SYNTHESIS OF PULSE TRAINS FROM SPECIFIED AUTOCORRELATION FUNCTIONS
3.1 INTRODUCTION

The problem of synthesizing signals which realize the desired auto-correlation function (ACF) can be divided into a number of inter-related problems, each of which has an independent practical significance The separate problems could be formulated as follows, [42]

1 Determine the class of functions which are realizable ACF's for arbitrary signals

2 Determine the subclass of ACF's for various signal structures such as discrete coded waveforms

3 Synthesis of signals whose ACF is a close approximation to the desired ACF not belonging to the class of realizable functions

4 Synthesis of signals (pulse trains) which satisfy a given set of requirements, e.g. range resolution, energy utilization, etc

So far, a comprehensive analytical treatment of these problems and their solution has not been formulated For example, a simple criterion for determining the realizability of an ACF has not yet been found

To appreciate the nature of the problem arising here consider the ACF

$$r(t) = \sum_{n=-\infty}^{\infty} S(nT + \tau)S^*(nT)$$  \hspace{1cm} (3.1)

or its equivalent expression

$$r(\tau) = \int_{-\frac{W}{2}}^{\frac{W}{2}} |S(f)|^2 \exp(j2\pi f \tau) df$$  \hspace{1cm} (3.2)

Where $|S(f)|^2$ is the power spectrum of the signal $S(nT)$ and is assumed to be band limited In other words $S(f)$ is zero outside some range $(-w/2, w/2)$ strictly speaking the requirement for a finite band width $W$ is incompatible with a finite duration signal However, an approximation to the finite spectrum condition can be reached if a major portion of the signal energy is concentrated within a specified frequency band
From Eq (3.2) it can be seen that the ACF and the power spectrum form a Fourier transform pair. Hence it follows that for \( r'(\tau) \) to be a realizable ACF its spectrum \( R(f) \), must be real and non-negative. Even if the given ACF is realizable the synthesis problem can not be solved uniquely. Since the phase information is lost in the power spectrum, it is not possible to determine \( S(f) \) itself which is necessary to find \( S(nT) \). Therefore, all signals whose spectra differ only in phase will have the same ACF. Thus the synthesis problem may be divided into the following two steps:

1. The power spectrum \( |S(f)|^2 \) is determined from the given ACF. (It is assumed that the ACF is realizable, i.e., \( R(0) = \text{FT}\{r(\tau)\} > 0 \).

2. From the determined power spectrum one signal having such a spectrum is derived by assigning an arbitrary phase function \( \theta(f) \), i.e.,

\[
S(nT) = \text{IFT}\{|S(f)|\exp(j\theta(f))\}
\]

For digital application, however, only finite length sequence can be processed. The next section will, therefore, be devoted to the problem of factorising the power spectrum using ZT techniques.

### 3.2 SYNTHESIS OF PULSE TRAINS IF THE ACF IS KNOWN IN MAGNITUDE AND PHASE

If the ACF is given in phase and magnitude at discrete points its ZT can be written as Eq (2.36)

\[
R(Z) = Z^NS(Z)S^*(1/Z)
\]

As mentioned previously the ZT provides a convenient method to represent a signal in the form of its zero pattern which is obtained by factorizing its polynomial in \( Z \). In factorized form the equivalent representation of a polynomial \( S(z) \) of order \( N \) in power of \( Z^{-1} \) can be written as

\[
S(Z) = S(0)\prod_{i=1}^{N} (1 - \frac{Z_i}{Z})
\]  

(3.3)
Fig. 3.1  (a) Factorized ACF of 11 element Barker Code
(b) Resulting sequence when choosing the zero patterns 1, 2, 3, ..., 10.
Where \( Z_i \) are the zeros of \( S(Z) \), i.e., \( S(Z_i) = 0, i = 1, 2, \ldots, N \). Similarly, \( S^*(1/Z) \) can be represented as

\[
S^*(1/Z) = S^*(0) \prod_{i=1}^{N} (1 - Z_i Z_i^*)
\] (3.4)

\[
S^*(1/Z) = S(0)(-1)^N \left( \prod_{i=1}^{N} Z_i^* \right) \prod_{i=1}^{N} (Z - \frac{1}{Z_i})
\]

\[
S^*(1/Z) = S^*(N) \prod_{i=1}^{N} (Z - \frac{1}{Z_i})
\]

\[
R(Z) = S(0).S^*(N) \prod_{i=1}^{N} (1 - \frac{Z}{Z_i})(Z - \frac{1}{Z_i})
\]

Where the unessential delay factor \( Z^N \) has been neglected. Since \( R(Z) \) essentially represents a power spectrum, the above equation can be regarded as the factorized power spectrum, [43].

