, on the solutions, the I-forms, eqs.(1.43) and (1.44), must be proportional. Again multiplication of eq.(1.43) by the factor $R(x, \dot{x}, t)$, which acts as the integrating factor for eq.(1.44), gives on the solutions that

$$dI = R(\phi + S\dot{x})dt - RSdx - Rd\dot{x} = 0, \tag{1.45}$$

where $\phi \equiv \frac{P}{Q}$. A mutual comparison of eqs.(1.43) and (1.45) provides, on the solutions, the following expressions

$$\frac{\partial I}{\partial t} = R(\phi + S\dot{x}), \;\; \frac{\partial I}{\partial x} = -RS, \;\; \frac{\partial I}{\partial \dot{x}} = -R. \tag{1.46}$$

and the compatibility conditions $\frac{\partial^2 I}{\partial t \partial x} = \frac{\partial^2 I}{\partial x \partial t}$, $\frac{\partial^2 I}{\partial t \partial \dot{x}} = \frac{\partial^2 I}{\partial \dot{x} \partial t}$ and $\frac{\partial^2 I}{\partial x \partial \dot{x}} = \frac{\partial^2 I}{\partial \dot{x} \partial x}$, between eqs.(1.46), require that

$$D[S] = -\frac{\partial \phi}{x}\phi_x + S\frac{\partial \phi}{\partial \dot{x}} + S^2, \tag{1.47}$$

$$D[R] = -R(S + \frac{\partial \phi}{\partial \dot{x}}), \tag{1.48}$$

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial \dot{x}} + R\frac{\partial S}{\partial \dot{x}}, \tag{1.49}$$

where $D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}$. The solution of eq.(1.47), after substituting expression of ϕ, gives S and once S is known then eq.(1.48) gives the value of R, where the functions R and S have to satisfy eq.(1.49), an extra constraint. Once a compatible solution satisfying all the three equations is obtained, then the functions R and S fix the integral of motion I by the relation

$$\begin{aligned} I = &\int R(\phi + \dot{x}S)dt - \int (RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt)dx \\ &- \int \Big[R + \frac{d}{d\dot{x}}[\int R(\phi + \dot{x}S)dt - \int (RS + \frac{d}{dx}\int R(\phi + \dot{x}S)dt)dx]\Big]d\dot{x}. \end{aligned} \tag{1.50}$$

Hence eq.(1.50) defines an integral for every independent set (S, R).

1.3.9 Darboux integrability method

The Darboux method [51], to obtain invariants, is one of the well known method for finding first integrals of polynomial ODEs. To emphasize this approach, let us consider a system of two first-order ODEs of the form

$$\frac{dx_1}{dt} = X_1(x_1, x_2, t), \;\; \frac{dx_2}{dt} = X_2(x_1, x_2, t). \tag{1.51}$$

The solution of which, assuming the values $x_1(0), x_2(0)$ at $t = t_0$, defines a certain curve in space which passes through the point $P_0(t_0, x_1(0), x_2(0))$ and called as an integral curve of the system. Now consider planar polynomial differential systems $\dot{x} = Q(x, y)$ and $\dot{y} = P(x, y)$ where $P(x, y) = \sum_{i=0}^{m} P_i(x, y)$ and $Q(x, y) = \sum_{i=0}^{m} Q_i(x, y)$ are co-prime polynomials in $\mathcal{C}$ such that $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous components of degree i. The planar differential system, described by eq.(1.51), can alternatively be expressed by vector field $D = Q(x, y)\frac{\partial}{\partial x} + P(x, y)\frac{\partial}{\partial y}$ or a differential form $\omega = Pdx - Qdy$ and the corresponding phase-flow being given by the solutions of following first-order ODE

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}. \tag{1.52}$$

The tangents to the trajectories of a planar polynomial differential system are defined everywhere. If $f(x, y) = 0$ is the equation of an invariant curve, then its tangent must coincide with the tangents of the trajectories. In other words, the gradient of $f, \nabla f = (\frac{\partial f}{\partial x}\frac{\partial f}{\partial y})$, and (Q, P) must be orthogonal over the curve $f(x, y) = 0$, i.e.

$$\dot{f} = \left(Q\frac{\partial f}{\partial x} + P\frac{\partial f}{\partial y}\right)_{f=0} = 0. \tag{1.53}$$

An invariant curve $f(x, y) = 0$ is called an algebraic curve of degree m when $f(x, y)$ is a polynomial of degree m. A curve $f(x, y) = 0$ is an *invariant* algebraic curve if $D[f]/f$ is a polynomial where D represents vector field associated with differential equation and the polynomial $\lambda_f = D[f]/f$ is called the cofactor of the invariant algebraic curve.

