A study on the invariants for classical dynamical systems

1 Introduction

For a better understanding of complicated physical phenomena such as weather changes, earthquakes, cardiac arrhythmias, etc., scientists have found it useful to introduce mathematical models whose time evolution might exhibit some features very similar to those of the original one. The evolution of typical dynamical systems is often described by nonlinear ordinary and partial differential equations. Characteristically, these nonlinear dynamical systems show regular as well as chaotic trajectories [1, 2, 3, 4, 5] in phase space, depending on the number of dependent variables involved, the nature and the range of the external forces and the parameters involved, and the energy of the system. Since lot of parameters are involved in equation of motion, which make it difficult to find an exact analytical close form solution of it.

It is, in fact, one of the important problems in nonlinear dynamics to identify when a given system displays regular motion. In other words, under what conditions the given system, be it Hamiltonian or non-Hamiltonian, becomes completely integrable and when it is nonintegrable [1, 2, 3, 4, 5] exhibiting irregular or chaotic motion. Then naturally, the question which arises in this regard is: what is meant by integrability and when does it occur? The answer to the former question is somewhat vague as the concept of integrability is itself in a sense not well defined and there seems no unique definition for it yet. The latter is even more difficult to answer, as no well defined criteria seem to exist to identify integrable cases. Integrability can be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative information to understand the dynamics of the system globally.

There are various circumstances in which interest in integrability can arise. For example, when studying a specific physical problem, one will usually be interested in all information that can be obtained on the system. Integrability is one such property that can be successfully used [1] to get more predictive power. It will then be important to know whether the system is integrable, and if it is, one wants to know as many quantities as possible whose values are conserved during the time evolution of the system. The global quantity mentioned above is a function(al) from the space of dependent variables (phase space) to the real (or complex) numbers. This function is in the literature variously called as a constant of motion, integral of motion, conserved quantity, second invariant etc. Depending on the number of degrees of freedom present in the system there can be several, even an infinity of constants of motion. As the invariants are analytic functions, therefore, one can learn a lot about nonlinear dynamical systems as analytic results are much easier to use, to interpret and to generalize, and can also be further utilized to develop suitable schemes in order to deal with non integrable systems by treating the integrable case as basic zeroth order exact solution.

Recent investigations show that the integrability nature of a dynamical systems can be methodically investigated using the following two broad notions [1, 5]. The first one uses essentially the literal meaning: integrable - integrated with the required number of integration constants; nonintegrable - proven not to be integrable. This loose definition of integrability can be related to the existence of single valued, analytic solutions, for differential equations lead to the notion of integrability in the complex plane.

The second notion, particularly applicable to Hamiltonian systems, is to look for a sufficient number of single valued, analytic, involutive integrals for a Hamiltonian system with N-degrees of freedom, so that the associated Hamilton's equations of motion, in principle, can be integrated by quadratures in the sense of Liouville.

There are, of course, many approaches, which one can try to search invariants if the system seems to be integrable. However, in practical problems the prospects of integrability should be tested by other means, e.g. numerically and by singularity analysis. If the system does not fail either test, the next step would be to search for the invariant(s).

Searching constants of motion/invariants for dynamical systems is quite vital as these mathematical structures have multitude of applications in different branches of sciences [4, 5]. Dynamical invariant operators proved useful in solving Schrodinger equation for certain class of time dependent potentials [6, 7, 8]. There is an intimate relation between invariants and underlying symmetries for physical systems with the existence of action variables which, in turn, rules out the possibility of occurrence of chaos and then the corresponding equations of motion merely reduce to quadratures. Invariants are also utilized to check the accuracy of numerical simulations of dynamical systems [9].

In past, many concerted efforts had been made to devise different methods to isolate invariants and find their possible utilities and interpretations for both time dependent and time independent systems [1, 4, 6, 7, 8]. Recently Struckmeier and Riedel [9] proposed a novel technique for the construction of quadratic invariants for non-autonomous systems. In fact, quadratic invariants are widely studied because of their resemblance with system's Hamiltonian whose kinetic energy part is also quadratic in momenta. Naturally one can be curious to look for higher order invariants and their possible implications. Such studies can also be applied to check the veracity of the existing methods for construction of invariants. In this direction, many authors have also paid their attention to investigate cubic and quartic invariants and their applications [10, 11, 12, 13, 14]. The higher order invariants are particularly interesting for establishing super integrability of dynamical systems [15]. With this motivation in the present work, we extended the approach of Struckmeier and Riedel [9] and obtained invariants for number of time dependent systems.

