Chapter – 2

A Parametric Approach to Fuzzy Queue
CHAPTER – II
A PARAMETRIC APPROACH TO FUZZY QUEUE

This chapter discusses two fuzzy queuing models with fuzzy parameter. The first model shows the method of constructing the membership functions of system characteristics of FM/FM/1 queuing system where the arrival rate and service rate are trapezoidal fuzzy numbers. The discussion of first model is confined to two fuzzy variables, nevertheless the procedure can be extended to the system with more than two fuzzy variables. The procedure for the solution is illustrated with a numerical example. The second model shows the method for constructing the membership functions of finite capacity queuing system. A pair of parametric non-linear programmes is developed to describe the family of crisp queues with finite capacity through which the membership functions of the system performance measures are derived. A parametric approach is used to obtain a fuzzy solution and a supporting example is also given.

2.1. Parametric study on Fuzzy Queuing Model

Queuing models have wider applications in service organization as well as manufacturing firms, in that various customers are serviced by various types

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of servers according to specific queue discipline [31]. Within the context of traditional queuing theory, the inter arrival times and service times are required to follow certain distributions. In usual practice the arrival rate, service rate are frequently described by linguistic terms such as fast, slow or moderate can be best described by the fuzzy sets.

Li and Lee [58] have derived analytical results for two fuzzy queuing systems based on Zadeh’s extension principle [78,79]. However, as commented by Negi and Lee [63] their approach is fairly complicated and is generally unsuitable for computational purposes. Negi and Lee [63], therefore, propose two approaches, the α-cut and two variable formulations. Unfortunately, their approach provides only possible numbers rather than intervals, in other words, the membership functions of the performance measures are not completely described.

This model follows α-cut approach to decompose a fuzzy queue into a family of crisp queues. When α-varies the parametric programming technique is used to describe the family of crisp queues. The solutions from the parametric programs derives the membership functions of the crisp queues. To demonstrate the validity of the proposed approach the fuzzy queue FM/FM/1, is taken into consideration where FM denotes fuzzified exponential time.
2.2. Problem Formulation

Consider a general queuing system with one server. The inter arrival time $\tilde{A}$ and service time $\tilde{S}$ are approximately known and are represented by the following fuzzy sets.

$$\tilde{A} = \{(a, \mu_{\tilde{A}}(a))/a \in x\}, \quad \tilde{S} = \{(s, \mu_{\tilde{S}}(s))/s \in y\}$$

Where $x$ and $y$ are the crisp universal sets of the inter arrival time and service time, and $\mu_{\tilde{A}}(a)$ and $\mu_{\tilde{S}}(s)$ are the corresponding membership functions.

The $\alpha$-cuts (or $\alpha$-level sets of $\tilde{A}$ and $\tilde{S}$ are

$$A(\alpha) = \{a \in x/\mu_{\tilde{A}}(a) \geq \alpha\}, \quad S(\alpha) = \{s \in y/\mu_{\tilde{S}}(s) \geq \alpha\}$$

Where $A(\alpha)$ and $S(\alpha)$ are crisp sets. Using $\alpha$-cuts, the inter arrival time and service time can be represented by different levels of confidence intervals. Consequently, a fuzzy queue can be reduced to a family of crisp queues with different $\alpha$-level sets \{A(\alpha)/0 < \alpha \leq 1\} and \{S(\alpha)/0 < \alpha \leq 1\}. These two sets form nested structures for expressing the relationship between ordinary sets and fuzzy sets [40]. Let the confidence intervals of the fuzzy sets $\tilde{A}$ and $\tilde{S}$ are $[\ell_{A(\alpha)}, u_{A(\alpha)}]$ and $[\ell_{S(\alpha)}, u_{S(\alpha)}]$ respectively. When both inter arrival time and service time are fuzzy numbers, based on Zadeh’s extension principle [79], the membership function of the performance measure $P(x, y)$ is defined as

$$\bigg\{ \begin{array}{l} \displaystyle \text{Sup}_{\mu_{P(\tilde{A}, \tilde{S})}(z) = x \in X, \ y \in Y} \left\{ \min (\mu_{\tilde{A}}(x), \mu_{\tilde{S}}(y))/z = P(x, y) \right\} \end{array} \bigg\} \quad \ldots (2.1)$$
Our approach is to construct the membership function $\mu_{p(\tilde{A},\tilde{S})}$ which is based on deriving the $\alpha$-cuts of $\mu_{p(\tilde{A},\tilde{S})}$. The corresponding, Parametric programming technique for finding lower and upper bounds of the $\alpha$-cut of $\mu_{p(\tilde{A},\tilde{S})}(z)$ are

$$\ell_{p(\alpha)} = \min p(x,y) \Rightarrow \ell_{A(\alpha)} \leq x \leq u_{A(\alpha)} \text{ and } \ell_{S(\alpha)} \leq y \leq u_{S(\alpha)}$$

$$u_{p(\alpha)} = \max p(x,y) \Rightarrow \ell_{A(\alpha)} \leq x \leq u_{A(\alpha)} \text{ and } \ell_{S(\alpha)} \leq y \leq u_{S(\alpha)}$$

If both $\ell_{p(\alpha)}$ and $u_{p(\alpha)}$ are invertible with respect to $\alpha$, then a left shape function $L(z) = \ell^{-1}_{p(\alpha)}$ and a right shape function $R(z) = u^{-1}_{p(\alpha)}$ can be obtained from which the membership function $\mu_{p(\tilde{A},\tilde{S})}$ is constructed.

$$\mu_{p(\tilde{A},\tilde{S})}(z) = \begin{cases} L(z), & z_1 \leq z \leq z_2 \\ 1, & z_2 \leq z \leq z_3 \\ R(z), & z_3 \leq z \leq z_4 \end{cases}$$

where $z_1 \leq z_2 \leq z_3 \leq z_4$ and $L(z_1) = R(z_4) = 0$.

2.3. THE (FM/FM/1): ($\infty$/FCFS) QUEUES

This queue adopts a first-come first-served queue discipline and consider an infinite source population where both the inter arrival time and the service time follow exponential distributions with ratio $\tilde{\lambda}$ and $\tilde{\mu}$ respectively, which are fuzzy variables rather than crisp values.
The expected number of customers in the system

\[ L_S = \frac{\lambda}{\mu - \lambda} \]

The average waiting in the system

\[ W_S = \frac{1}{\mu - \lambda} \]

The expected number of customers in the queue

\[ L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \]

The average waiting time of a customer in the queue

\[ W_q = \frac{\lambda}{\mu(\mu - \lambda)} \]

The probability that the number of customers in the system is greater than (or) equal to \( K \),

\[ P_K = P(n \geq K) = \left( \frac{\lambda}{\mu} \right)^K \]

### 2.4. NUMERICAL Example

Consider a FM/FM/1 queue, where both the arrival rate and service rate are fuzzy numbers represented by \( \tilde{\lambda} = [3,4,5,6] \) and \( \tilde{\mu} = [19,20,21,22] \). The interval of confidence at possibility level \( \alpha \) as \( [3 + \alpha, 6 - \alpha] \) and \( [19 + \alpha, 22 - \alpha] \).

