4.1. Introduction

Banach core of a sequence, analogous to the Knopp core, is inherently connected to the concept of Banach limit. The Banach core (or $B$-core) of a real bounded sequence $x$ is defined to be the closed interval $[-q(-x), q(x)]$, where

$$q(x) = \limsup_p \sup_n t_{pn}(x)$$

is a sublinear functional on $\ell_\infty$.

Like Shcherbakov [52], it is natural to extend this definition for $B$-core, i.e. for every complex bounded sequence $x$

$$B - \text{core}\{x\} = \bigcap_{z \in \mathbb{C}} B^*_x(z),$$

where

$$B^*_x(z) := \{w \in \mathbb{C} : |w - z| \leq \limsup_p \sup_n |t_{pn}(x) - z|\}.$$ 

Note that $q(x) \leq L(x)$ for all $x \in \ell_\infty$, where $L(x) = \limsup x$. Hence it follows that

$$B - \text{core}\{x\} \subseteq K - \text{core}\{x\}.$$ 

From Section 1.6, we note that

$$B - \text{core}\{x\} \subseteq st - \text{core}\{x\}, \text{ for all } x \in \ell_\infty.$$
In this chapter we determine necessary and sufficient conditions for the inclusions

\[ B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}, \]

and

\[ st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\}, \]

further, we extend these results to the following inclusions

\[ B - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\}, \]

and

\[ st - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\}, \]

where \( T \) is a normal matrix, and also we prove some core theorems analogous to Orhan [44].

4.2. Some Previous Results

Analogue and extensions of Knopp core theorem have been established by various authors (cf. Choudhary [4], Das [10], Maddox [38], Mursaleen [41]). In [44] Orhan has proved the following:

Theorem 4.2.1. \( B - \text{core}\{Ax\} \subseteq K - \text{core}\{x\} \) if and only if \( A \) is almost regular and

\[ (4.2.1.1) \lim \sup_p \sup_n \sum_k | \frac{1}{p+1} \sum_{i=0}^p a_{n+i,k} | = 1. \]

Theorem 4.2.2. \( B - \text{core}\{Ax\} \subseteq B - \text{core}\{x\} \) if and only if \( A \) is \( f \)-regular and (4.2.1.1) holds.
4.3. Main Results

**Theorem 4.3.1.** If \( \|A\| < \infty \), then for every \( x \in \ell_\infty \)

\[
(4.3.1.1) \quad B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}
\]

if and only if

\[
(4.3.1.2) \quad A \text{ is almost regular, and}
\]

\[
\lim \sum_{k \in E} | t(n, k, p) | = 0 \text{ uniformly in } n, \text{ whenever } \delta(E) = 0 \text{ for } E \subseteq N;
\]

\[
(4.3.1.3) \quad \limsup_p \sup_n \sum_k | t(n, k, p) | = 1.
\]

**Proof.** **Necessity.** Let \( B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} \) and \( x \) be statistically convergent to the number \( \ell \). Then

\[
\{\ell\} = st - \text{core}\{x\} \supseteq B - \text{core}\{Ax\},
\]

i.e.

\[
q(Ax) \leq st - \lim x = \ell.
\]

Since \( \|A\| < \infty \) if and only if \( A \in (\ell_\infty, \ell_\infty) \) by Lemma 2.1.3, i.e. \( Ax \in \ell_\infty \) for \( x \in \ell_\infty \). Hence

\[
st - \liminf x \leq -q(-Ax) \leq q(Ax) \leq st - \limsup x.
\]

But \( st - \liminf x = st - \limsup x = \ell \). So that

\[
q(Ax) = -q(-Ax) = \ell,
\]

i.e. \( B - \text{core}\{Ax\} = \{\ell\} \). This implies that \( A \in (st \cap \ell_\infty, f)_{\text{reg}} \) and hence by Theorem 2.2.3 we have that \( A \) is almost regular and

\[
\lim \sum_{k \in E} | t(n, k, p) | = 0 \text{ uniformly in } n, \text{ whenever } \delta(E) = 0,
\]

i.e. condition (4.3.1.2).