The equivalent expressions above allow the study of pulse trains using their zero patterns in the complex \( Z \)-plane. The conditions \( S(0) \neq 0 \), and \( S(N) \neq 0 \), are clearly equivalent to

\[
S(0) \neq 0 \text{ and } S^*(0) \neq 0
\]

It is easy to verify that if \( S^0(Z) \) denotes the polynomial

\[
S^0(Z) = Z^N S^*(1/Z)
\]

then

\[
(S^0(Z))^o = S(Z)
\] (3.5)

and

\[
(S(Z) P(Z))^o = S^0(Z) P^0(Z)
\] (3.6)

In addition, it follows from Eq (3.3) and Eq (3.4) that for

\[
S(Z) = |S^*(1/Z)|
\] (3.7)

Moreover, if a polynomial \( S(Z) \) of degree \( N \) has \( P_1 \) zeros inside the unit circle, \(|Z| = 1 \) (counting multiples), \( P_2 \) on the unit circle, and \( P_3 \) zeros outside, where \( P_1 + P_2 + P_3 = N \), it is referred to as of the type \( (P_1, P_2, P_3) \). Since it has been assumed
that $S(0) \neq 0$, it is clear that $Z_j$ is a zero of $S(Z)$ if $1/Z_j^*$ is a zero of $S^*(1/Z)$. The zeros $Z_j$ and $1/Z_j^*$ have the same angle in the $Z$-plane but reciprocal magnitude as indicated in Fig. 3.1. It is clear that $S(Z)$ is of the type $(P_1, P_2, P_3)$ if $S^*(1/z)$ is of the type $(P_1, P_2, P_3)$.

The class of polynomials, $P$, for which $P(Z)$ and $P^*(1/Z)$ have the same set of zeros are known as self-inversive polynomials, [46]. It is apparent that a polynomial $P(Z)$ is self-inversive if its zeros are symmetric with respect to inverse on the unit circle from the foregoing it should be clear that:

1. A self-inversive polynomial of degree $N$ is of type $(P, N-2P, P)$ for $P \geq 0$.

2. Since the polynomial $R(z)$ consists of the product of the two factors $S(z)$ and $S^*(1/Z)$ it is of type $(P_1+P_3, 2P_2, P_1+P_3)$, where $2(P_1+P_2+P_3) = 2N$. Consequently, $R(Z)$ is self-inversive and its zeros must occur in reciprocal conjugate pairs.

Thus for a finite pulse train to be an ACF it has to satisfy condition (ii). The design technique for pulse trains from a given realizable ACF can now be summarized as follows:

1. Factorization of the ZT polynomial which represents the ACF
2. Selection of a suitable zero pattern to obtain the signal after multiplication

The design procedure is probably best illustrated by an example. Consider the 11-element Barker Code where ACF has the ZT representation


This polynomial can now be factorized on a digital computer using a standard root-finding algorithm. The resulting zeros are given in Fig. 3.1(a). All the zeros occur in reciprocal conjugate pairs. In addition, as a consequence of the coefficients of $R(z)$ being real, all complex zeros must occur in conjugate pairs. The selection of zeros for $S(z)$ and multiplying them out completes the design procedure. The resulting sequence choosing the zeros labeled as 1, 2, 3, , 10 is shown in Fig. 3.1(b).
3.3 SYNTHESIS OF PULSE TRAINS IF ONLY THE MAGNITUDE OF THE ACF IS KNOWN.

In the previous section the magnitude and phase of the ACF at discrete points was required to find a solution to the synthesis problem. However, from practical considerations only the magnitude of the function is usually known, since the phase does not affect the accuracy and resolution of the range measurements. In this section the design procedure is extended to the case where only the magnitude of a realizable ACF is given at discrete points. The basic underlying idea of the method presented here is due to Vakman and is also implied by Voclicker, in a different context.