The parametric programs to derive the membership functions of \( L_S \)

\[ \ell_{L_S}(\alpha) = \min \frac{x}{y - x} \text{ such that } 3 + \alpha \leq x \leq 6 - \alpha \text{ and } 19 + \alpha \leq y \leq 22 - \alpha \ldots (2.2a) \]

\[ u_{L_S}(\alpha) = \max \frac{x}{y - x} \text{ such that } 3 + \alpha \leq x \leq 6 - \alpha \text{ and } 19 + \alpha \leq y \leq 22 - \alpha \ldots (2.2b) \]
When \( x \) reaches its lower bound and \( y \) reaches its upper bound, \( \frac{x}{y-x} \) attains its minimum. Consequently, the optional solution for (2.2a) is 
\[
\ell_{L_3}(\alpha) = \frac{3 + \alpha}{19 - 2\alpha}.
\]
On the contrary, to maximize \( \frac{x}{y-x} \), \( x \) increases to its upper bound and \( y \) decreases to its lower bound. In this case the optimal solution for (2.2b) is 
\[
u_{L_3}(\alpha) = \frac{6 - \alpha}{13 + 2\alpha}.
\]
\( \ell_{L_3}(\alpha) \) is invertible

\[
z = \frac{3 + \alpha}{19 - 2\alpha}
\]
\[
\alpha = \frac{19z - 3}{1 + 2z}
\]

\( 0 \leq \alpha \leq 1, \)

\[
0 \leq \frac{19z - 3}{1 + 2z} \leq 1
\]

\[
0 \leq \frac{19z - 3}{1 + 2z} \quad \text{and} \quad \frac{19z - 3}{1 + 2z} \leq 1
\]

\[
z \geq \frac{3}{19} \quad \text{and} \quad z \leq \frac{4}{17}
\]

Therefore \( \frac{3}{19} \leq z \leq \frac{4}{17} \)

\( \nu_{L_3}(\alpha) \) is invertible

\[
z = \frac{6 - \alpha}{13 + 2\alpha}
\]
\[ \alpha = \frac{6 - 13z}{2z + 1} \]

\[ 0 \leq \frac{6 - 13z}{2z + 1} \leq 1 \quad \text{then} \quad 0 \leq \frac{6 - 13z}{2z + 1} \quad \text{and} \quad \frac{6 - 13z}{2z + 1} \leq 1 \]

\[ z \leq \frac{6}{13} \quad \text{and} \quad z \geq \frac{1}{3} \]

Therefore \[ \frac{1}{3} \leq z \leq \frac{6}{13} \]

The membership function \( \mu_{L_S}(z) \) of as

\[
\mu_{L_S}(z) = \begin{cases} 
(19z - 3) / (1 + 2z), & \frac{3}{19} \leq z \leq \frac{4}{17} \\
1, & \frac{4}{17} \leq z \leq \frac{1}{3} \\
(6 - 13z) / (1 + 2z), & \frac{1}{3} \leq z \leq \frac{6}{13} 
\end{cases}
\]

### Table 2.1

The \( \alpha \)-cuts of \( \mu_{L_S}(z) \) at 11 distinct \( \alpha \) Value

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \ell_{x(\alpha)} )</th>
<th>( u_{x(\alpha)} )</th>
<th>( \ell_{y(\alpha)} )</th>
<th>( u_{y(\alpha)} )</th>
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<th>( u_{L_S}(\alpha) )</th>
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<td>0.333</td>
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The parametric programs to derive the membership functions of $W_S$

$$\ell_{W_S}(\alpha) = \min \left( \frac{1}{y-x} \right)$$

Such that $3 + \alpha \leq x \leq 6 - \alpha$ and $19 + \alpha \leq y \leq 22 - \alpha$ \hspace{1cm} (2.3a)

$$u_{W_S}(\alpha) = \max \left( \frac{1}{y-x} \right)$$

Such that $3 + \alpha \leq x \leq 6 - \alpha$ and $19 + \alpha \leq y \leq 22 - \alpha$ \hspace{1cm} (2.3b)

When $x$ reaches its lower bound and $y$ reaches its upper bound, \(\frac{1}{y-x}\) attains its minimum. Consequently, the optimal solution for (2.3a) is

$$\ell_{W_S}(\alpha) = \frac{1}{19 - 2\alpha}.$$ 

On the contrary when $x$ reaches its upper bound and $y$ reaches its lower bound then \(\frac{1}{y-x}\), attains its maximum. Hence the optimal solution for (2.3b) is

$$u_{W_S}(\alpha) = \frac{1}{13 + 2\alpha}.$$
\( \ell_{W_S}(\alpha) \) is invertible

\[
\begin{align*}
z &= \frac{1}{19 - 2\alpha} \\
\alpha &= \frac{19z - 1}{2z}
\end{align*}
\]

\( 0 \leq \alpha \leq 1, \)

\[
0 \leq \frac{19z - 1}{2z} \leq 1
\]

\[
0 \leq \frac{19z - 1}{2z} \quad \text{and} \quad \frac{19z - 1}{2z} \leq 1
\]

\[
z \geq \frac{1}{19} \quad \text{and} \quad z \leq \frac{1}{17}
\]

\( u_{W_S}(\alpha) \) is invertible

\[
\begin{align*}
z &= \frac{1}{13 + 2\alpha} \\
\alpha &= \frac{1-13z}{2z}
\end{align*}
\]

\( 0 \leq \alpha \leq 1, \quad 0 \leq \frac{1-13z}{2z} \quad \text{and} \quad \frac{1-13z}{2z} \leq 1\)

\[
z \leq \frac{1}{13} \quad \text{and} \quad z \geq \frac{1}{15}
\]

Therefore, the membership function of \( \mu_{W_S}(z) \) is defined as

\[
\mu_{W_S}(z) = \begin{cases} 
\frac{(19z - 1)}{2z} , & \frac{1}{19} \leq z \leq \frac{1}{17} \\
1 , & \frac{1}{17} \leq z \leq \frac{1}{15} \\
\frac{(1-13z)}{2z} , & \frac{1}{15} \leq z \leq \frac{1}{13}
\end{cases}
\]
Table – 2.2
The $\alpha$-cuts of $\mu_{\tilde{W}_S}$ at 11 distinct $\alpha$ value

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\ell_{x(\alpha)}$</th>
<th>$u_{x(\alpha)}$</th>
<th>$\ell_{y(\alpha)}$</th>
<th>$u_{y(\alpha)}$</th>
<th>$\ell_{W_S}(\alpha)$</th>
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<td>0.075</td>
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<td>0.066</td>
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Membership function of $\tilde{W}_S$

Fig. 2.2
The parametric programme to derive the membership functions of $L_q$

$$\ell_{L_q}(\alpha) = \min \frac{x^2}{y(y-x)}$$

... (2.4a)

such that $3 + \alpha \leq x \leq 6 - \alpha$ and $19 + \alpha \leq y \leq 22 - \alpha$

$$u_{L_q}(\alpha) = \max \frac{x^2}{y(y-x)}$$

... (2.4b)

such that $3 + \alpha \leq x \leq 6 - \alpha$ and $19 + \alpha \leq y \leq 22 - \alpha$

$$\ell_{L_q}(\alpha) = \frac{(3 + \alpha)^2}{(22 - \alpha)(22 - \alpha - 3 - \alpha)} = \frac{\alpha^2 + 6\alpha + 9}{2\alpha^2 - 63\alpha + 418}$$