Also, we know that \( st - \text{core}\{x\} \subseteq K - \text{core}\{x\} \). Hence

\[
B - \text{core}\{Ax\} \subseteq K - \text{core}\{x\}
\]
and Theorem 4.2.1 gives that

$$\limsup \sup_{n} \sum_{k} |t(n, k, p)| = 1,$$

i.e. condition (4.3.1.3).

**Sufficiency.** Let conditions (4.3.1.2) and (4.3.1.3) hold and that $w \in B - \text{core}\{Ax\}$. Then for any $z \in \mathcal{C}$ we have

$$|w - z| \leq \limsup_{n} \sup_{p} |z - t_{n}(Ax)|$$

$$\leq \limsup_{n} \sup_{p} |z - \sum_{k} t(n, k, p)x_k|$$

$$\leq \limsup_{n} \sup_{p} |\sum_{k} t(n, k, p)(z - x_k)|$$

$$+ \limsup_{n} \sup_{p} |z| |1 - \sum_{k} t(n, k, p)|$$

$$= \limsup_{n} \sup_{p} |\sum_{k} t(n, k, p)(z - x_k)|,$$ by (4.3.1.3).

Hence

(4.3.1.4)

$$|w - z| \leq \limsup_{n} \sup_{p} |\sum_{k} t(n, k, p)(z - x_k)| .$$

Let $r = st - \limsup_{k} |z - x_k|$ and $E := \{k : |z - x_k| > r + \varepsilon\}$ for $\varepsilon > 0$. Then $\delta(E) = 0$ and we have

(4.3.1.5)

$$|\sum_{k} t(n, k, p)(z - x_k)| \leq \sup_{k} |z - x_k| \sum_{k \in E} |t(n, k, p)|$$

$$+ (r + \varepsilon) \sum_{k \notin E} |t(n, k, p)| .$$
Therefore by conditions (4.3.1.2) and (4.3.1.3), we get

\[ \limsup_{n} \sup_{k} \sum_{p} t(n, k, p) (z - x_k) \leq r + \varepsilon. \]

Hence by (4.3.1.4) we have

\[ |w - z| \leq r + \varepsilon \]

and since \( \varepsilon \) is arbitrary

\[ |w - z| \leq r = \limsup_{k} |z - x_k|, \]

i.e. \( w \in S^*_k(z) \). Hence \( w \in st - \text{core}\{x\} \), so that

\[ B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}. \]

This completes the proof of the theorem.

Our next result is an analogue of Theorem 3.2.3 as well as Theorem 3.2.4, which is a slight generalization of the previous result.

**Theorem 4.3.2.** Let \( T = (t_{jk}) \) be a normal matrix. Let \( A = (a_{nj}) \) be any matrix. In order that whenever \( Tx \) is bounded \( Ax \) should exist and be bounded and satisfy

\[ B - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\} \]

it is necessary and sufficient that

\( (c_{nk}) = C = AT^{-1} \) exists;

\( C \) is almost regular, and

\[ \lim \sum_{p} k \in E |b(n, k, p)| = 0 \] uniformly in \( n \), whenever \( \delta(E) = 0 \) for \( E \subseteq \mathcal{N}; \)

\( \limsup_{p} \sup_{n} \sum_{k} |b(n, k, p)| = 1, \) where

\[ b(n, k, p) = \frac{1}{p + 1} \sum_{i=0}^{p} c_{n+i,k} \]
(4.3.2.5) for any fixed $n$,

$$
\sum_{k=0}^{m} \sum_{j=m+1}^{\infty} a_{nj}t_{jk}^{-1} \to 0 \text{ as } m \to \infty.
$$

**Proof.** **Necessity.** Let (4.3.2.1) hold and $A_n(x)$ exist for every $n$ whenever $Tx \in \ell_\infty$. Then by Lemma 2.1.5 it follows that conditions (4.3.2.2) and (4.3.2.5) hold. Further by the same Lemma, we obtain $Ax = Cy$, where $y = Tx$. Since $Ax \in \ell_\infty$, we have $Cy \in \ell_\infty$. Therefore (4.3.2.1) implies that

$$
 B - \text{core}\{Cy\} \subseteq st - \text{core}\{y\}.
$$

Hence using Theorem 4.3.1, we see that conditions (4.3.2.2) and (4.3.2.4) hold.