It is an established fact that the Hamiltonian formulation for a physical system in real phase space proves suitable to solve equations of motion and to understand the underlying dynamics. But in some cases, the formulation of the concerned problem in complex phase space can be a better path to get them solved. One can track the utility of complex Hamiltonians in the study of nuclear models, atomic, molecular and nuclear scattering phenomena, chemical reactions, population biology, delocalized transitions in type-II superconductors and laser physics [16, 17, 18, 19]. The complex Hamiltonians now a days become more potent with the advent of $\mathcal{PT}$-Symmetric quantum mechanics [20].

To find some signatures of complex systems in classical mechanics, recently Kaushal and co-workers [10, 11] studied complex invariants for both time dependent and time independent systems within the framework of an extended complex phase space

(ECPS) characterized by $x = x_1 + ip_2$ and $p = p_1 + ix_2$. Recently the ECPS approach is further applied to find higher order complex invariants for a number of systems [21, 22]. Some workers have also solved Schrodinger equation for a variety of one and two dimensional complex Hamiltonian systems within the framework of ECPS [23, 24, 25, 26, 27].

Since in the ECPS the degrees of freedom of a system get doubled, therefore this complexifying scheme is better suited to study one dimensional systems. A $\mathcal{PT}$-Symmetric form of a complex Hamiltonian in the ECPS can be found by invoking $\mathcal{PT}$ invariance condition $\mathcal{PT}(x_1, p_1, x_2, p_2; i) \to (-x_1, p_1, -x_2, p_2; -i)$.

2 Methodology

In past, many methods [9, 11, 17, 24, 25, 26, 27] have been developed to obtain constants of motion which are in involution, that span from elementary algebraic methods to symmetry considerations evaluated through symplectic group transformations or Noether's theorem. Recently various researchers have applied some new methods for construction of invariants [28, 29, 30, 31, 32, 33, 34, 35]. But none of such methods have a universal character, and in most of the cases one or more adhoc assumptions are to be made for obtaining concrete results.

There are, of course, many approaches, which one can try to search invariants if system seems to be integrable. However, while dealing practical problems the prospects of integrability should be tested by other means, e.g. numerically and by singularity analysis [16, 17]. If the system does not fail either test, the next step would be to search for the invariant(s).

Now we describe the methods used in the present thesis for the investigation of invariants.

2.1 Rationalization method

Whittaker [5] introduced the rationalization method for the construction of invariants, second order in momenta, of time independent (TID) systems. Subsequently, this method has been used by many researchers for finding invariants of both TID and time dependent (TD) systems in one and two dimensions. A brief description of this method is as follows.

Consider a three-dimensional TD dynamical system whose Hamiltonian is given as

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z, t). \tag{1}$$

Considering the existence of a constant of motion I (say, fourth order in momenta) for the system, eq.(1), of the form

$$I = a_0 + a_i\xi_i + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{3!}a_{ijk}\xi_i\xi_j\xi_k + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l, \tag{2}$$

where $i, j, k, l = 1, 2, 3, 4$, $\xi_1 = \dot{x}_1$, $\xi_2 = \dot{x}_2$, $\xi_3 = \dot{x}_3$ and $a_0, a_i, a_{ij}, a_{ijk}, a_{ijkl}$, are functions of coordinates x_1, x_2 and x_3 only.
The invariance condition of the function I implies

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H]_{PB} = 0, \tag{3}$$

where $[..]_{PB}$ is Poisson bracket. On rationalizing the expression, obtained after using eq.(1) and (2) in eq.(3), with respect to the powers of ξ_i, ξ_j, ξ_k and their all possible products, we get a system of over-determined coupled first order differential equations for unknown coefficient functions $a_0, a_i, a_{ij}, a_{ijk}$ and a_{ijkl}. The mutually consistent solutions of these partial differential equations for potential V give the desired invariant. As this method gives exact invariants for a system, one can utilize it to find higher order invariants for both real and complex Hamiltonian systems in two or higher dimensions. We used this method in the chapter 4 for construction of higher order real and complex classical invariants of a number of dynamical systems.