$\ell_{L_q}(\alpha)$ is invertible

Let $z = \frac{\alpha^2 + 6\alpha + 9}{2\alpha^2 - 63\alpha + 418}$

$$\alpha^2 (2z - 1) - \alpha(63z + 6) + (418z - 9) = 0$$

$$\alpha = \frac{(63z + 6) \pm \sqrt{(63z + 6)^2 - 4(2z - 1)(418z - 9)}}{2(2z - 1)}$$

$$\alpha = \frac{(63z + 6) \pm 25\sqrt{z^2 + 4z}}{2(2z - 1)}$$

$0 \leq \alpha \leq 1,$

$$0 \leq \frac{(63z + 6) \pm 25\sqrt{z^2 + 4z}}{2(2z - 1)} \leq 1$$

$$0 \leq \frac{(63z + 6) \pm 25\sqrt{z^2 + 4z}}{2(2z - 1)}$$

$$-(63z+6) \leq 25\sqrt{z^2 + 4z}$$
3344z^2 − 1744z + 36 ≤ 0

Let \( z = \frac{1744 \pm \sqrt{3041536 - 481536}}{6688} \)
\[ = \frac{1744 \pm \sqrt{2560000}}{6688} \]
\[ = \frac{1744 \pm 1600}{6688} \]
\[ z = \frac{1}{2}, \frac{9}{418} \]

\( \alpha \leq 1, \)
\[ \frac{(63z + 6) \pm 25\sqrt{z^2 + 4z}}{2(2z - 1)} \leq 1 \]
\[ (63z + 6) \pm 25\sqrt{z^2 + 4z} \leq (4z - 2) \]
\[ 625z^2 + 2500z \leq 34812^2 + 944z + 64 \]
\[ 0 \leq 2856z^2 - 1556z + 64 \]
\[ 2856z^2 - 1556z + 64 \leq 0 \]
\[ z = \frac{1556 \pm \sqrt{2421136 - 731136}}{5712} \]
\[ z = \frac{1}{2}, \frac{16}{357} \]

When \( x \) reaches its upper bound and \( y \) reaches its lower bound then
\[ \frac{x^2}{y(y-x)} \] attains its maximum.

\[ u_{14}(\alpha) = \frac{(6 + \alpha)^2}{(19 + \alpha)(19 + \alpha - 6 + \alpha)} = \frac{36 + \alpha^2 - 12\alpha}{247 + 51\alpha + 2\alpha^2} \]

\( u_{14}(\alpha) \) is invertible.
Let $z = \frac{\alpha^2 - 12\alpha + 36}{2\alpha^2 + 51\alpha + 247}$

$2\alpha^2 z + 51\alpha z + 247z = \alpha^2 - 12\alpha + 36$

$\alpha^2 (2z - 1) + \alpha(51z + 12) + (247z - 36) = 0$

$\alpha = \frac{-(51z + 12) \pm \sqrt{(51z + 12)^2 - 4(2z - 1)(247z - 36)}}{2(2z - 1)} = \frac{-(51z + 12) \pm 5\sqrt{25z^2 + 100z}}{2(2z - 1)}$

$0 \leq \alpha \leq 1,$

$0 \leq \frac{-(51z + 12) \pm 5\sqrt{25z^2 + 100z}}{2(2z - 1)}$

$(51z + 12) \leq 5\sqrt{25z^2 + 100z}$

$1976z^2 - 1276z + 144 \leq 0$

$z = \frac{1276 \pm \sqrt{(1276)^2 - 4(1276)(144)}}{2(1276)} = \frac{1}{2}, \frac{36}{247}$

$\alpha \leq 1,$

$\frac{-(51z + 12) \pm 5\sqrt{25z^2 + 100z}}{2(2z - 1)} \leq 1$

$-(51z + 12) \pm 5\sqrt{25z^2 + 100z} \leq 2(2z - 1)$

$24z^2 - 14z + 1 \geq 0$

$z = \frac{14 \pm \sqrt{196 - 96}}{48} = \frac{1}{2}, \frac{1}{12}$

The membership function of $L_q$ is
\[ \mu_{L_q}(z) = \begin{cases} \\ \frac{(63z + 6) \pm 25\sqrt{z^2 + 42}}{2(2z-1)}, & \frac{9}{418} \leq z \leq \frac{16}{357} \\
1, & \frac{16}{357} \leq z \leq \frac{1}{12} \\
-(51z + 12) \pm 5\sqrt{25z^2 + 100z}}{2(2z-1)}, & \frac{1}{12} \leq z \leq \frac{36}{247} \\ \end{cases} \]

Table – 2.3

The \( \alpha \)-cuts of \( \mu_{L_q} \) at 11 distinct \( \alpha \) value

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \ell_x(\alpha) )</th>
<th>( u_x(\alpha) )</th>
<th>( \ell_y(\alpha) )</th>
<th>( u_y(\alpha) )</th>
<th>( \ell_{L_q}(\alpha) )</th>
<th>( u_{L_q}(\alpha) )</th>
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</tr>
<tr>
<td>0.8</td>
<td>3.8</td>
<td>5.2</td>
<td>19.8</td>
<td>21.2</td>
<td>0.039</td>
<td>0.094</td>
</tr>
<tr>
<td>0.9</td>
<td>3.9</td>
<td>5.1</td>
<td>19.9</td>
<td>21.1</td>
<td>0.042</td>
<td>0.088</td>
</tr>
<tr>
<td>1.0</td>
<td>4.0</td>
<td>5.0</td>
<td>20.0</td>
<td>21.0</td>
<td>0.045</td>
<td>0.083</td>
</tr>
</tbody>
</table>
The parametric programme to derive the membership functions of $W_q$

\[ \ell_{W_q}(\alpha) = \min \frac{x}{y(y-x)} \quad \text{... (2.5a)} \]

Such that $3 + \alpha \leq x \leq 6 - \alpha$ and $19 + \alpha \leq y \leq 22 - \alpha$

\[ u_{W_q}(\alpha) = \max \frac{x}{y(y-x)} \quad \text{... (2.5b)} \]

such that $3 + \alpha \leq x \leq 6 - \alpha$ and $19 + \alpha \leq y \leq 22 - \alpha$

When $x$ reaches its lower bound and $y$ reaches its upper bound $\frac{x}{y(y-x)}$ attains its minimum.

When $x$ reaches its upper bound and $y$ reaches its lower bound then $\frac{x}{y(y-x)}$ attains its maximum.

\[ \ell_{W_q}(\alpha) = \frac{(3 + \alpha)}{(22 - \alpha)(22 - \alpha - 3 - \alpha)} = \frac{(3 + \alpha)}{2\alpha^2 - 63\alpha + 418} \]
$\ell_{w_i}(\alpha)$ is invertible, let $z = \frac{(3 + \alpha)}{2\alpha^2 - 63\alpha + 418}$

$\alpha^2 2z - \alpha(63z + 1) + (418z - 3) = 0$

$\alpha = \frac{(63z + 1) \pm \sqrt{(63z + 1)^2 - 4 \times 2z (418z - 3)}}{4z} = \frac{(63z + 1) \pm \sqrt{625z^2 + 150z + 1}}{4z}$

$0 \leq \alpha \leq 1,$

$0 \leq \frac{(63z + 1) \pm \sqrt{625z^2 + 150z + 1}}{4z} \leq (63z + 1) \pm \sqrt{625z^2 + 150z + 1}$

$z(3344z - 24) \leq 0$

$z = 0 \& z = \frac{3}{418}$

$\frac{(63z + 1) \pm \sqrt{625z^2 + 150z + 1}}{4z} \leq 1$

$(63z + 1) \pm \sqrt{625z^2 + 150z + 1} \leq 4z$

$0 \leq z(2856z - 32)$

$z = 0 \& z = \frac{4}{357}$

When $x$ reaches its upper bound and $y$ reaches its lower bound then

$\frac{x}{y(y-x)}$ attains its maximum.
\[ \therefore u_{w_1}(\alpha) = \frac{6 - \alpha}{(19 + \alpha)(19 + \alpha - 6 + \alpha)} = \frac{(6 - \alpha)}{2\alpha^2 + 51\alpha + 247} \]