1.3.10 Painlevé method

This method is basically used to identify and characterize the nature of the singularities admitted by general solutions of ODEs and PDEs in complex plane [16, 24, 52, 53, 54] of the independent variables. For an ODE to be Painlevé-type, it is necessary that it has no movable branch points, either algebraic or logarithmic. To explain the method, in brief, we consider a nth order ODE as

$$\frac{d^n w}{dz^n} = F(z, w, dw/dz, ..., d^{n-1}w/dz^{n-1}), \tag{1.54}$$

or equivalently, n first-order equations as

$$\frac{dw_i}{dz} = F_i(z, w_1, w_2, ..., w_n), \qquad i = 1, 2, ..., n, \tag{1.55}$$

where F and F_i are analytic in z and rational in their arguments. So, to check the necessary conditions for the existence of the P-type solution and integrability of eq.(1.54), the ARS conjecture essentially consists of determination of leading-order behavior of the Laurent series in the neighborhood of the movable singular point z_0, determination of resonances, i.e. the power at which arbitrary constants of the solution of eq.(1.54) can enter into the Laurent series expansions and, verification that a sufficient number of arbitrary constants exist without the introduction of movable critical points. So if a system qualifies P-test then it will be integrable which can further be classified by finding sufficient number of invariants.

Although one can find many more methods which can give exact or approximate invariants, but for the sake of bravity, we presented a few widely used methods. To establish the interest of a general reader in the field of theory of invariants, next, we describe a few interesting applications of invariants in different fields of physics and mathematics.

1.4 Applications of invariants

Invariants when defined in a broader sense play an important role in the domain of a variety of fields such as astrophysics, accelerator physics, plasma physics, biophysics, laser physics, solid state physics, fields theories, quantum optics and squeezed states, fibre optics, molecular and femto-chemistry and so on [55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65]. A knowledge of invariants offers an alternative route for calculating the particle production in cosmological models. In one way or the other the knowledge of invariants also helps in investigating the solutions of a particular class of nonlinear differential equations by reducing them to quadratures. The role of classical dynamical invariants is also found to be important in some special types of relativistic time dependent systems. The invariants are also capable of reducing order of differential equations [1], testing stability of differential equations (onset of chaos) [66, 67] and to assess the accuracy of numerical simulations of TD systems [28]. The higher order invariants [30, 31, 68, 69] provide the internal symmetry of physical systems particularly in molecular dynamics [57]. Also complex invariants [58] have been discussed in the past, in context of understanding fermion masses and quark mixing [4], and CP-conserving two-Higgs-doublet model [54] scalar potentials in the Particle physics [1] etc. In what follows, we describe some of the important applications in which invariants had played a distinctive role.

1.4.1 Integration by quadrature

If a sufficient number of integrals of motion for a Hamiltonian system are known then the associated Hamilton's equation of motion, in principle, can be integrated by quadratures [1]. To prove the above argument, consider an ordinary differential equation

$$\ddot{x} + \omega^2 x = 0, \tag{1.56}$$

which can further be expressed as a set of coupled first order equations as

$$\dot{x} = y, \tag{1.57}$$

$$\dot{y} = -\omega^2 x. \tag{1.58}$$

Now on multiplying eq.(1.57) by $\omega^2 x$ and eq.(1.58) by y and adding the resultant expressions, the following equation is obtained

$$y\dot{y} + \omega^2 x\dot{x} = 0, \tag{1.59}$$

which can also be written as $\frac{d}{dt}(\frac{y^2}{2} + \frac{\omega^2 x^2}{2}) = 0$ and immediately gives the following constant quantity

$$I_1 = \frac{1}{2}(y^2 + \omega^2 x^2). \tag{1.60}$$

This constant quantity I_1 is known as constant of motion or first integral. Here in the present case I_1 represents the mechanical energy of the system. Further this quantity reduces eq.(1.56) from a second order to a first order system. From eq.(1.60) one can explicitly express y in terms of x and I_1, i.e. $y = \sqrt{2(I_1 - \frac{1}{2}\omega^2 x^2)}$ and hence eq.(1.57) becomes

$$\frac{dx}{dt} = \sqrt{2(I_1 - \frac{1}{2}\omega^2 x^2)}. \tag{1.61}$$

Thus the original second order differential equation reduced to a first order one and is solvable as an explicit integral or quadrature, i.e.