2.2 Struckmeier and Riedel method

Recently, for construction of exact invariants for TD classical Hamiltonians systems, Struckmeier and Riedel (SR) gave a formulation by considering a system of a non-relativistic ensemble of N-particles of the same species moving in an explicitly TD and velocity-independent potential, whose Hamiltonian H takes the form

$$H = \sum \frac{1}{2}[p_x^2 + p_y^2 + p_z^2] + V(x, y, z, t), \tag{4}$$

where x, y and z represent the N component vectors of the spatial coordinates of all particles and for each particle i, from the canonical equations, the equations of motion are given as

$$\dot{x} = p_x; \quad \dot{p}_x = -\frac{\partial V(x, y, z, t)}{\partial x}, \tag{5}$$

and likewise for the y and z degrees of freedom. The solution functions $x(t)$, $y(t)$, $z(t)$ and $p_x(t), p_y(t), p_z(t)$ define a path within the $6N$-dimensional phase space that completely describes the system's time evolution.

A quantity $I = I(x(t), y(t), z(t), p_x(t), p_y(t), p_z(t), t)$ constitutes an invariant of the particle motion if its total time derivative vanishes i.e.

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \sum \left[\frac{\partial I}{\partial x}\dot{x} + \frac{\partial I}{\partial y}\dot{y} + \frac{\partial I}{\partial z}\dot{z} + \frac{\partial I}{\partial p_x}\dot{p}_x + \frac{\partial I}{\partial p_y}\dot{p}_y + \frac{\partial I}{\partial p_z}\dot{p}_z \right] = 0. \quad (6)$$

Now a special ansatz for I being at most quadratic in the velocities is given as

$$I = f_2(t)(P_x^2 + P_y^2 + P_z^2) + f_1(x,t)p_x + g_1(y,t)p_y + h_1(z,t)p_z + f_0(x,y,z,t), \quad (7)$$

where the set of functions $f_2(t)$, $f_1(x,t)$, $g_1(y,t)$, $h_1(z,t)$ and $f_0(x,y,z,t)$ that render I invariant are to be determined. The set of unknown functions $f_1(x,t)$, $g_1(y,t)$, $h_1(z,t)$ and $f_0(x,y,z,t)$ are obtained in terms of $f_2(t)$ by rationalizing the the total time derivative of eq.(7), after inserting equations of motion (5), with respect to the powers of velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ and their combinations.

This form of invariants has successfully been used to assess accuracy of numerical simulations of TD systems. Real and complex forms of invariant, eq.(7), are studied in the present thesis in chapters 2-4.

3 Organization of the thesis

The present thesis contains total five chapters. The introductory first chapter contains a detailed background of the work carried out. Here we introduced the concept of integrability, its definition and importance in nonlinear dynamics. We also gave here the meaning, the types, the methods of construction and applications of invariants.

The second chapter encompasses the investigation of invariants of five systems namely a two-dimensional coupled quartic Hamiltonian system, Toda potential, one dimensional general quartic polynomial potential, Morse potential and the Hulthen's potential using SR method. Here we also generalized the SR method to obtain fourth order invariants of a couple of systems i.e. one dimensional harmonic oscillator and a one dimension a general time dependent potential.

The subject matter of third chapter is to isolate dynamical invariants of four complex classical systems in $z\bar{z}$-space viz linearly confining system, a coupled non-linear oscillator system, a shifted Harmonic oscillator system and a general inverse potential is presented.

Fourth Chapter comprises the details of the extended complex phase space (ECPS) approach. Here Cubic Invariant of a simple harmonic oscillator, quartic invariants of a shifted harmonic oscillator, $\mathcal{PT}$-symmetric shifted harmonic oscillator and simple harmonic oscillator are investigated using rationalization method. Here we also found TD invariants of general nonlinear quartic oscillator within the framework of ECPS utilizing the SR method.

In the concluding fifth chapter, we summarized the thesis by highlighting the major finding of the present work. A brief note on the future scope of the present work is also given.

4 Work carried out

In the present thesis work, five studies are carried out which spanned from chapter 2-4. In what follows, a very brief account of the potentials and their corresponding invariants is presented.