\( u_{w_1}(\alpha) \) is invertible,

Let \( z = \frac{(6 - \alpha)}{2\alpha^2 + 51\alpha + 247} \)

\[ z(2\alpha^2 + 51\alpha + 247) = 6 - \alpha \]

\[ \alpha^2 2z + \alpha(51z + 1) + 247z - 6 = 0 \]

\[ \alpha = \frac{- (51z + 1) \pm \sqrt{(51z + 1)^2 - 4.2z(247z - 6)}}{4z} = \frac{- (51z + 1) \pm \sqrt{625z^2 + 150z + 1}}{4z} \]

\[ 0 \leq \alpha \leq 1, \]

\[ 0 \leq \frac{- (51z + 1) \pm \sqrt{625z^2 + 150z + 1}}{4z} \]

\[ (51z + 1) \leq \sqrt{625z^2 + 150z + 1} \]

\[ z(1976z - 48) \leq 0 \]

\[ z = 0 \& z = \frac{6}{247} \]

\[ \frac{- (51z + 1) \pm \sqrt{625z^2 + 150z + 1}}{4z} \leq 1 \]

\[ 0 \leq z(2400z - 40) \]
\[ z = 0 \text{ and } z = \frac{1}{60} \]

\[ \therefore \text{ The membership function of } W_q \]

\[ \mu_{W_q}(z) = \begin{cases} 
\frac{(63z + 1) \pm (625z^2 + 150z + 1)^{1/2}}{4z}, & \frac{3}{418} \leq z \leq \frac{4}{357} \\
1, & \frac{4}{357} \leq z \leq \frac{1}{60} \\
\frac{- (51z + 1) \pm (625z^2 + 150z + 1)^{1/2}}{4z}, & \frac{1}{60} \leq z \leq \frac{6}{247} 
\end{cases} \]

**Table – 2.4**

The \( \alpha \)-cuts of \( \mu_{W_q} \) at 11 distinct \( \alpha \) value

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \ell_{s(\alpha)} )</th>
<th>( u_{s(\alpha)} )</th>
<th>( \ell_{y(\alpha)} )</th>
<th>( u_{y(\alpha)} )</th>
<th>( \ell_{W_q}(\alpha) )</th>
<th>( u_{W_q}(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>3.0</td>
<td>6.0</td>
<td>19.0</td>
<td>22.0</td>
<td>0.007</td>
<td>0.024</td>
</tr>
<tr>
<td>0.1</td>
<td>3.1</td>
<td>5.9</td>
<td>19.1</td>
<td>21.9</td>
<td>0.007</td>
<td>0.023</td>
</tr>
<tr>
<td>0.2</td>
<td>3.2</td>
<td>5.8</td>
<td>19.2</td>
<td>21.8</td>
<td>0.007</td>
<td>0.022</td>
</tr>
<tr>
<td>0.3</td>
<td>3.3</td>
<td>5.7</td>
<td>19.3</td>
<td>21.7</td>
<td>0.008</td>
<td>0.021</td>
</tr>
<tr>
<td>0.4</td>
<td>3.4</td>
<td>5.6</td>
<td>19.4</td>
<td>21.6</td>
<td>0.008</td>
<td>0.020</td>
</tr>
<tr>
<td>0.5</td>
<td>3.5</td>
<td>5.5</td>
<td>19.5</td>
<td>21.5</td>
<td>0.009</td>
<td>0.020</td>
</tr>
<tr>
<td>0.6</td>
<td>3.6</td>
<td>5.4</td>
<td>19.6</td>
<td>21.4</td>
<td>0.009</td>
<td>0.019</td>
</tr>
<tr>
<td>0.7</td>
<td>3.7</td>
<td>5.3</td>
<td>19.7</td>
<td>21.3</td>
<td>0.009</td>
<td>0.018</td>
</tr>
<tr>
<td>0.8</td>
<td>3.8</td>
<td>5.2</td>
<td>19.8</td>
<td>21.2</td>
<td>0.010</td>
<td>0.018</td>
</tr>
<tr>
<td>0.9</td>
<td>3.9</td>
<td>5.1</td>
<td>19.9</td>
<td>21.1</td>
<td>0.010</td>
<td>0.017</td>
</tr>
<tr>
<td>1.0</td>
<td>4.0</td>
<td>5.0</td>
<td>20.0</td>
<td>21.0</td>
<td>0.011</td>
<td>0.016</td>
</tr>
</tbody>
</table>
The parametric program to derive the membership function of 

\[ P(n \geq K) = \left( \frac{\lambda}{\mu} \right)^K \]

\[ \tilde{\lambda} = [3, 4, 5, 6], \quad \tilde{\mu} = [19, 20, 21, 22] \]

\[ \mu_{\tilde{\lambda}}(s) = \begin{cases} 
  s-3 & 3 \leq s \leq 4 \\
  1 & 4 \leq s \leq 5 
\end{cases} \]

\[ \mu_{\tilde{\mu}}(s) = \begin{cases} 
  s-19 & 19 \leq s \leq 20 \\
  1 & 20 \leq s \leq 21 
\end{cases} \]

The interval of confidence at possibility level \( \alpha \) is \([3 + \alpha, 6 - \alpha]\) and \([19 + \alpha, 22 - \alpha]\).
\[ P(n \geq K) = \min \left( \frac{x}{y} \right)^K \]

Such that \( 3 + \alpha \leq x \leq 6 - \alpha \) and \( 19 + \alpha \leq y \leq 22 - \alpha \)

\[ u_{P(n \geq K)}(\alpha) = \max \left( \frac{x}{y} \right)^K \]

Such that \( 3 + \alpha \leq x \leq 6 - \alpha \) and \( 19 + \alpha \leq y \leq 22 - \alpha \)

When \( x \) reaches its lower bound and \( y \) reaches its upper bound then

\[ \left( \frac{x}{y} \right)^K \] attains its minimum.

\[ K = 1 \]

\[ \therefore \ P_{(n \geq K)}(\alpha) = \frac{3 + \alpha}{22 - \alpha} \]

Let \( z = \frac{3 + \alpha}{22 - \alpha} \)

\[ \alpha = \frac{22z - 3}{1 + z} \]

\[ \therefore \ P_{(n \geq K)}(\alpha) \text{ is invertible} \]

\[ 0 \leq \alpha \leq 1, \]

\[ 0 \leq \frac{22z - 3}{1 + z} \leq 1 \]

\[ 0 \leq \frac{22z - 3}{1 + z} \text{ and } \frac{22z - 3}{1 + z} \leq 1 \]

\[ z \geq \frac{3}{22} \text{ and } z \leq \frac{4}{21} \]
\[ \frac{3}{22} \leq z \leq \frac{4}{21} \]

When \( x \) reaches its upper bound and \( y \) reaches its lower bound then

\[ \left( \frac{x}{y} \right)^K \]

attains maximum.

\[ u_{P(n \geq K)}(\alpha) = \frac{6 - \alpha}{19 + \alpha} \]

Let \( z = \frac{6 - \alpha}{19 + \alpha} \)

\[ \alpha = \frac{6 - 19z}{1 + z} \]