$$\int dt = \int \frac{dx}{\sqrt{2(I_1 - \frac{1}{2}\omega^2 x^2)}}, \tag{1.62}$$

whose integration provides

$$t + I_2 = \frac{1}{\omega}\sin^{-1}(\frac{x\omega}{\sqrt{2I_1}}), \tag{1.63}$$

where I_2 is integration constant. Hence we finally get

$$x(t) = \frac{\sqrt{2I_1}}{\omega} sin(\omega t + I_2\omega), \tag{1.64}$$

the desired solution as obtained by other methods. One more application in this line can be found in two body central force problems where invariants, total energy and total angular momentum, again reduce second order equation of motion to quadratures [60].

1.4.2 Invariants and Separability

As we have seen above, one of the most conclusive proofs of integrability is the explicit construction of a sufficient number of independent integrals of motion. Then making use of these integrals one can reduce the problem to quadratures and obtain explicit solutions. For Hamiltonian systems, this can be done in principle by constructing the angle variables, while treating the *N* involutive integrals as action variables. However, in many cases, the problem turn out to be separable, either for the equation of motion or for the underlying Hamilton-Jacobi equation, some times through nontrivial transformations or variables. As examples, one can cite the well known two body Kepler problem (a superintegrable system) as it has three involutive conservative quantities namely the total energy E, the square of the angular momentum L^2 and the z-component of the angular momentum, L_z and this superintegrability is argued to be the main reason for the Kepler problem to be separable in different coordinate systems. Also the coupled quartic oscillator, for four cases [2], have been found separable under coordinate transformations, in addition to being integrable. We illustrate the separability aspects of a coupled quartic oscillators described by the Hamiltonian as

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + Ax^2 + By^2 + \alpha x^4 + \beta y^4 + \delta x^2 y^2, \tag{1.65}$$

whose equations of motion are given as

$$\ddot{x} + 2(A + 2\alpha x^2 + \delta y^2)x = 0, \;\; \ddot{y} + 2(B + 2\beta y^2 + \delta x^2)x = 0. \tag{1.66}$$

There are four specific choice of parameters in eq.(1.66) for which second integrals of motion exist so that the system becomes integrable in each of these cases. These four cases along with their second integrals of motion are given below.

Case (i): $A = B$, $\alpha = \beta$, $\delta = 6\alpha$

$$I = p_x p_y + 2\Big[A + 2\alpha(x^2 + y^2)\Big]. \tag{1.67}$$

Case (ii): $\alpha = \beta$, $\delta = 2\alpha$

$$I = (xp_y - yp_x)^2 + \frac{2}{\alpha}(B - A)\left[\frac{p_x^2}{2} + Ax^2 + \alpha(x^2 + y^2)x^2\right]. \tag{1.68}$$

Case (iii): $A = 4B$, $\alpha = 16\beta$, $\delta = 12\beta$

$$I = (yp_x - xp_y)p_y + 2(B + 4\beta x^2 + 2\beta y^2)xy^2. \tag{1.69}$$

Case (iv): $A = 4B$, $\alpha = 8\beta$, $\delta = 6\beta$

$$\begin{aligned} I = {} & p_y^4 + 4y^2(B + 6\beta x^2 + \beta y^2)p_y^2 - 16\beta xy^3 p_x p_y + 4\beta y^4 p_x^2 + 4B^2 y^4 \\ & + 4\beta\left[2B + \beta(2x^2 + y^2)\right](2x^2 + y^2)y^4. \end{aligned} \tag{1.70}$$

It has been found that the first three cases are separable under coordinate transformations, in addition to being integrable. The fourth case is also separable but after a complicated transformation involving momenta. A brief detail of the above four cases respectively given below.

Case (i): Under the linear transformation $u = x + y$, $v = x - y$, the equations of motion (1.66) become separable as

$$\ddot{u} + 2Au + 4\alpha u^3 = 0, \;\; \ddot{v} + 2Av + 4\alpha v^3 = 0. \tag{1.71}$$

Each of the above equation correspond to the independent cubic anharmonic oscillator are solvable in terms of Jacobian elliptic functions.

Case (ii): This case is separable in polar coordinates $x = rcos\theta$, $y = rsin\theta$ and the Hamiltonian (1.65) is independent of θ i.e. $H = (p_r^2 + 2Ar^2 + 2\alpha r^4 + I_2^2/r^2)/2$, where the angular momentum $I_2 = (xp_y - yp_x)$ is an integral of motion. The resultant equations of motion can be again solved in terms of elliptic functions.