In the first study, we took five real TD potentials of physical importance and found their corresponding quadratic invariants using SR method. The list of potentials is given as

(i). A two-dimensional coupled quartic potential

$$V_a = a_1(t)(x^2+y^2)+a_2(t)xy+a_3(t)(x^4+y^4)+a_4(t)x^2y^2+a_5(t)x^3y+a_6(t)xy^3,$$

(ii). The Toda potential

$$V_b = e^{\alpha(t)x-y} + e^{-\alpha(t)x-y},$$

(iii). A one-dimensional quartic potential

$$V_c = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3 + a_4(t)x^4,$$

(iv). The Morse potential

$$V_d = V_0(t)(e^{-2ax} - 2e^{-ax},$$

(v). The Hulthen's potential

$$V_e = \frac{e^{-a(t)x}}{1 - e^{-a(t)x}},$$

and the corresponding invariants of the above five cases are given respectively as

$I_{f2a} = \frac{g_{x,y}}{2f_2}(x^2+y^2) + \frac{1}{f_2}[(f_2 p_x - \frac{1}{2}\dot{f}_2 x)^2 + (f_2 p_y - \frac{1}{2}\dot{f}_2 y)^2$

$$+ 2f_2^2 xy(a_2 + a_4 xy + a_5 x^2 + a_6 y^2) + 2f_2^2 a_3(x^4 + y^4)],$$

$$I_{f_2 b} = \tfrac{1}{f_2}[(f_2\dot{x} - \tfrac{1}{2}\dot{f}_2 x)^2 + (f_2\dot{x} - \dot{f}_2 x)^2 + \tfrac{g_{x,y}}{2}(x^2 + y^2) + 2f_2^2(e^{\alpha x - y} + e^{-\alpha x - y})],$$

$$I_{f_2 c} = \tfrac{1}{f_2}[(f_2 p - \tfrac{1}{2}\dot{f}_2 x)^2 + \tfrac{g_x}{2}x^2 + 2f_2^2(a_0 + a_1 x + a_3 x^3 + a_4 x^4)],$$

$$I_{f_2 d} = \tfrac{1}{f_2}[(f_2 p - \tfrac{1}{2}\dot{f}_2 x)^2 + 2f_2^2 V(x,t) + \tfrac{1}{2}g_x x^2],$$

$$I_{f_2 e} = \tfrac{1}{f_2}[(f_2 p - \tfrac{1}{2}\dot{f}_2 x)^2 + 2f_2^2 V + \tfrac{g_x}{2}x^2].$$

In the second study, we considered two real TD potentials and obtained their corresponding quartic invariants using SR method. The list of potentials is given as

(i). A dynamical harmonic oscillator

$$V_a = \frac{1}{2}\omega^2(t)x^2.$$

(ii). A general time dependent potential

$$V_b = a(t)x^m + b(t)x^n,$$

and their invariants are listed as

$$I_{f_4 a} = f_4[\{p^2 + \omega^2 x^2\}^2 - \tfrac{4}{3}\omega\dot{\omega}x^3 p + \tfrac{1}{3}\dot{\omega}^2 x^4 + \tfrac{1}{3}\omega\ddot{\omega}x^4] + \dot{f}_4\{-xp^3 - \tfrac{5}{3}\omega^2 x^3 p + \tfrac{7}{6}\omega\dot{\omega}x^4\}$$
$$+ \ddot{f}_4\{\tfrac{1}{2}x^2p^2 + \tfrac{2}{3}\omega^2 x^4\} - \tfrac{1}{6}\dddot{f}_4 x^3 p + \tfrac{x^4}{24}\{\tfrac{1}{2}\tfrac{\dddot{f}_4^2}{\ddot{f}_4} + \tfrac{g_x}{\ddot{f}_4}\},$$

$$I_{f_4 b} = f_4[\{p^2 + 2(ax^m + bx^n)\}^2 - 4\{\tfrac{\dot{a}x^{m+1}}{m+1} + \tfrac{\dot{b}x^{n+1}}{n+1}\}p + 4\{\tfrac{\ddot{a}x^{m+2}}{(m+1)(m+2)} + \tfrac{\ddot{b}x^{n+2}}{(n+1)(n+2)}\}]$$
$$+ \dot{f}_4\{-xp^3 - \tfrac{3m+4}{m+1}ax^{m+1}p + \tfrac{3n+4}{n+1}bx^{n+1}p + \tfrac{3m+8}{(m+1)(m+2)}\dot{a}x^{m+2} + \tfrac{3n+8}{(n+1)(n+2)}\dot{b}x^{n+2}\}$$
$$+ \ddot{f}_4\{\tfrac{1}{2}x^2p^2 + \tfrac{m+2}{m+1}ax^{m+2} + \tfrac{n+2}{n+1}bx^{n+2}\} - \tfrac{\dddot{f}_4}{6}x^3 p + \tfrac{x^4}{24}\{\tfrac{\dddot{f}_4^2}{2\ddot{f}_4} + \tfrac{g_x}{\ddot{f}_4}\}.$$