\( u_{P(n \geq K)}(\alpha) \) is invertible

\[ 0 \leq \alpha \leq 1, \]

\[ 0 \leq \frac{6 - 19z}{1 + z} \leq 1 \]

\[ 0 \leq \frac{6 - 19z}{1 + z} \text{ and } \frac{6 - 19z}{1 + z} \leq 1 \]

\[ z \geq \frac{1}{4} \text{ and } z \leq \frac{6}{19} \]

\[ \therefore \frac{1}{4} \leq z \leq \frac{6}{19} \]

\[ \therefore \mu_{P(n \geq 1)}(x) = \begin{cases} \frac{22z - 3}{1 + z} & \frac{3}{22} \leq z \leq \frac{4}{21} \\ \frac{1}{4} & \frac{1}{19} \leq z \leq \frac{1}{4} \\ \frac{6 - 19z}{1 + z} & \frac{1}{4} \leq z \leq \frac{6}{19} \end{cases} \]
\( K = 2 \)

\[
\varepsilon_{P(n\geq 2)}(x) = \min \left( \frac{x}{y} \right)^2
\]

Such that \( 3 + \alpha \leq x \leq 6 - \alpha \) and \( 19 + \alpha \leq y \leq 22 - \alpha \)

\[
u_{P(n\geq 2)}(x) = \max \left( \frac{x}{y} \right)^2
\]

Such that \( 3 + \alpha \leq x \leq 6 - \alpha \) and \( 19 + \alpha \leq y \leq 22 - \alpha \)

When \( x \) reaches its lower bound and \( y \) reaches its upper bound then

\[
\left( \frac{x}{y} \right)^2 \text{ attains its minimum.}
\]

\[
\varepsilon_{P(n\geq 2)}(\alpha) = \frac{(3 + \alpha)^2}{(22 - \alpha)^2}
\]

When \( x \) reaches its upper bound and \( y \) reaches its lower bound then

\[
\left( \frac{x}{y} \right)^2 \text{ attains its maximum.}
\]

\[
U_{P(n\geq 2)}(\alpha) = \frac{(6 - \alpha)^2}{(19 + \alpha)^2}
\]

Let \( z = \frac{(3 + \alpha)^2}{(22 - \alpha)^2} \)

\[
z = \frac{9 + \alpha^2 + 6\alpha}{484 + \alpha^2 - 44\alpha}
\]

\[
\alpha^2(z-1) - \alpha(44z+6) + (484z-9) = 0
\]
\begin{align*}
\alpha &= \frac{(44z + 6) \pm \sqrt{(44z + 6)^2 - 4(z - 1)(484z - 9)}}{2(z - 1)} = \frac{(44z + 6) \pm 50\sqrt{z}}{2(z - 1)}
\end{align*}

\( \mathcal{L}_{P(n \geq 2)}(\alpha) \) is invertible

\[ 0 \leq \alpha \leq 1, \]

\[ 0 \leq \frac{(44z + 6) \pm 50\sqrt{z}}{2(z - 1)} \leq 1 \]

\[ 1936z^2 - 1972z + 36 \geq 0 \]

\[ z = \frac{1972 \pm \sqrt{3888784 - 4 \times 1936 \times 36}}{3872} = 1, \ \frac{9}{484} \]

\[ \frac{(44z + 6) \pm 50\sqrt{z}}{2(z - 1)} \leq 1 \]

\[ (44z + 6) \pm 50\sqrt{z} \leq 2(z - 1) \]

\[ z = \frac{457 \pm \sqrt{(457)^2 - 4(441)(16)}}{882} = 1, \ \frac{16}{441} \]

i.e. \[ \frac{9}{484} \leq z \leq \frac{16}{441} \]

When \( x \) reaches its upper bound and \( y \) reaches its lower bound then

\( \left( \frac{x}{y} \right)^2 \) attains its maximum.

\[ u_{P(n \geq 2)}(\alpha) = \frac{(6 - \alpha)^2}{(19 + \alpha)^2} = \frac{36 + \alpha^2 - 12\alpha}{361 + \alpha^2 + 38\alpha} \]

Let \[ z = \frac{36 + \alpha^2 - 12\alpha}{361 + \alpha^2 + 38\alpha} \]

\[ \alpha^2(z - 1) + \alpha(38z + 12) + (361z - 36) = 0 \]
\[
\alpha = \frac{- (38z + 12) \pm \sqrt{(38z + 12)^2 - 4(z - 1)(361z - 36)}}{2(z - 1)} = \frac{- (38z + 12) \pm 50\sqrt{z}}{2(z - 2)}
\]

\[u_{P(\alpha \geq 2)}(\alpha) \text{ is invertible}
\]

\[
0 \leq \frac{- (38z + 12) \pm 50\sqrt{z}}{2(z - 2)} \leq 1
\]

\[0 \leq \alpha \leq 1,
\]

\[
0 \leq \frac{- (38z + 12) \pm 50\sqrt{z}}{2(z - 2)}
\]

\[
0 \leq -(38z+12) \pm 50\sqrt{z}
\]

\[ (38z + 12) \leq 50\sqrt{z}
\]

\[1444z^2 + 144 + 912z \leq 2500 z
\]

\[1444z^2 - 1588z + 144 \leq 0
\]

\[
z = \frac{1588 \pm \sqrt{2521744 - 4 \times 1444 \times 144}}{2888} = 1, \frac{288}{2888}
\]

\[\alpha \leq 1,
\]

\[
- \frac{(38z + 12) \pm 50\sqrt{z}}{2(z - 2)} \leq 1
\]

\[-(38z + 12) \pm 50\sqrt{z} \leq (2z-2)
\]

\[16z^2-17z+1 \leq 0
\]

\[
z = \frac{17 \pm \sqrt{289 - 64}}{32} = 1, \frac{1}{16}
\]

i.e. \[\frac{1}{16} \leq z \leq \frac{288}{2888}\]
\[
\mu_{P(n \geq 2)}(z) = \begin{cases} 
\frac{(44z + 6) \pm 50\sqrt{z}}{2(z-1)} & 9 \leq z \leq \frac{16}{441} \\
1 & \frac{16}{441} \leq z \leq \frac{1}{16} \\
-\frac{(38z + 12) \pm 50\sqrt{z}}{2(z-1)} & \frac{1}{16} \leq z \leq \frac{288}{2888}
\end{cases}
\]

2.5. Parametric non-linear programming for analyzing fuzzy queues with finite capacity

The finite-capacity queuing systems have been extensively studied by many researches like Shi [65], Gouweleeuw and Tijms [32], Bretthaner and Cote [5]. An important practical problem is the design of finite buffers to achieve a prespecified loss probability [32]. By traditional queuing theory, the inter arrival times and service times are required to follow certain distribution. However, in many practical applications, the statistical information may be obtained subjectively, i.e. the arrival pattern and service pattern are more suitably described by linguistic terms such as fast, slow (or) moderate rather than by probability distributions. Thus, if the crisp queues with finite capacity can be extended to fuzzy queue with finite capacity queues, queuing models would have even wider applications.

Kao et al. [11], therefore adopt parametric programming to construct the membership function of system performance measures for fuzzy queues and successfully applied to four simple fuzzy queues with one (or) two fuzzy variables, namely, M/F/I, F/M/I, F/F/I and FM/FM/I, where F denotes fuzzy time and FM denotes fuzzified exponential time. It seems that their approach
can be applied to the fuzzy finite-capacity queues. However, since the fuzzy finite-capacity queuing systems are more complicated than the above four fuzzy queues, the solution procedure for the fuzzy finite-capacity queue are not explicitly known and it deserves further investigation.

In this model, the basic idea is to employ the $\alpha$-cuts and Zadeh’s extension principle [79] to transform the fuzzy finite-capacity queue to a family of crisp finite-capacity queues. As the $\alpha$-varies the conventional finite capacity queues are then described and solved by parametric nonlinear programming. To demonstrate the validity of the proposed solution method, two examples inspired by management problems in real life are solved.