Case (iii): Similarly, for the parametric restrictions $A = 4B$, $\alpha = 16\beta$, $\delta = 12\beta$, if we transform the cartesian coordinates to parabolic coordinates, $x = (\epsilon^2 - \eta^2)/2$, $y = \epsilon\eta$, the resulting Hamiltonian becomes

$$H = \frac{1}{\epsilon^2 - \eta^2}\left[\frac{1}{2}(p_\epsilon^2 + p_\eta^2) + B(\epsilon^6 + \eta^6) + \beta(\epsilon^{10} + \eta^{10})\right], \tag{1.72}$$

and hence the associated Hamilton-Jacobi equation is separable.

Case (iv): Finally, for the parametric choice $A = 4B$, $\alpha = 8\beta$ and $\delta = 6\beta$, the equations of motion become separable under a more general transformation involving both the coordinates and momenta

$$u = \frac{{p_y}^2 + c}{x^2} + 2y^2 + 4x^2, \;\; v = \frac{{p_y}^2 - c}{y^2} + 2y^2 + 4x^2, \tag{1.73}$$

where c is the square-root of the second constant of motion. Then the equations of motion separate and one can reduce it to quadratures.

1.4.3 Cosmological applications

The importance of dynamical invariants in cosmology and astrophysics is well known. In particular, Berger, Misner and Ray [38, 39, 40] have studied in detail the relationship between particle number present in a cosmological model and an adiabatic invariant. The knowledge of invariants, in fact, offers an alternative route for calculating the particle production in cosmological models. Here, however, we discuss the work of Oppenheimer and Volkoff. They studied the gravitational equilibrium of a neutron star using the equation of state $p = p(\rho)$, assuming the system to be a perfect, cold Fermi gas and derived the following classical results from Einstein's field equations in general relativity as

$$\frac{dm}{dr} = 4\pi r^2\rho, \quad \frac{dp}{dr} = -\frac{\rho m}{r^2}(1+\frac{p}{\rho})(1+\frac{4\pi p r^3}{m})(1-\frac{2m}{r})^{-1}, \tag{1.74}$$

where the symbols have their usual meanings. These equations are sometimes called Tolman-Oppenheimer-Volkoff (TOV) equations and are useful in providing detailed information about the structure of stellar objects such as white dwarfs, quark, and diquark stars. To obtain the space invariant K [59] the above equations are expressed as follow

$$\rho(r) = \frac{m'}{4\pi r^2}; \quad \frac{dp}{dr} = A(r) + B(r)p + C(r)p^2, \tag{1.75}$$

where

$$A(r) = \frac{-\rho m}{r^2}(1-\frac{2m}{r})^{-1},$$
$$B(r) = \frac{-\rho}{r^2}(1-\frac{2m}{r})^{-1}(4\pi r^3 + \frac{m}{\rho}),$$
$$C(r) = -4\pi r(1-\frac{2m}{r})^{-1}.$$

The second equation of eq.(1.75) is Riccati form. Now with the help of the Riccati transformation, i.e. $p = -Z'/CZ$, eq.(1.75) takes the following form

$$Z'' - (B + C'/C)Z' + ACZ = 0. \tag{1.76}$$

Further, the transformation $Z(r) = u(r)exp(-F(r)/2)$ can be used to eliminate the Z' term in eq.(1.76) and recast this in the 'Schrodinger-like' form as

$$u''(r) + q^2(r)u(r) = 0, \tag{1.77}$$

where $F(r) = -\int (B + C'/C)dr$ and

$$q^2(r) = -\frac{1}{2}f' - \frac{1}{4}f^2 + AC$$
$$= AC + \frac{1}{2}B' - \frac{1}{4}B^2 - \frac{BC'}{2C} + \frac{C''}{2C} - \frac{3}{4}(\frac{C'}{C})^2. \quad (1.78)$$

Next, to solve the eq.(1.77) we make an ansatz $u(r) = Nu_0(r)sin(\phi(r))$, where $\phi(r)$ is the phase function to be determined later and $u_0(r)$ is a real function. Using this ansatz in eq.(1.77) and subsequently equating the coefficients of the (mutually orthogonal) sine and cosine functions in the resultant equation to zero, one obtain

$$u_0'' + [q^2(r) - \phi'^2]u_0 = 0; \quad \phi'' u_0 + 2\phi' u_0' = 0. \quad (1.79)$$