Before proceeding any further it is necessary to recall the convolution theorem derived from basic Fourier transform theory:

\[ r(t) * r*(-t) \leftrightarrow |R(f)|^2 \quad (3.8) \]

\[ |r(t)|^2 \leftrightarrow R(f)^* R*(-f) \quad (3.9) \]

The above relation show the duality between the ACF of the spectrum \( R(f) \) and the ACF of the time signal \( r(t) \). As \( r(f) \) is assumed to be band-limited it can be represented by its Nyquist samples. However, due to the convolution process in the frequency domain the squared envelope, \( |r(t)|^2 \), will have twice the band-width of \( r(t) \), [29]. In other words, if \( |r(t)|^2 \) is sampled at Nyquist rate, \( r(t) \) is sampled at twice that rate. Hence it is assumed that the squared envelope of the ACF is known at integer multiples of \( T/2 = T' \).

Since \( |r(t)|^2 \) is of finite duration it is completely defined by frequency domain samples taken at intervals \( 1/T' \). Such a signal has a finite Fourier representation of \( N \) terms:

\[ m(t) = |r(t)|^2 = \sum_{k=0}^{N-1} c(k) \exp(j2\pi k \frac{t}{T'}) \quad (3.10) \]

where \( T_s \) is the duration of the signal and \( N = WT_s \).

The samples of the envelope are thus given by...
\[ m(\frac{z}{\tau}) = \sum_{k=0}^{N-1} c(k) \exp(j2\pi k \frac{n}{N}) \] ................................. (3.11)

\[ n = 0, 1, 2, ..., (N-1) \]

By substituting the symbol \( Z \) for \( \exp(j2\pi n/T) \) the finite Fourier series can be rewritten as a polynomial in powers of \( Z \)

\[ m(Z) = C(0) + C(1)Z + ... + C(N-1)Z^{N-1} \]

\[ m(Z) = \sum_{k=0}^{N-1} c(k)Z^k \] ................................. (3.12)

This transformation can be regarded as the dual of the ZT discussed in Chapter 2.

Consider now the polynomial of order \( L \) representing the ACF, \( r(t) \).

\[ r(Z) = R(0) + R(1)Z + ... + R(L)Z^L \] ................................. (3.13)

clearly, for \( r(Z) \) to be realizable all \( R(n) \) must be real and non-negative (\( R(n) \geq 0 \)), since the coefficients of the polynomial are the power spectral samples. The squared modulus of \( r(t) \), \( |r(t)|^2 \) can thus be represented as a polynomial multiplication.

\[ m(Z) = [R(0) + R(1)Z + ... + R(L)Z^L][R^*(0) + R^*(1)Z + ... + R^*(L)Z^L] \]

\[ m(Z) = Z^L [R(0) + R(1)Z + ... + R(L)Z^L] [ R^*(0) + R^*(1)Z + ... + R^*(L)Z^L] \]

\[ m(Z) = Z^L r(Z) r^*(1/Z) \]

Where the coefficients of \( m(Z) \) are given by

\[ C(k) = \sum_{n=0}^{L-|k|} R(n)R(n+k) \]

\[ k = 0, \pm1, ..., \pm L \]

Since \( r(z) \) is a polynomial with real coefficients

\[ m(Z) = Z^L r(Z) r(1/Z) \]

Thus the operation of convolution in the frequency domain reduces to a multiplication of two polynomials. This clearly reflects the duality of time and frequency as pointed out earlier.
Since \( m(z) \) has the form of an ACF it is possible to proceed in a similar manner to that described in section 3.1 in order to find the power spectral components \( R(n) \) given \( m(z) \). The properties of \( m(z) \) are revealed by studying the zeros in the complex \( Z \)-plane. The coefficients of \( m(z) \) specify the ACF of the spectrum of \( r(z) \). Its \( 2^L \) zeros must therefore occur in reciprocal conjugate pairs. In addition, since all coefficients are real (and in particular non-negative) they all occur in complex conjugate pairs. Thus if \( Z_j \) is a zero of \( m(Z) \), then \( Z_j^* \), \( 1/Z_j \) and \( 1/Z_j^* \) also must be zeros of \( m(z) \). This relationship is illustrated in Fig. 3.1(a). Consequently, if \( m(z) \) is to represent a realizable power spectrum ACF the following conditions must be satisfied.

i) \( m(Z) \) is finite and its zeros occur in complex conjugate reciprocal pairs.

ii) The coefficients, \( R(n) \) of \( r(Z) \) must be real and non-negative.