2.6. Fuzzy queues with finite capacity

Consider a queuing system in which the arrivals follows poison process with a fuzzy rate $\tilde{\lambda}$, and the service time follows exponential distribution with fuzzy rate $\tilde{\mu}$. Customers are served according to a first-come-first-served (FCFS) discipline, and the size of the service population is infinite. The maximum number of customers in the system is limited to $k$.

This model will hereafter be denoted by FM/FM/1/FCFS/K/$\infty$.

In this model the arrival rate and service rate are represented by the following fuzzy sets

\[
\tilde{\lambda} = \left\{ (x, \mu_{\tilde{\lambda}}(x)) \mid x \in X \right\} \quad \text{... (2.6a)}
\]

\[
\tilde{\mu} = \left\{ (y, \mu_{\tilde{\mu}}(y)) \mid y \in Y \right\} \quad \text{... (2.6b)}
\]
Where \( X \) and \( Y \) are the crisp universal set of the arrival rate and service rate and \( \mu_\lambda(x) \) and \( \mu_\mu(y) \) are the corresponding membership functions. Let \( p(x,y) \) denote the system performance measure of interest. When \( \tilde{\lambda} \) and \( \tilde{\mu} \) are fuzzy numbers, \( P(\tilde{\lambda}, \tilde{\mu}) \) is also a fuzzy number. According to Zadeh’s extension principle, the membership function of the performance measure \( P(\tilde{\lambda}, \tilde{\mu}) \) is defined as

\[
\mu_{P(\tilde{\lambda}, \tilde{\mu})}(z) = \sup_{x \in X, y \in Y} \min \left\{ \mu_\lambda(x), \mu_\mu(y) \right\} \Big/ z = p(x, y) \quad \ldots (2.7)
\]

Without Loss of generality, assume that the performance measure of interest is \( W_s \), i.e. \( P(x,y) = W_s \). Let \( \rho = \frac{x}{y} \) denote the traffic intensity of the system. From the knowledge of traditional queuing theory, the expected waiting time in the system with finite capacity is

\[
W_s = \frac{\rho[1-(k+1)\rho^k + k\rho^{k+1}]}{\lambda(1-\rho^{k+1})(1-\rho)} \quad \ldots (2.8)
\]

\[
\mu_{w_q}(z) = \sup_{x \in X, y \in Y} \min \left\{ \mu_\lambda(x), \mu_\mu(y) \right\} \Big/ z = \frac{\rho[1-(k+1)\rho^k + k\rho^{k+1}]}{\lambda(1-\rho^{k+1})(1-\rho)} \quad \ldots (2.9)
\]

Membership functions for the performance measure of \( w_q \) can be obtained from Little’s formula [31] in the same manner.

\[
\mu_{w_q}(z) = \sup_{x \in X, y \in Y} \min \left\{ \mu_\lambda(x), \mu_\mu(y) \right\} \Big/ z = \frac{\rho[1-(k+1)\rho^k + k\rho^{k+1}]}{\lambda(1-\rho^{k+1})(1-\rho)} - \frac{1}{\mu} \quad \ldots (2.10)
\]
Although the membership functions in (2.9) and (2.10) are theoretically correct, they are not in the usual forms for practical use and it is very difficult to imagine their shapes.

In this model, a pair of parametric non-linear programme is developed to find the \( \alpha \)-cut of \( P(\lambda, \mu) \) based on the extension principle.

### 2.7. Solution procedure

Our approach is to construct the membership function \( \mu_{P(\lambda, \mu)}(z) \) using \( \alpha \)-cuts. The \( \alpha \)-cuts of \( \tilde{\lambda} \) and \( \tilde{\mu} \) are defined as

\[
\lambda(\alpha) = \{ x \in X \mid \mu_{\tilde{\lambda}}(x) \geq \alpha \} \quad \text{... (2.11a)}
\]

\[
\mu(\alpha) = \{ y \in Y \mid \mu_{\tilde{\mu}}(y) \geq \alpha \} \quad \text{... (2.11b)}
\]

The \( \alpha \)-cuts of \( \lambda(\alpha) \) and \( \mu(\alpha) \) are crisp sets rather than fuzzy sets. By using \( \alpha \)-cut, the imbedded fuzzy Markov chain in the FM/FM/1/FCFS/K/\( \infty \) can be decomposed into a family of ordinary Markov chains. The arrival rate and service rate can also be represented by different levels of confidence intervals [82]. Consequently, the FM/FM/1/FCFS/K/\( \infty \) queue can be reduced to a family of crisp M/M/1/FCFS/K/\( \infty \) queues with different \( \alpha \) level sets \( \{ \lambda(\alpha)/0 < \alpha \leq 1 \} \) and \( \{ \mu(\alpha)/0 < \alpha \leq 1 \} \). These two sets represent sets of movable boundaries, and they form a nested structure for expressing the relationship between ordinary sets and fuzzy set [40].
Suppose the fuzzy arrival rate \( \tilde{\lambda} \) and fuzzy service rate \( \tilde{\mu} \) of the FM/FM/1/FCFS/K/\( \infty \) queuing system are fuzzy numbers. The \( \alpha \)-level sets of \( \tilde{\lambda} \) and \( \tilde{\mu} \) defined in (2.11a) and (2.11b) are crisp intervals which can be expressed in another form

\[
\lambda(\alpha) = \left[ \min_{x \in X} \{ x / \mu_{\tilde{\lambda}}(x) \geq \alpha \}, \max_{x \in X} \{ x / \mu_{\tilde{\lambda}}(x) \geq \alpha \} \right] = [x^L_\alpha, x^U_\alpha] \quad \cdots (2.12a)
\]

\[
\mu(\alpha) = \left[ \min_{y \in Y} \{ y / \mu_{\tilde{\mu}}(y) \geq \alpha \}, \max_{y \in Y} \{ y / \mu_{\tilde{\mu}}(y) \geq \alpha \} \right] = [y^L_\alpha, y^U_\alpha] \quad \cdots (2.12b)
\]

These intervals indicate where the constant arrival rate and service rate lie at possibilities level \( \alpha \). By the convexity of a fuzzy number, the bounds of these intervals are functions of \( \alpha \) and can be obtained as \( x^L_\alpha = \min \mu^{-1}_{\tilde{\lambda}}(\alpha), \)

\( x^U_\alpha = \max \mu^{-1}_{\tilde{\lambda}}(\alpha), \)

\( y^L_\alpha = \min \mu^{-1}_{\tilde{\mu}}(\alpha), \)

\( y^U_\alpha = \max \mu^{-1}_{\tilde{\mu}}(\alpha) \) respectively. Clearly, the membership function of \( P(\tilde{\lambda}, \tilde{\mu}) \) defined in (2.7) is also parameterized by \( \alpha \).

Consequently, we can use its \( \alpha \)-cut to construct its membership function.