In third study we extended the SR approach in complex $z\bar{z}$ space and derived invariants for TD classical systems. The potentials taken up in this study are as follow

(i). A linearly confining potential

$$V_a = \omega(t)(z\bar{z})^{1/2} - \beta(t)(z\bar{z})^{-1/2},$$

(ii). A shifted harmonic oscillator potential

$$V_b = \frac{1}{2}a(t)(z + \bar{z}) - i\frac{1}{2}a(t)(z - \bar{z}) + \frac{1}{2}\omega^2(t)(z\bar{z}),$$

(iii). A coupled nonlinear oscillator

$$V_c = \frac{1}{2}\omega^2 z\bar{z} + \frac{1}{2}\beta(z\bar{z}^2 + \bar{z}z^2),$$

(iv). An inverse potential

$$V_d = \omega\frac{z}{\bar{z}} + \beta\frac{\bar{z}}{z},$$

and their invariants respectively are

$I_{f_2a} = \frac{1}{f_2}[(f_2\dot{z}-\frac{1}{2}\dot{f}_2z)^2+(f_2\dot{\bar{z}}-\frac{1}{2}\dot{f}_2\bar{z})^2]+f_2[\omega(z\bar{z})^{1/2}-\beta(z\bar{z})^{-1/2}]+\frac{g}{2f_2}(z^2+\bar{z}^2)$,

$I_{f_2b} = \frac{1}{f_2}[(f_2\dot{z}-\frac{1}{2}\dot{f}_2z)^2+(f_2\dot{\bar{z}}-\frac{1}{2}\dot{f}_2\bar{z})^2]+2f_2[\omega^2 z\bar{z}+a(z+\bar{z})-ia(z-\bar{z})]+ \frac{g}{2f_2}(z^2+\bar{z}^2)$,

$I_{f_2c} = \frac{1}{f_2}[(f_2\dot{z}-\frac{1}{2}\dot{f}_2z)^2+(f_2\dot{\bar{z}}-\frac{1}{2}\dot{f}_2\bar{z})^2]+2f_2z\bar{z}[\omega^2+\beta(z+\bar{z})z\bar{z}]+\frac{g}{2f_2}(z^2+\bar{z}^2)$,

$I_{f_2d} = \frac{1}{f_2}[(f_2\dot{z}-\frac{1}{2}\dot{f}_2z)^2+(f_2\dot{\bar{z}}-\frac{1}{2}\dot{f}_2\bar{z})^2]+4f_2[\omega\frac{z}{\bar{z}}+\beta\frac{\bar{z}}{z}]+\frac{g}{2f_2}(z^2+\bar{z}^2)$.

Fourth study contains the construction of quartic, cubic and quadratic invariants of some TID and TD systems whose potentials are given as

(i). A shifted harmonic oscillator

$$V_a = ax + bx^2,$$

(ii). A $\mathcal{PT}$-symmetric shifted harmonic oscillator with real and imaginary parts

$$V_{1b} = -a_2p_2 + b_1(x_1^2 - p_2^2), V_{2b} = a_2x_1 + 2b_1x_1p_2,$$

(iii). A simple harmonic oscillator

$$V_c = \frac{1}{2}\omega^2x^2,$$

the corresponding quartic invariants of the above potentials are written as

$$\begin{aligned} I_a = {} & (p_1+ix_2)^2\{\tfrac{1}{3}(\psi_1+i\psi_2)(x_1^3+ip_2^3+3ix_1p_2(x_1-ip_2)) \\ & +(\psi_3+i\psi_4)(x_1^2+p_2^2)\}+\tfrac{8}{15}(c_1+ic_2)\{(x_1-ip_2)^5-10x_1p_2^4 \\ & +10ip_2x_1^4\}-\tfrac{4}{3}(d_1+id_2)\{x_1^4-p_2^4+4i(p_2x_1^3+p_2^3x_1)\} \\ & +\tfrac{2}{3}(e_1+ie_2)\{(x_1-ip_2)^3+6ix_1p_2(x_1-ip_2)\} \\ & +(\alpha_1+i\alpha_2)(x_1-ip_2)(p_1+ix_2)^4, \end{aligned}$$