If these conditions are met then at least one and in general a whole set of ACF's having the same magnitude can be found. The steps in the design procedure can be summarized and best illustrated by using the 7-element Barker code as an example.

1. Given the samples of \( r(t) \), where \( r(t) \) is assumed to be band limited and sampled at twice the Nyquist rate, Fig. 3.2(a), one computes the DFT of the sequence \( |r(KT)|^2 \), i.e.

\[
\text{DFT}[|r(0)|^2, |r(T)|^2, ..., |r(13T)|^2, ..., |r(T')|^2]
\]

This gives, except for a scale factor, the \( 2L+1 \) Fourier coefficients of the periodically repeated square envelope of the ACF, Fig. 3.2(b).

2. Factorize the polynomial whose coefficients are these Fourier components. The \( 2^L \) roots should occur in reciprocal complex conjugate pairs, Fig. 3.2(c)

3. Select \( L \) roots from each reciprocal conjugate pair and its conjugate. Now, multiply out to obtain a set of \( L+1 \) Fourier coefficients which are, neglecting a scale factor, the DFT of the samples of \( r(t) \). Verify that \( r(t) \) is indeed an ACF. This is done simply by making sure that the coefficients obtained are all real and
Fig. 3.2  
(a) Samples of the ACF magnitude
(b) Spectrum samples of squared envelope of (a)
(c) Zero pattern of (b)
(d) Spectrum samples of the ACF
(e) ACF in magnitude and phase.
non-negative. If the test fails, select a new zero pattern and repeat the procedure from step 3, until a realizable ACF is obtained, Fig. 3.2(d). From the \( L \) roots only \( L/2 \) can be chosen independently, since the zero must be selected in complex conjugate pairs. Hence, there are in general \( 2^{L/2} \) possible zero patterns. However, not all zero combinations will result in realizable ACF's. The zeros chosen in this case are labelled 1, 2, ..., 13, Fig. 3.2(c).

4. The final stage of the synthesis procedure is to take the IDF of the Fourier Transform coefficients to obtain the sampled values of \( r(t) \) Fig. 3.2(e). Once the ACF has been found in magnitude and phase, it is then straightforward to synthesize a pulse train which realizes this ACF by following the procedure outlined in section 3.2.

### 3.4 DISCRETE PHASE APPROXIMATION TO LINEAR FM SIGNALS

The Linear FM (LFM) or (Chirp) waveform is probably the principal type of signal transmitted by a radar or sonar system. The main advantage in using such waveform lies in their ease of generation and insensitivity to small doppler shifts. The description and properties of analogue chirp techniques are well documented in the literature and will not be repeated here, [22], [25], [41].

In general the complex envelope of a LFM signal can be expressed as

\[
S(t) = \exp (j\mu t^2/2) \tag{3.15}
\]

where

\[
\mu = 2\pi w/T_s
\]

\[W = f_2 - f_1 \] = Frequency Change During Sweep

\[T_s \] = time duration of the sweep.
As implied by the term LFM, the instantaneous frequency is swept linearly from $f_1$ at $t=0$, to a maximum value of $f_2$ at $t=T$. The complex envelope can be written in terms of the pulse compression ratio (time bandwidth product) denoted as $m_c$

$$S(t) = \exp[j\pi(wt)^2/m_c]$$

If the waveform is sampled at uniform time interval of $T$ seconds

$$S(nT) = \exp[j\pi(wnT)^2/m_c]$$

and with

$$m_c = NTW$$

$$S(nT) = \exp(j\pi WTN^2/N)$$

$$n = 0, 1, 2, \ldots, (N-1)$$

The total phase change over the signal duration is

$$\phi = \pi W T (N-1)^2/N$$

Furthermore, if $T$ is set equal to the Nyquist rate, then

$$T = 1/W$$

and

$$S(nT) = \exp(j\pi n^2/N)$$

(3 17)