The membership function stated in (2.9), is not in the usual form and is very difficult to imagine its shape, \( \mu_{\text{ws}}(z) \) is the minimum of \( \mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y) \). To tackle this either we need \( \mu_{\tilde{\lambda}}(x) = \alpha \) and \( \mu_{\tilde{\mu}}(y) \geq \alpha \) or \( \mu_{\tilde{\mu}}(y) = \alpha \) and \( \mu_{\tilde{\lambda}}(x) \geq \alpha \) such that

\[
z = \frac{\rho[1-(k+1)p^k+k\rho^{k+1}]}{\lambda(1-p^{k+1})(1-\rho)}
\]

to satisfy \( \mu_{\text{ws}}(z) = \alpha \). This can be accomplished via the parametric NLP techniques. For the former case,
the corresponding parametric non-linear programme for finding the lower and upper bounds of the $\alpha$-cut of $\mu_w$ are

\[
W_{s_a}^{L_1} = \min_{x,y \in \mathbb{R}^+} \frac{y^{k+1} - (k + 1)x^ky + kx^{k+1}}{(y-x)(y^{k+1} - x^{k+1})} \quad \text{... (2.13a)}
\]

so that $x_a^L \leq x \leq x_a^U$, $y \in \mu(\alpha)$

\[
W_{s_a}^{U_1} = \max_{x,y \in \mathbb{R}^+} \frac{y^{k+1} - (k + 1)x^ky + kx^{k+1}}{(y-x)(y^{k+1} - x^{k+1})} \quad \text{... (2.13b)}
\]

so that $x_a^L \leq x \leq x_a^U$, $y \in \mu(\alpha)$

and for the latter case:

\[
W_{s_a}^{L_2} = \min_{x,y \in \mathbb{R}^+} \frac{y^{k+1} - (k + 1)x^ky + kx^{k+1}}{(y-x)(y^{k+1} - x^{k+1})} \quad \text{... (2.13c)}
\]

so that $y_a^L \leq y \leq y_a^U$, $x \in \lambda(\alpha)$

\[
W_{s_a}^{U_2} = \max_{x,y \in \mathbb{R}^+} \frac{y^{k+1} - (k + 1)x^ky + kx^{k+1}}{(y-x)(y^{k+1} - x^{k+1})} \quad \text{... (2.13d)}
\]

so that $y_a^L \leq y \leq y_a^U$, $x \in \lambda(\alpha)$

According to the definition of $\lambda(\alpha)$ and $\mu(\alpha)$ in (2.12a) and (2.12b), $x \in \lambda(\alpha)$ and $y \in \lambda(\alpha)$ can be replaced by $x \in [x_a^L, x_a^U]$ and $y \in [y_a^L, y_a^U]$ respectively.

Since all $\alpha$-cuts form a nested structure with respect to $\alpha$, i.e. Given $0 < \alpha_2 < \alpha_1 \leq 1$, we have $[x_{a_1}^L, x_{a_1}^U] \subseteq [x_{a_2}^L, x_{a_2}^U]$ and $[y_{a_1}^L, y_{a_1}^U] \subseteq [y_{a_2}^L, y_{a_2}^U]$. Therefore, (2.13a) and (2.13c) and (2.13b) and (2.13d) are the same,
respectively. Thus based on (2.9), to find the membership function of \( \mu_{w_S} \), it is enough to find the lower bound \( W^L_S \) and upper bound \( W^U_S \) of the \( \alpha \)-cuts of \( \mu_{w_S} \) which can be written as

\[
W^L_S = \min_{x, y \in \mathbb{R}^+} \frac{y^{k+1} - (k+1)x^ky + kx^{k+1}}{(y - x)(y^{k+1} - x^{k+1})} \quad \ldots \quad (2.14a)
\]

so that \( x^L_a \leq x \leq x^U_a \) and \( y^L_a \leq y \leq y^U_a \)

\[
W^U_S = \max_{x, y \in \mathbb{R}^+} \frac{y^{k+1} - (k+1)x^ky + kx^{k+1}}{(y - x)(y^{k+1} - x^{k+1})} \quad \ldots \quad (2.14b)
\]

so that \( x^L_a \leq x \leq x^U_a \) and \( y^L_a \leq y \leq y^U_a \)

This pair of mathematical programs involves the systematic study of how the optional solutions change when \( x^L_a, x^U_a, y^L_a, y^U_a \) vary over the interval \( \alpha \in [0,1] \) thus they fall into the category of parametric NLP [28]. As \( k \) increases, the objective function becomes more complicated and hence the difficulty in solving the pair of mathematical programs also increases. To overcome this difficulty, software package Mathematica for windows is adopted to obtain the result.

If both \( W^L_S \) and \( W^U_S \) are invariable with respect to \( \alpha \), then a left-shape function \( L(z) = (W^L_S z^{-1})^{-1} \) and a right shape function \( R(z) = (W^U_S z^{-1})^{-1} \) can be obtained, from which the membership function \( \mu_{w_S} \) is constructed.
\[
\mu_{W_S}(z) = \begin{cases} 
L(z), & z_1 \leq z \leq z_2 \\
1, & z_2 \leq z \leq z_3 \\
R(z), & z_3 \leq z \leq z_4 
\end{cases} \quad \ldots (2.15)
\]

2.8. Numerical Examples

To illustrate how the proposed method can be applied to analyze fuzzy queues with finite capacity, we investigate two examples often encountered in real life. The first is relatively simple, thus a closed-form membership function can be derived by taking the inverse of its \(\alpha\)-cuts. The second is more complicated, thus a closed-form solution cannot be derived. Numerical solutions for different \(\alpha\)-values are calculated to approximate the membership function.

In the solution procedure of the following two examples, several calculations, for example, the derivation of the inverse function, are somewhat cumbersome. The software package Mathematica for windows is adopted for alleviating the computational burden.

2.8.1. Numerical Example 1

Consider an automatic car wash facility operating with only one bay. Suppose the facility has only one parking space, if the parking lot is full, newly arriving cars balk to other facilities. Cars arrive at this facility in accordance with a Poisson process, and the service time follows an exponential distribution. Both the arrival rate and service rate are trapezoidal fuzzy number represented by \(\tilde{\lambda} = [4,4.5,5.5, 6.5] \) and \(\tilde{\mu} = [8,9,11,12] \) per hour, respectively.
The manager of the facility wishes to determine the impact of waiting customer in the system on losing customers to the competition.

It is clear that the system can be described by the FM/FM/1/FCFS/2/∞ model and the performance measures can be constructed by the proposed procedure stated in section 2.7. It is easy to find

\[
[x^L_a, x^U_a] = [\min \mu^{-1}_x(\alpha), \max \mu^{-1}_x(\alpha)] = [(8 + \alpha)/2, (13 - 2\alpha)/2]
\]

and

\[
y^L_a, y^U_a = [\min \mu^{-1}_y(\alpha), \max \mu^{-1}_y(\alpha)] = [8 + \alpha, 12 - \alpha]
\]

Thus, following (2.14), the parametric nonlinear programs for deriving the membership function of \( \tilde{w}_S \) are

\[
W_{S_a}^L = \min_{x, y \in \mathbb{R}^+} \frac{y^3 - 3x^2y + 2x^3}{(y - x)(y^3 - x^3)}
\]

so that \((8 + \alpha)/2 \leq x \leq (13 - 2\alpha)/2\) and \((8 + \alpha) \leq y \leq (12 - \alpha)\)

\[
W_{S_a}^U = \max_{x, y \in \mathbb{R}^+} \frac{y^3 - 3x^2y + 2x^3}{(y - x)(y^3 - x^3)}
\]

so that \((8 + \alpha)/2 \leq x \leq (13 - 2\alpha)/2\) and \((8 + \alpha) \leq y \leq (12 - \alpha)\)

Consequently, the shortest \( W_s \) occurs when \( x \) is at its lower bound \((8 + \alpha)/2\) and \( y \) is at its upper bound \((12 - \alpha)\), therefore

\[
W_{S_a}^L = \frac{80}{3\alpha^2 - 80\alpha + 916}
\]

i.e.