$$\begin{aligned} I_b = {} & (\alpha_1+i\alpha_2)[\{\tfrac{4b_1}{3}(x_1^3+ip_2^3+3ix_1^2p_2+3x_1p_2^2)+2ia_2(x_1^2+p_2^2) \\ & +(x_1-ip_2)(p_1+ix_2)^2\}(p_1+ix_2)^2+\tfrac{8b_1^2}{15}\{(x_1-ip_2)^5-10x_1p_2^4 \\ & +10ip_2x_1^4\}+\tfrac{4ib_1a_2}{3}(x_1^4-p_2^4+4ip_2x_1^3+4ip_2^3x_1) \\ & -\tfrac{2a_2^2}{3}\{(x_1-ip_2)^3+6ix_1^2p_2+6x_1p_2^2\}], \end{aligned}$$

$$\begin{aligned} I_c = {} & (\alpha_1+i\alpha_2)[\tfrac{2}{3}\omega^2(x_1^3+ip_2^3+3ix_1^2p_2+3x_1p_2^2)(p_1+ix_2)^2 \\ & +\tfrac{8}{15}\omega^4\{(x_1-ip_2)^5-10x_1p_2^4+10ip_2x_1^4\} \\ & +(x_1-ip_2)(p_1+ix_2)^4], \end{aligned}$$

and the cubic invariant of a simple harmonic oscillator is obtained as

$$I_c = 4\omega^2\alpha_1(p_1+ix_2)\big(x_1^3-ip_2^3+3x_1p_2^2+3ip_2x_1^2+(x_1-ip_2)(p_1+ix_2)^2/\omega^2\big).$$

Following are a couple of potentials taken up in the fourth study to obtain quadratic invariants:

(iv). A TD harmonic oscillator

$$V_a = \frac{1}{2}w^2(t)x^2,$$

(v). A general nonlinear quartic potential

$$V_b = a_0 + ax + bx^2 + cx^3 + dx^4,$$

(vi). The $\mathcal{PT}$-symmetric case of the general nonlinear quartic potential, whose real and imaginary parts are given as

$$V_{1c} = a_{0r} - a_2p_2 + b_1(x_1^2 - p_2^2) + c_2p_2^3 - 3c_2x_1^2p_2 + d_1(x_1^2 - p_2^2)^2 - 4d_1x_1^2p_2^2,$$

$$V_{2c} = a_{0i} + a_2x_1 + 2b_1x_1p_2 + c_2x_1^3 - 3c_2x_1p_2^2 + 4d_1x_1p_2(x_1^2 - p_2^2),$$

whose respective invariants are written as

$$\begin{aligned}
I_a &= 2\Lambda x^*p^2 - \tfrac{\dot{\Lambda}}{2}(x_1^2 + p_2^2)p + \tfrac{\ddot{\Lambda}}{12}(x_1^3 + ip_2^3) + \tfrac{\ddot{\Lambda}}{4}ix^*x_1p_2 \\
&\quad + 16i\Lambda\omega^2x^*x_1p_2 + \tfrac{\Lambda\omega^2}{3}(x_1^3 + ip_2^3), \\
I_b &= 2\Lambda x^*p^2 - \tfrac{\dot{\Lambda}}{2}(x_1^2 + p_2^2)p + \tfrac{\ddot{\Lambda}}{12}x_1^3 - \tfrac{i\ddot{\Lambda}^*}{12}p_2^3 + \tfrac{ix_1p_2\ddot{\Lambda}}{4}x^* \\
&\quad + 16\Lambda a(x_1^2 + p_2^2) + \tfrac{32\Lambda}{3}b(x_1^3 + ip_2^3) + 16\Lambda c(x_1^4 - p_2^4) + \tfrac{64\Lambda}{5}d(x_1^5 - ip_2^5) \\
&\quad - 32ix_1p_2\Lambda bx - 32ix_1p_2\Lambda^*c^*(x_1^2 + p_2^2) + 64ix_1p_2\Lambda d(x_1^3 + p_2^3), \\
I_c &= 2\Lambda x^*p^2 - \tfrac{\dot{\Lambda}}{2}(x_1^2 + p_2^2)p + \tfrac{\ddot{\Lambda}}{12}x_1^3 - \tfrac{i\ddot{\Lambda}^*}{12}p_2^3 + \tfrac{ix_1p_2\ddot{\Lambda}}{4}x^* + 16i\Lambda a_2(x_1^2 + p_2^2) \\
&\quad + \tfrac{32\Lambda}{3}b_1(x_1^3 + ip_2^3) + 16i\Lambda c_2(x_1^4 - p_2^4) + \tfrac{64\Lambda}{5}d_1(x_1^5 - ip_2^5) \\
&\quad - 32ix_1p_2\Lambda b_1x - 32x_1p_2\Lambda^*c_2(x_1^2 + p_2^2) + 64ix_1p_2\Lambda d_1(x_1^3 + p_2^3).
\end{aligned}$$