Since each segment is coded into one of $N$ possible phases these sequences are some time referred to as polyphase codes. In particular, the sequences whose phase follows a quadratic progression will be called Quadratic Phase (QP) codes. An interesting property of QP codes is their periodic ACF which is zero for all non-zero time lags, [50], provided the sequence is coded as in eqn (3 17) if $N$ is even and is modified to

$$S(nT) = \exp(j\pi n(N+1)/N)$$

(3 18)

for $N$ odd
Higher order poly-phase codes with zero circular ACF \( k \neq 0 \) have been described by Frank Zadoir and Heimiller, [16], [17] & [18]. The length \( N \) of such codes, however, is restricted to perfect squares. While the QP sequences and Frank codes have ideal cyclic auto-correlation, they do not have of course perfect a periodic auto-correlation.

Another property of practical importance is the simple generation of QP pulse trains if the sequence length is chosen properly. This can be demonstrated by expanding the expression \( n^2/N \) as

\[
n^2/N = q + q_1 + \alpha_n/N
\]

where \( q = 0, 1, 2, \ldots \)

\( q_1 = 0 \) or \( 1 \)

\( \alpha_n = \text{remainder} \)

\[
\exp(jqn) = \begin{cases} 
-1; & \text{q is odd} \\
1; & \text{q is even}
\end{cases}
\]

and

\[
\exp(j\pi/2) = j
\]

The QP code can be also written as

\[
S(nT) = (\pm 1)(1/j)\exp(j\pi\alpha_n/N) \tag{3.19}
\]

The number of different samples to be generated is thus a function of the number of distinct remainders of \( n^2/N \). Roy and Lowenschurrs, [51] have shown that for a proper choice of \( N \) the number of different samples can be kept very small indeed. For example for \( N = 16 \) only three different values must be generated, \( \exp(j\pi/16) \), \( \exp(j\pi/4) \) and 1. Incidentally, this property has also been exploited in the Blusteen algorithm which computes the DFT using a Chirp Filter, [52].
3.4.1 Properties of the Compressed Pulse Train

The exact expression for the ACF can be obtained by substituting eqn (3.16) into Eqn (3.1)

\[ r(kT) = \sum_{n=0}^{N-1-k} \exp(j\pi \omega \frac{kT}{N} \cdot [n^2 - (n + k)^2] \]

\[ = \exp(j\pi \omega T \frac{k^2}{N} \sum_{n=0}^{N-1-K} \exp(-j2\pi \omega \frac{kT}{N}) \]

\[ k = 0, 1, 2, ..., (N-1) \]

The summation in the last expression is of the form of a geometric progression and can be written in closed form. By rewriting the sum term as \( \sum_{n=0}^{N-1-k} r^n \), Where \( r = \exp(-j2\pi \omega TK/N) \) it is recognised that the series containing a total of (N-K) terms has a sum of, \( S = \frac{r^{N-k} - 1}{r - 1} \).

Therefore,

\[ r(kT) = \exp(-j\pi \omega k \frac{T}{N}) \frac{\exp[-j2\pi \omega TK(N - k)] - 1}{\exp(-j2\pi \omega T \frac{k}{N}) - 1} \]

\[ r(kT) = \exp[-j\pi \omega k T \frac{(N-1)}{N}] \frac{\sin[\pi \omega T \frac{(N-k)}{N}]}{\sin(\pi \omega T \frac{k}{N})} \]

Since only the magnitude of the ACF is of interest

\[ |r(kT)| = \frac{\left| \sin[\pi \omega Tk \frac{(N-k)}{N}] \right|}{\left| \sin(\pi \omega T \frac{k}{N}) \right|} \]

If \( T \) is equal to the Nyquist sampling rate, i.e. \( WT = 1 \), Eqn (4.16) becomes

\[ |r(kT)| = \frac{\left| \sin[\pi k \frac{1-N}{N}] \right|}{\left| \sin(\pi k \frac{1}{N}) \right|} \]

Because \( r(KT) \) exhibits complex conjugate symmetry with respect to \( k = 0 \), it is sufficient to consider only positive time lags. The nature of the function (3.21) in the vicinity of \( k = 0 \) has the form of a sine function with a peak value of \( N \). Because of the periodicity of the expression this characteristic will be repeated at intervals \( 1/N \). For even length sequences the function is symmetrical with respect to \( N/2 \). The effect of the term \( (1-K/N) \) can be explained, considering that
\[ |\sin[\pi K(1-K/N)]| = |\sin(\pi K^2/N)| \]

Thus it modulates the frequency of the ripples in a 'chirp like' fashion. Whenever a sequence is used as a modulating function it is always of interest to consider its spectrum. Since the power spectrum contain the relevant information, it is sufficient to consider the amplitude spectrum only. For convenience the spectrum is assumed to be zero outside some range \((0,W)\). This is no restriction, since any band limited signal can be brought into this form by a suitable frequency translation.