The second equation of eq.(1.79) immediately yields $\phi' = k/u_0^2$ or

$$\phi(r) = k \int u_0^{-2} dr + \delta, \quad (1.80)$$

where k and δ are the constants of integration. The first expression of eq.(1.79) can be expressed in the form

$$u_0''(r) + q^2(r)u_0 = k^2/u_0^3, \quad (1.81)$$

which is called as Milne's equation. At this stage one can eliminate $q^2(r)$ from eqs.(1.77) and (1.81) and subsequently integrate the resultant equation with respect to the variable r. This immediately provides us the space invariant

$$K = k^2(u/u_0)^2 + (u_0 u' - u_0' u)^2. \quad (1.82)$$

The role of space invariant becomes particularly crucial when one studies the stability of stellar objects by using the TOV equations of state [59].

1.4.4 Calculation of Feynman propagator

The existence of an invariant for a dynamical system simplifies the calculation of Feynman propagator. In fact the Feynman propagator [70] provides an alternative route from classical to quantum description of a system. Lawande et al.[71] have studied in detail the role played by the invariants in the propagator theory for a large class of potentials. The explicit path-integral calculations have shown that the propagators, in general, admit expansion in terms of eigenfunctions of the invariant operator. This allows the Feynman

propagator to be expressed in terms of the eigenfunctions of the Ermakov invariant in an exact manner.

A variety of dynamical systems have been studied by Lawnade and his coworkers by extending the propagator theory to several dimensions of applications. Here, we just demonstrate the use of the Ermakov invariant in the calculation of the propagator for a system described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - (\frac{\ddot{\rho}\alpha}{\rho} - \ddot{\alpha})x + (\frac{\ddot{\rho}}{2\rho})x^2 - \frac{1}{\rho^2}F((x-\alpha)/\rho), \tag{1.83}$$

which possesses a second order invariant

$$I(x,p,t) = \frac{1}{2}[\rho(p-\dot{\alpha}) - \dot{\rho}(x-\alpha)]^2 + F((x-\alpha)/\rho). \tag{1.84}$$

Here $\rho(t), \alpha(t)$ and $F((x-\alpha)/\rho)$ are arbitrary functions of their arguments. It is possible to write eq.(1.83) in the form

$$L = \frac{d\chi}{dt} + L_0, \tag{1.85}$$

where L_0 is a new Lagrangian given by

$$L_0 = \frac{1}{2}\rho^2[(\frac{d}{dt})((x-\alpha)/\rho)]^2 - \frac{1}{\rho^2}F((x-\alpha)/\rho), \tag{1.86}$$

and

$$\chi = (\frac{\dot{\rho}}{2\rho})x^2 + ((\dot{\alpha}\rho - \dot{\rho}\,\alpha)/\rho)x - \frac{1}{2}\int \rho^2[(\frac{d}{dt})(\alpha/\rho)]^2 dt. \tag{1.87}$$

The Feynman Propagator, $K(x'',t'';x',t')$, defined as quantum mechanical amplitude for finding the position x'' at a time t'' if the particle had been at x' at an earlier time t', is expressed by

$$K(x'',t'',x',t') = \int exp[\frac{i}{\hbar}\int_{t'}^{t''} Ldt]Dx(t), \tag{1.88}$$

where $Dx(t)$ is the usual Feynman differential measure.

After doing some calculations, finally the propagator K can be written as

$$K = (\rho'\rho'')^{-1/2} exp[\frac{i}{\hbar}\{\chi(t'') - \chi(t')\}]\bar{K}_0(\xi'', \tau'', \xi', \tau'), \tag{1.89}$$

where

$$\bar{K}_0 = \int exp[\frac{i}{\hbar}\int_{\tau'}^{\tau''} \bar{L}_0 d\tau] D\xi(t),$$

with

$$\bar{L}_0 = \bar{L}_0(\xi, d\xi/d\tau) = \frac{1}{2}(d\xi/d\tau)^2 - F(\xi); \quad (\xi = x/\rho).$$

Thus, it becomes clear that the propagator for TD system is related to the propagator for an associated TID system corresponding to the Langrangian $\bar{L}_0$ in the new space-time variables (ξ, τ). Further, note that the invariant eq.(1.84) is basically the Hamiltonian H_0 associated with the Langrangian $\bar{L}_0$. Khandekar and Lawande have also obtained the Feynman propagator in an exact and closed form for non-central potentials.