**Sufficiency.** Let the conditions (4.3.2.2)-(4.3.2.5) hold. Then obviously the conditions of Lemma 2.1.5 are satisfied and so $Cy \in \ell_\infty$, hence $Ax \in \ell_\infty$. Now using Theorem 4.3.1, we obtain

$$
 B - \text{core}\{Cy\} \subseteq st - \text{core}\{y\}
$$

and consequently

$$
 B - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\},
$$

since $y = Tx$ and $Cy = Ax$.

This completes the proof of the theorem.

We can also easily prove the following results analogous to Theorems 4.2.1 and 4.2.2.

**Theorem 4.3.3.** Let $A$ and $T$ be same as in Theorem 4.3.2. Then

(4.3.3.1) \hspace{1em} $B - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}$

if and only if (4.3.2.2) and (4.3.2.5) hold

(4.3.3.2) $C$ is almost regular;
(4.3.3.3) \( \limsup_p \sup_n \sum_k \left| \frac{1}{p+1} \sum_{i=0}^p c_{n+i,k} \right| = 1. \)

**Theorem 4.3.4.** Let \( A \) and \( T \) be same as in Theorem 4.3.2. Then

(4.3.4.1) \( B - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\} \)

if and only if (4.3.2.2), (4.3.2.5) and (4.3.3.3) hold

(4.3.4.2) \( C \) is \( F \)-regular.

**Remark 4.3.5.** If we take \( T = I \), the unit matrix, then Theorem 4.3.2 reduces to Theorem 4.3.1 and Theorem 4.3.3 and 4.3.4 reduce to that of Orhan [44].

**Theorem 4.3.6.** If \( \|A\| < \infty \), then for every \( x \in \ell_\infty \)

(4.3.6.1) \( st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\} \)

if and only if

(4.3.6.2) \( A \in (f, st \cap \ell_\infty)_{\text{reg}} \)

(4.3.6.3) \( st - \lim_n \sum_{k \in E} a_{nk} = 1 \), whenever \( N \setminus E \) is finite for \( E \subseteq N \).

**Proof.** Necessity. Let (4.3.6.1) hold and \( x \) be almost convergent to \( L \). Then

\[ \{L\} = B - \text{core}\{x\} \supseteq st - \text{core}\{Ax\}. \]

Since \( \|A\| < \infty \) implies \( Ax \in \ell_\infty \) for \( x \in \ell_\infty \), we have

\[ -q(-x) \leq st - \liminf Ax \leq st - \limsup Ax \leq q(x). \]

But \(-q(-x) = q(x) = L\), so that

\[ st - \lim Ax = f - \lim x = L. \]

Hence \( A \in (f, st \cap \ell_\infty)_{\text{reg}} \), i.e. condition (4.3.6.2) holds.
To prove (4.3.6.3), let us define \( x = (x_k) \in \ell_\infty \) by

\[
x_k = \begin{cases} 
1, & \text{if } k \in E, \\
0, & \text{otherwise,}
\end{cases}
\]

where \( E \subseteq \mathbb{N} \) such that \( \mathbb{N} \setminus E \) is finite. Then

\[
B - \text{core}\{x\} = \{1\}.
\]

Since \( Ax \in \ell_\infty \), \( Ax \) has at least a statistical cluster point. Therefore by proposition 1.4.2, \( st - \text{core}\{Ax\} \neq \emptyset \). Since

\[
st - \text{core}\{Ax\} \subseteq B - \text{core}\{x\} = \{1\},
\]

we have \( st - \text{core}\{Ax\} = \{1\} \) and 1 is the only statistical cluster point of \( Ax \). Hence

\[
st - \lim Ax = 1,
\]

i.e.

\[
st - \lim_n \sum_{k \in E} a_n x_k = 1, \quad \text{whenever } \mathbb{N} \setminus E \text{ is finite.}
\]

Therefore (4.3.6.3) holds.