\[
z = \frac{80}{3\alpha^2 - 80\alpha + 916}
\]

\[
3\alpha^2z - 80\alpha z + (916z - 80) = 0
\]
\[
\alpha = \frac{80z \pm \sqrt{(80z)^2 - 4.3z(916z - 80)}}{6z} = \frac{80z \pm \sqrt{960z - 4592z^2}}{6z}
\]

\[
\alpha \geq 0, \quad 80z \pm \sqrt{960z - 4592z^2} \geq 0
\]

\[
z(10992z - 960) \geq 0
\]

\[
z = 0, \quad z = \frac{960}{10992} = .0873
\]

\[
\alpha \leq 1, \quad \frac{80z \pm \sqrt{960z - 4592z^2}}{6z} \leq 1
\]

\[
z(10068z - 960) \geq 0
\]

\[
z = 0 \text{ (or) } z = \frac{960}{10068} = .0953
\]

On the contrary, in maximizing the response, the largest values of \(x\) and the smallest value of \(y\) are desired, which results in

\[
W_{Sa}^U = \frac{4(21 - \alpha)}{4\alpha^2 + 6\alpha + 633}
\]

i.e. \[
z = \frac{4(21 - \alpha)}{4\alpha^2 + 6\alpha + 633}
\]

\[
4z\alpha^2 + \alpha(6z + 4) + (633z - 84) = 0
\]

\[
\alpha = \frac{-(6z + 4) \pm \sqrt{(6z + 4)^2 - 4.4z(633z - 84)}}{8z}
\]

\[
\alpha = \frac{-(6z + 4) \pm \sqrt{-10092z^2 + 1392z + 16}}{8z}
\]

\[
\alpha \geq 0, \quad -(6z + 4) \pm \sqrt{-10092z^2 + 1392z + 16} \geq 0
\]
\[ z(10128z - 1344) \geq 0 \]

\[ z = 0 \text{ (or) } z = \frac{1344}{10128} = .1327 \]

\[ \alpha \leq 1, -(6z + 4) \pm \sqrt{-10092z^2 + 1392z + 16} \leq 8z \]

\[ z(10288z - 1280) \geq 0 \]

\[ z = 0 \text{ (or) } z = \frac{1280}{10288} = .1244 \]

Thus inverse function of \( W_{\alpha}^L \) and \( W_{\alpha}^U \) exist, which gives the membership function \( \mu_{w_s}(z) \) as

\[
\mu_{w_s}(z) = \begin{cases} 
\frac{80z + \sqrt{960z - 4592z^2}}{6z} & \quad \frac{960}{10992} \leq z \leq \frac{960}{10068} \\
1 & \quad \frac{960}{10068} \leq z \leq \frac{10068}{1280} \\
\frac{-(6z + 4) + \sqrt{16 + 1392z - 10092z^2}}{8z} & \quad \frac{1280}{10288} \leq z \leq \frac{1344}{10128}
\end{cases} \quad \ldots (2.17)
\]

Fig. 2.5 a

Membership function for \( \mu_{w_s} \) in Example 1
2.8.2. Numerical Example 2

Consider an automobile emission inspection station with one inspection stall and with room to accommodate only one car. It is reasonable to assume that cars wait in such a way that when the stall becomes vacant, the car at the head of the line pulls up to it. The station can accommodate at most nine cars waiting at the same time; that is, the maximum number of cars in the station is 10. Cars arrive at this facility in accordance with Poisson process, and the service time following an exponential distribution. Both the arrival rate and service rate are trapezoidal fuzzy number represented by $\tilde{\lambda} = [5, 6, 7, 8]$ and $\tilde{\mu} = [10, 11, 12, 13]$ per hour respectively. The chief inspector wishes to know the waiting time in the system during the peak hour.

It is clear that this system can be described by the FM/FM/1/FCFS/10/$\infty$ model, the proposed procedure stated in section 2.7 can be used to construct the membership functions of the performance measures.

The corresponding $\alpha$-cuts of these two fuzzy numbers are $[x_\alpha^L, x_\alpha^U] = [5 + \alpha, 8 - \alpha]$ and $[y_\alpha^L, y_\alpha^U] = [10 + \alpha, 13 - \alpha]$ at a specified possibility level $\alpha$, the lower and upper bounds of the $\alpha$-cuts of $\mu_{ws}$, according to (2.14), can be solved as

$$W_{ws}^L = \min_{x, y \in R^+} \frac{y^{11} - 11x^{10}y + 10x^{11}}{(y - x)(y^{11} - x^{11})}$$ ...

such that $5 + \alpha \leq x \leq 8 - \alpha$ and $10 + \alpha \leq y \leq 13 - \alpha$
\[ W_{S_a}^L = \max_{x, y \in \mathbb{R}^+} \frac{y^{11} - 11x^{10}y + 10x^{11}}{(y - x)(y^{11} - x^{11})} \] ... (2.18b)

such that \( 5 + \alpha \leq x \leq 8 - \alpha \) and \( 10 + \alpha \leq y \leq 13 - \alpha \)

Owing to the complicated form of the objective function, it is impossible to represent the optimal solution \( w_{S_a}^L \) and \( w_{S_a}^U \) in terms of \( \alpha \). Consequently a closed form membership function for \( \mu_{w_s} \) cannot be obtained. However, as the number of \( \alpha \) values considered increases, the shape of \( \mu_{w_s} \) can be seen. Here we enumerate 51 values of \( \alpha = 0, 0.02, 0.04, 0.06, ..., 1.00 \).

Fig.2.5 depicts the rough shape of \( \mu_{w_s} \) constructed from these 51 \( \alpha \) values. The rough shape turns out rather fine and looks like a continuous function.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_a^L )</th>
<th>( x_a^U )</th>
<th>( y_a^L )</th>
<th>( y_a^U )</th>
<th>( w_S^L )</th>
<th>( w_S^U )</th>
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<td>8.00</td>
<td>12.00</td>
<td>0.1250</td>
<td>0.4747</td>
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<tr>
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<td>5.50</td>
<td>9.00</td>
<td>11.00</td>
<td>0.1535</td>
<td>0.2768</td>
</tr>
</tbody>
</table>

Table – 2.5

\( \alpha \)-cuts at 11 distinct \( \alpha \) values: 0, 0.1, 0.2, ..., 1.0
The $\alpha$-cuts of the arrival rates, service rates and waiting time in the system at 11 distinct $\alpha$ values for the problem in example 2.

![Fig. 2.5 b](image)

**Fig. 2.5 b**  
Membership function for $\mu_{w_s}$ in Example 2

2.9. Discussion

When the arrival rate (or) service rate are imprecise, the performance measures are also imprecise. By applying the $\alpha$-cut approach, the range of a performance measure at different possibility levels can be derived. Consider the waiting time in system in Example 1. At one extreme end for possibility level $\alpha = 1$, the range of waiting time in the system is approximately $[0.0953, 0.1244]$ refer to (2.17), indicating that it is definitely possible that the expected waiting time in the system falls between $[0.0953, 0.1244]$, although it is imprecise. At another end for possibility level $\alpha = 0$, the range of the waiting time in the system is approximately $[0.0873, 0.1327]$ refer to (2.17). This range indicates that the expected waiting time in the system will never exceed 0.1327 and never fall below 0.0873. For the fuzzy expected waiting time in the
system \( \mu_{w_s} \) in example 2, the range of waiting time in the system at \( \alpha = 1 \) is [0.1535, 0.2768] (refer to Table 2.5), indicating that the waiting time in the system falls between 0.1535 and 0.2768. Moreover, the range of waiting time in the system at \( \alpha = 0 \) is [0.1250, 0.4747] (refer to Table 2.5), indicating that the waiting time in the system will never exceed 0.4747 (or) never fall below 0.1250. The above information will be definitely useful for designing queuing system and also helpful for system designers and practitioners.