1.4.5 Solutions of ordinary differential equations

Solving nonlinear ODEs is one of the classical area of research in the theory of dynamical systems. Various analytical methods have been proposed to deal with nonlinear ODEs. Prelle and Singer (PS) [18] have devised a method for solving first order ODEs that presents the solution in terms of elementary functions if such a solution exists. Recently, Duarte et al.[49] modified PS technique and applied it to second order ODEs. Their approach was based on the conjecture that if an elementary solution exists for the given second order ODE then there exist at least one elementary first integral $I(x, \dot{x}, t)$ whose derivatives are all rational functions of t, x and $\dot{x}$.

To emphasize the PS theory, consider an example of a simple harmonic oscillator, for which the equation of motion is given as

$$\ddot{x} = -x, \tag{1.90}$$

so one get two solutions for S and R from the eqs.(1.47)-(1.49), namely

$$\begin{aligned} S_1 &= \frac{x}{\dot{x}}, \quad R_1 = \dot{x} \ \ and \\ S_2 &= -\frac{\dot{x}}{x}, \quad R_2 = \frac{x}{x^2 + \dot{x}^2}. \end{aligned} \tag{1.91}$$

Thus, the eq.(1.50) provides two invariants for the two sets of solutions, (S_1, R_1) and (S_2, R_2) as

$$I_1 = \dot{x}^2 + x^2, \qquad I_2 = -t - tan^{-1}(\frac{\dot{x}}{x}). \tag{1.92}$$

The above equations can easily be solved to find general solution for the simple harmonic oscillator and the solution is given as

$$x = \sqrt{I_1} cos(t + I_2). \tag{1.93}$$

Similarly one can deduce general solutions of higher order ODEs of dynamical systems.

1.4.6 Invariants in quantum mechanics

The role of integrals of motion in physics can hardly be overlooked [72]. These help to analyze and classify the behavior of various classical and quantum systems. In particular, knowledge of integrals of motion simplifies significantly the process of solving dynamical equations governing the system evolution. In quantum cases, this was shown by Lewis and Riesenfield [73] and their method was generalized and applied to different problems. A quantum integral of motion is identified usually as an operator whose average value $< \psi(t)|\hat{I}|\psi(t) >$ does not depend on time for any state $|\psi(t) >$ obeying the Schrödinger equation. $\hat{I}$ satisfies the equation $i\hbar\frac{\partial \hat{I}}{\partial t} = [\hat{H}, \hat{I}]$. for example, in cases of quantum harmonic oscillator with a TD frequency $\omega(t)$ or a charged particle moving in a TD homogeneous magnetic field, one has a linear integral of motion of the form

$$\hat{A}(t) = \rho(t)\hat{p} - \dot{\rho}\hat{x},$$

where $\rho(t)$ is a solution to the the equation $\ddot{\rho} + \omega^2(t)\rho = 0$. Depending on the choice of concrete solution $\rho(t)$, eigenstates of the operator $\hat{A}(t)$ may be either coherent states, or squeezed states, or propagators in various representations.
Quadratic integrals of motions such as $\hat{A}^2(t)$ have been used in introducing even and odd coherent states, which are so popular nowadays due to their interpretation as examples of 'Schrödinger cat states'. Using quadratic integrals of motion such as

$$\hat{A}^\dagger(t)\hat{A}(t) = |\rho|^2\hat{p}^2 + |\dot{\rho}|^2\hat{x}^2 - Re(\dot{\rho}\rho^*)(\hat{p}\hat{x} + \hat{x}\hat{p}),$$

known as the Lewis invariant in quantum mechanics, (also identified as Courant-Snyder invariants in particle beam physics), one can find TD solutions to the Schrödinger equation, which are generalizations of the Fock states. Such integrals of motion are very useful for calculating the Berry phase and can also be generalized to the relativistic cases.

From the list of applications, it is evident that the role of invariants can hardly be ignored in theory and applications in physical problems. In other cases although the invariants as such do not appear directly, nevertheless their scope in problems is specified in the sense that an appropriate remodeling of the problem might accommodate them and subsequently make the understanding of the phenomenon better. Not only this if invariants exist and are obtained for a system, then they can offer a deeper insight into the problem. At the end of this introductory chapter, we can safely argue that invariants are very vital in the analysis of a variety of physical systems. It is an active research field. With this motivation, we also found real and complex invariants of a number of dyanmical systems. The work is presented in the following three chapters.