**Sufficiency.** Suppose that the conditions (4.3.6.2) and (4.3.6.3) hold and \( w \in st - \text{core}\{Ax\} \). Then for any \( z \in \mathcal{C} \) we have

\[
| w - z | \leq st - \lim_n \sup | z - A_n(x) |
\]

\[
= st - \lim_n \sup | z - \sum_k a_n x_k |
\]

\[
\leq st - \lim_n \sup \left| \sum_k a_n (z - x_k) \right|
\]

\[
+ st - \lim_n \sup | z | \left| 1 - \sum_k a_n \right|
\]

\[
= st - \lim_n \sup \left| \sum_k a_n (z - x_k) \right|, \quad \text{by (4.3.6.3)}.
\]
Therefore for an index set $N = \{n_i\}$ such that $\delta(N) = 1$,

\[(4.3.6.4) \quad |w - z| \leq \limsup_i \sum_k a_{n_i k} (z - x_k) .\]

Now proceeding as in the proof of sufficiency part of Theorem 2.3.4, we obtain

\[(4.3.6.5) \quad \limsup_i \sum_k a_{n_i k} (z - x_k) = \limsup_i \sum_k a_{n_i k} (z - t_{p_n}(x)).\]

Now, let $r = \limsup_p \sup_n |t_{p_n}(x) - z|$ and $E = \{k : |t_{p_n}(x) - z| > r + \varepsilon\}$ for $\varepsilon > 0$. Then $\delta(E) = 0$ as $E$ is finite. Therefore

\[|\sum_k a_{n_i k} (z - x_k)| = |\sum_k a_{n_i k} (z - t_{p_n}(x))|\]

\[\leq \sup_n |z - t_{p_n}(x)| \sum_{k \in E} a_{n_i k} + (r + \varepsilon) |\sum_{k \notin E} a_{n_i k}| .\]

From (4.3.6.2) and (4.3.6.3), we get

\[\limsup_i |\sum_k a_{n_i k} (z - x_k)| \leq r + \varepsilon\]

and so by (4.3.6.4) we have

\[|w - z| \leq r + \varepsilon.\]

Since $\varepsilon$ is arbitrary,

\[|w - z| \leq r = \limsup_p \sup_n |t_{p_n}(x) - z|,\]

i.e. $w \in B^*_n(z)$. Hence $w \in B - \operatorname{core}(x)$, so that

\[st - \operatorname{core}(Ax) \subseteq B - \operatorname{core}(x).\]

**Remark 4.3.7.** In Example 4.4.3, we shall see that the condition (4.3.6.3) cannot be replaced by

\[(4.3.7.1) \quad st - \lim_n \sum_{k \in E} a_{nk} = 1 \quad \text{for any set } E \subseteq \mathcal{N} \text{ such that } \delta(E) = 1.\]
Our next theorem is an analogue of Theorem 3.3.9.

**Theorem 4.3.8.** Let \( T = (t_{jk}) \) be a normal matrix. Let \( A = (a_{nj}) \) be any matrix. In order that whenever \( T x \) is bounded \( A x \) should exist and be bounded and satisfy

\[
\text{st} - \text{core}\{Ax\} \subseteq B - \text{core}\{Tx\}
\]

it is necessary and sufficient that

\[
(c_{nk}) = C = AT^{-1} \text{ exists;}
\]

\[
C \in (f, st \cap \ell_{\infty})_{\text{reg}};
\]

\[
\text{st} - \lim_n \sum_{k \in E} c_{nk} = 1 \text{ whenever } N \setminus E \text{ is finite;}
\]

\[
\text{for any fixed } n, \quad \sum_{k=0}^{m} \left| \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \right| \to 0 \text{ as } m \to \infty.
\]

**Remark 4.3.9.** If \( T = I \), the unit matrix, then Theorem 4.3.8 is reduced to Theorem 4.3.6.

4.4. Examples

**Example 4.4.1.** In support of our main result Theorem 4.3.1, we will show that the condition

\[
\lim_p \sum_{k \in E} | t(n, k, p) | = 0 \text{ uniformly in } n,
\]

of (4.3.1.2) can not be changed whenever \( \delta(E) = 0 \).