The magnitude of the ACF is shown in fig. 3.3 for a QP sequence of length \(N = 128\) when sampled at the Nyquist rate, \(WT = 1\). The characteristics of the QP pulse trains when sampled at a lower or faster rate than the Nyquist rate (under or over sampling) are depicted in Fig. 3.3 & 3.4. These graphs reveal some interesting properties. First, if sampled at the Nyquist rate \((WT = 1)\), the ACF consists of a sharp narrow spike with low residue side lobes. Secondly if \(WT < 1\), over sampling occurs and side lobes near the main peak appear. This is not surprising since increased sampling rate implies a closer approximation to the analogue FM signal whose maximum side-lobes are immediately adjacent to the main lobe, as shown in Fig. 3.4. In other words, the sampling points do not miss these large side-lobes as is the case for \(WT = 1\). Thirdly, for under-sampling \(WT > 1\), the aliased versions of the ACF will produce significant range ambiguity at time shifts \(K = N/WT\) from the compressed pulse i.e. for \(WT = 2\), \(K = N/2\) as illustrated in Fig. 3.7. The cause of the spurious response peaks can be avoided provided the waveform is sampled at the Nyquist rate. The resulting sequences have low side-lobes and low side-lobe energy and thus are suitable in a multiple-target environment. However, there may be specific cases where \(WT\) is made slightly greater than one, to attain somewhat better range resolution. By comparison of Fig. 3.6 with 3.3, it can be seen that a small increase in \(WT\) tends to reduce the side-lobes close to the main
lobe at the expense of an increase towards the end of the response. The important properties of QP pulse trains where \( WT = 1 \) may now be summarized as follows

i) If sampled at the Nyquist rate QP codes have virtually all the properties of LFM signals. Their ACF consists of a single spike with low level side-lobes.

ii) The codes have zero periodic ACF's for \( K \neq 0 \).

iii) For \( WT < 1 \), large close-in side-lobe appears, whereas for \( WT > 1 \) spurious response peaks occur further away from the main-lobe.

iv) The major side-lobes occur in two bands approximately centered at time shifts \( K = \sqrt{(N/2)} \) and \( K = [N - \sqrt{(N/2)}] \) respectively. The width of the bands is about \( 2[\sqrt{N} - \sqrt{(N/2)}] \). Moreover, for sequences of even length \( N \), the side-lobe structure is symmetrical with respect to \( K = N/2 \).

v) The maximum side-lobes increase approximately as \( 0.5\sqrt{N} \) and the energy ratio \( E_{r} < 5\% \) for \( N > 40 \).

vi) Relative simple generation of such pulse trains with a suitable choice of \( N \).

Hence QP sequences have good range resolution properties which make them suitable for a dense-target environment. For example, the peak range side-lobe for \( N = 128 \) is -27.5 dB down on the main response and the rms side-lobes are about -36.6 dB down.

In subsequent chapters an essentially different method of synthesizing discrete coded signals with desired auto-correlation properties is described. The method is based on numerical optimization techniques. Such an approach has a number of advantages. First, no restriction on the class of admissible phase functions is necessary. Secondly, no information of the signal's phase structure is usually required. Thirdly, these methods are flexible in a sense that it is possible to control particular side-lobes or side-lobe regions.
Fig. 3.3—(a) ACF, (b) amplitude spectrum of 128-element QP code for $WT = 0.5$.

Note: $\text{ACF} = \text{autocorrelation function}$. 
Fig. 3.4 ACF of a LFM signal.
Fig. 3.6 Effects of slight under-sampling on the
(a) ACF, and (b) the amplitude spectrum.
Fig. 3.7 Aliasing effects on the ACF (a), and amplitude spectrum (b), due to