In last study we extended the SR method into ECPS to investigate the exact invariants of complex TD systems. The following one-dimensional nonlinear complex quartic potential has been considered

$$V = (a_{1r} + ia_{1i})(x_1 + ip_2) + (a_{2r} + ia_{2i})(x_1 + ip_2)^2$$

$$+(a_{3r} + ia_{3i})(x_1 + ip_2)^3 + (a_{4r} + ia_{4i})(x_1 + ip_2)^4,$$

and the invariant for this case is obtained as

$$\begin{aligned}
I &= f_2(p_1^2 - x_2^2 + 2ip_1x_2) - \dot{f}_2(x_1 + ip_2)(p_1 + ix_2) + 2f_2[(a_{1r} + ia_{1i})(x_1 + ip_2) \\
&\quad + (a_{2r} + ia_{2i})(x_1 + ip_2)^2 + (a_{3r} + ia_{3i})(x_1 + ip_2)^3 + (a_{4r} + ia_{4i})(x_1 + ip_2)^4] \\
&\quad + \tfrac{\ddot{f}_2}{2}(x_1 + ip_2)^2 + (\alpha_1 + i\alpha_2)(p_1 + ix_2) - (\dot{\alpha}_1 + i\dot{\alpha}_2)(x_1 + ip_2) + (\beta + i\gamma).
\end{aligned}$$

5 Summary and conclusions

Keeping in view multitude applications of invariants in various fields of physics, here we extended the approaches described in [9, 10] to determine TD and TID invariants of different orders in momenta for a number of real and complex classical systems. The major findings of this thesis are as follow.

- In chapter two, to expand the catalogue of applications of the SR method, two studies have been carried out. In the first endeavor, we found quadratic invariants of five systems namely a two-dimensional coupled quartic Hamiltonian system, Toda potential, one dimensional general quartic polynomial potential, Morse potential and the Hulthen's potential. In the second work, we generalized the SR method to obtain fourth order invariants of a couple of systems i.e. one dimensional harmonic oscillator and a one dimensional general time dependent potential.

- It is well known that a complex Hamiltonian description of many physical problems is better route to obtain some additional features which were not possible otherwise. One can find innumerable number of such problems in literature. The importance of complex Hamiltonians becomes much more after the development of $\mathcal{PT}$- symmetric quantum mechanics. With a view to find utility of complex Hamiltonians in classical realms and to expand the domain of applicability of the SR method, in the third chapter, we extended the SR method in complex $z\bar{z}$-space with an aim to isolate dynamical invariants of complex classical systems. This type of scheme of coordinate transformation for deriving invariants has also been utilized in many past studies. Here we successfully obtained invariants of four physical systems, namely linearly confining system, a coupled nonlinear oscillator system, a shifted Harmonic oscillator system and a general inverse potential.

- In continuity to our endeavor of dealing of complex Hamiltonian systems, in the fourth chapter, we presented two studies on invariants using two different approaches by scaling the concerned Hamiltonians on an extended complex phase space characterized by $x = x_1 + ip_2$, $p = p_1 + ix_2$.. In the first one, keeping in view the significance of higher order invariants, quartic, cubic and quadratic complex invariants have been determined utilizing the rationalization method for one dimensional TID and TD classical systems. We found quartic invariants of a shifted harmonic oscillator and its $\mathcal{PT}$-symmetric variant and quartic and cubic invariants of a simple harmonic oscillator. We also obtained quadratic invariants of a TD harmonic oscillator and a general TD nonlinear quartic oscillator. In our last work, the SR method has again been developed within the framework of the ECPS to derive quadratic invariants of a TD nonlinear quartic oscillator. This particular study can have some interesting bearings in the realms of newly developing field of $\mathcal{PT}$-symmetric classical and quantum mechanics.