Let \( A = (a_{nk}) \) be defined by

\[
a_{nk} = \begin{cases} 
2 & \text{if } n \text{ is even and } k = n^2, \\
0 & \text{otherwise.}
\end{cases}
\]
Then

\[ \sum_{k} a_{nk} = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases} \]

We see that \( A \in (c, f)_{\text{reg}} \). Now, let \( E = \{ k = n^2 : k \in \mathbb{N} \} \). Then \( \delta(E) = 0 \), and

\[ \sum_{k \in E} a_{nk} = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore

\[ \lim_p \sum_{k \in E} |t(n, k, p)| = 1 \text{ uniformly in } n, \]

and

\[ \lim \sup \sup_p \sum_{n} \sum_{k} |t(n, k, p)| = 1. \]

Let \( x = (x_k) \) be defined by

\[ x_k = \begin{cases} 3 & \text{if } k \text{ is not square}, \\ 1 & \text{if } k \text{ is square}. \end{cases} \]

Then

\[ st - \text{core}\{x\} = \{3\} \]

and

\[ \sum_{k \in E} t(n, k, p)x_k = \begin{cases} 2 & \text{if } n \text{ is even}, \\ 0 & \text{otherwise}. \end{cases} \]

Therefore

\[ B - \text{core}\{Ax\} = \{1\}. \]

Hence

\[ B - \text{core}\{Ax\} \not\subseteq st - \text{core}\{x\}. \]
Example 4.4.2. Let \( A = (a_{nk}) \) be defined by

\[
a_{nk} = \begin{cases} 
\frac{2}{n}, & \text{if } n \text{ is even and } k \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

Then

\[
\sum_k a_{nk} = \begin{cases} 
2, & \text{if } n \text{ is even}, \\
0, & \text{otherwise}.
\end{cases}
\]

We see that \( A \in (st \cap \ell_\infty, f)_{\text{reg}} \). Hence \( A \in (c, f)_{\text{reg}} \). Moreover, for any set \( E \subseteq \mathbb{N} \) such that \( \delta(E) = 0 \),

\[
\lim_p \sum_{k \in E} |t(n, k, p)| = 0 \text{ uniformly in } n,
\]

and

\[
\limsup_p \sup_n \sum_k |t(n, k, p)| = 1.
\]

Hence for any bounded sequence, e.g. \( x = (x_k) = (1, 0, 1, 0, \cdots) \), we have

\[
\sum_k t(n, k, p)x_k = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
0, & \text{otherwise}
\end{cases}
\]

and so

\[ B - \text{core}\{Ax\} = \left\{ \frac{1}{2} \right\}. \]

Therefore we have

\[ \left\{ \frac{1}{2} \right\} = B - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} = [0, 1]. \]
Example 4.4.3. Let $A = (a_{nk})$ be defined by

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a nonsquare and } k = n^2 \text{ or } n^2 + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_k a_{nk} = \begin{cases} 1, & \text{if } n \text{ is a nonsquare,} \\ 0, & \text{otherwise,} \end{cases}$$

and $A \in (f, st \cap \ell_\infty)_{\text{reg}}$. Further, for any set $E \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus E$ is finite, we have

$$st - \lim_n \sum_{k \in E} a_{nk} = 1.$$ 

Then for any bounded sequence $x$ we have

$$st - \text{core}(Ax) \subseteq B - \text{core}(x).$$

Now, let $E = \{k \neq n^2 \text{ and } k \neq n^2 + 1 : k \in \mathbb{N}\}$. Then $\delta(E) = 1$ and we have

$$\sum_{k \in E} a_{nk} = 0, \text{ for all } n.$$ 

Hence

$$st - \lim_n \sum_{k \in E} a_{nk} = 0.$$ 

Further, for any bounded sequence, say, $x = (1, 0, 1, 0, \ldots)$, we have

$B - \text{core}(x) = \{\frac{1}{2}\}$ and...
\[ \sum_{k} a_{nk}x_k = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is a nonsquare,} \\ 0, & \text{otherwise.} \end{cases} \]

Therefore, \( st - \lim Ax = \frac{1}{2}, \) i.e.

\[ st - \text{core}(Ax) = \left\{ \frac{1}{2} \right\} = B - \text{core}\{x\}, \]

but (4.3.7.1) does not hold.