References

[1] J Hietarinta, Phys.Rep. **147**, 87 (1987)

[2] M Lakshmanan and R Sahadevan, Phys. Rep. **224**, 1 (1993).

[3] A. Ramani, B. Grammaticos and T. Bountis, Phys. Rep. **180**, 159 (1989).

[4] R S Kaushal,:"Classical and Quantum Mechanics of Noncentral Potentials" (Narosa Publishing House, New Delhi, 1998)

[5] M Lakshmanan and S Rajasekar "Nonlinear Dynamics: Integrability, Chaos and Patterns Formation" (Springer-Verlag, New York)(2003)

[6] H R Lewis Phys. Rev. Lett. **18**, 510 (1967); J. Math. Phys. **9**, 1976(1968)

[7] R S Kaushal and D Prashar, Phys. Rev. A, **55**, 2610 (1997)

[8] F Chand and S C Mishra, Pramana- J. Phys, **68**, 891(2007).

[9] J Struckmeier and C Riedel Phys. Rev. Lett. **85**, 3830(2000)

[10] R S Kaushal and Sweta Singh, Ann. Phys.(N.Y.)**288**, 253 (2001)

[11] Sweta Singh and R S Kaushal, Phys. Scr. **67**, 181 (2003)

[12] S C Mishra and F Chand, Pramana: J.Phys. **66**,601 (2006)

[13] F Chand and S C Mishra, Pramana: J.Phys. **67**,999 (2006)

[14] F Chand Can. J. Phys. **88**, 165(2010)

[15] I Marquette and P Winternitz J. Phys. A: Math. Theor. **41** 304031(2008)

[16] N Moiseyev, Phys.Rep. **302**, 211 (1998)

[17] A L Xavier Jr. and M A M de Aguiar, Ann. Phys.(N.Y.)**252**, 458 (1996); Phys.Rev. Lett. **79**,3323 (1997)

[18] N Hatano and D R Nelson, Phys.Rev.Lett. **77**, 570 (1996); Phys.Rev.B **56**, 8651 (1997)

[19] D R Nelson and N M Snerb, Phys.Rev.E **58**,1383 (1998)

[20] C M Bender and S Boettcher, Phys.Rev. Lett. **80**,5243 (1998)

[21] J S Virdi, F Chand, C N Kumar and S C Mishra, Can. J. Phys. **90**, 151(2012)

[22] J S Virdi, F Chand, C N Kumar and S C Mishra, Pramana-J. Phys. **79**, 173(2012)

[23] R S Kaushal, J.Phys.A:Math. and Gen.**34**, L709 (2001)

[24] R S Kaushal and Parthasarthi, J.Phys.A:Math. and Gen.**35**, 8743 (2002)

[25] F Chand, R M Singh, N Kumar and S C Mishra, J.Phys. A: Math.Theor.**40**, 10171 (2007)

[26] R M Singh, F Chand and S C Mishra, Commun. Theor. Phys.**51**, 397 (2009)

[27] R M Singh, Pramana: J.Phys. DOI:10.1007/s12043-014-0783-9 (2014)

[28] B Grammaticos, B Dorizzi, and A.Ramani, J.Math.Phys. **24**, 2289 (1983)

[29] B Grammaticos and B Dorizzi, J.Math.Phys. **25**, 2194 (1984)

[30] L S Hall, Physica **D8**, 90 (1983); Phys.Rev.Lett. **54**, 614 (1985);
X W Chen, Y M Li and Y.H.Zhao, Phys.Lett.A **337**, 274 (2005)

[31] H R Lewis and P.G.L.Leach, J.Math Phys. **23**, 2371 (1982)

[32] P G L Leach, J.Math Phys. **21**, 300 (1980)

[33] P.G.L.Leach, J.Math Phys. **20**, 1 (1979); **26**, 77 (1985)

[34] W Sarlet and F.Cantrijn, Phys.Lett.A **88**, 383 (1982)

[35] V I Inozemtsev, Phys.Lett.A **96**, 447 (1983)