CHAPTER III

SOME STATISTICAL CORE THEOREMS

3.1. Introduction

The Knopp core (or K-core) of a real bounded sequence $x$ is defined to be the closed interval $[\ell(x), L(x)]$, where

$$
\ell(x) = \liminf x; \quad L(x) = \limsup x .
$$

The well-known Knopp's core theorem states that (cf. Knopp [28], Maddox [38]):

In order that $L(Ax) \leq L(x)$ for every real bounded sequence $x$, it is necessary and sufficient that $A$ should be regular and $\lim_n \sum_k |a_{nk}| = 1$.

Note that $L(Ax) \leq L(x)$ means $K-core\{Ax\} \subseteq K-core\{x\}$.

Shcherbakov [52] has shown that for every bounded complex sequence $x$,

$$
K - core\{x\} = \bigcap_{z \in \mathbb{C}} K^*_x(z),
$$

where

$$
K^*_x(z) := \{w \in \mathbb{C} : |w - z| \leq \limsup_{k} |x_k - z| \} .
$$

If $x$ is a statistically bounded sequence, then the statistical core of $x$ (cf. Friday and Orhan [22]) is defined to be the closed interval $[st - \liminf x, st - \limsup x]$.

It is noted that

$$
\liminf x \leq st - \liminf x \leq st - \limsup x \leq \limsup x
$$
and consequently
\[ st - \text{core}\{x\} \subseteq K - \text{core}\{x\}. \]

Fridy and Orhan \cite{23} introduced and studied the equivalent form of statistical core and proved that
\[ st - \text{core}\{x\} = \bigcap_{z \in \mathbb{C}} S^*_z(z), \]
where
\[ S^*_z(z) := \{ w \in \mathbb{C} : |w - z| \leq st - \limsup_k |x_k - z| \} \]
for a statistically bounded complex sequence \( x \).

Analogous to the Knopp core theorem, in \cite{23} necessary and sufficient conditions were established for
\[ K - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} \]
for every bounded complex sequence \( x \).

The sufficient conditions were also derived for \( A \) to yield
\[ st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}. \]

Recently Li and Fridy \cite{34} obtained the necessary and sufficient conditions for \( A \) to yield
\[ st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\}, \]
and moreover
\[ st - \text{core}\{Ax\} \subseteq K - \text{core}\{x\} \]
through the concepts of statistical partition and superior partition of \( \mathbb{N} \).

In the present chapter we also consider the same inclusions as in \cite{34} and obtain necessary and sufficient conditions in a more natural way by using the matrix classes involving the space of statistically convergent sequences. Our conditions are stronger than that of Li and Fridy and proofs are easier and shorter.

Also we obtain necessary and sufficient conditions to establish the inclusions
\[ st - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}, \]
(3.2.3.4) \( \lim_{n} \sum_{k=1}^{\infty} |c_{nk}| = 1; \)

(3.2.3.5) for any fixed \( n \),

\[
\lim_{m} \sum_{k=0}^{m} \left| \sum_{j=m+1}^{\infty} n_{jk} t_{j}^{-1} \right| = 0.
\]

**Theorem 3.2.4** (Fridy and Orhan [23]). Let \( T = (t_{nk}) \) be a normal matrix and denote its triangular inverse by \( T^{-1} = (t_{nk}^{-1}) \). For an arbitrary matrix \( A \), in order that, whenever \( Tx \in \ell_{\infty} \), \( Ax \) should exist and be bounded and satisfy

(3.2.4.1) \( K - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\} \),

it is necessary and sufficient that the following conditions hold:

(3.2.4.2) \( C = (c_{nk}) = AT^{-1} \) exists;

(3.2.4.3) \( C \) is regular and \( \lim_{n} \sum_{k \in E} |c_{nk}| = 0 \), whenever \( \delta(E) = 0 \) for \( E \subseteq \mathbb{N} \);

(3.2.4.4) \( \lim_{n} \sum_{k=1}^{\infty} |c_{nk}| = 1; \)

(3.2.4.5) for any fixed \( n \),

\[
\lim_{m} \sum_{k=0}^{m} \left| \sum_{j=m+1}^{\infty} n_{jk} t_{j}^{-1} \right| = 0.
\]

**Theorem 3.2.5** (Fridy and Orhan [23]). If \( A \) and \( T \) satisfy conditions (3.2.4.2)-(3.2.4.5), then

(3.2.5.1) \( st - \text{core}\{Ax\} \subseteq st - \text{core}\{Tx\} \),

for every \( x \) such that \( Tx \in \ell_{\infty} \). But converse need not be true in general.

**Theorem 3.2.6** (Li and Fridy [34]). If \( A \) is a matrix for which \( \{\sum_{j=1}^{\infty} |a_{nj}|\}_{n=1}^{\infty} \) is statistically bounded, then

(3.2.6.1) \( st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} \),

for every \( x \in \ell_{\infty} \) if and only if
\[ (3.2.6.2) \quad st - \lim_n \sum_{j \in E} a_{nj} = 1, \text{ for every } E \subseteq \mathbb{N} \text{ such that } \delta(E) = 1; \]

\[ (3.2.6.3) \quad st - \limsup_n \sum_{i=1}^\ell | \sum_{j \in K_i} a_{nj} | \leq 1, \text{ whenever } \{K_1, K_2, \ldots, K_\ell\} \text{ is a } st\text{-partition of } \mathbb{N}. \]

**Theorem 3.2.7** (Li and Fridy [34]). If \( A \) is a matrix for which \{\sum_{j=1}^\infty | a_{nj} |\}_{n=1}^\infty \) is statistically bounded, then

\[ (3.2.7.1) \quad st - core\{Ax\} \subseteq K - core\{x\} \]

for every \( x \in \ell_\infty \) if and only if

\[ (3.2.7.2) \quad st - \lim_n \sum_{j \in \mathbb{N} \setminus E} a_{nj} = 1, \text{ whenever } \mathbb{N} \setminus E \text{ is finite}; \]

\[ (3.2.7.3) \quad st - \limsup_n \sum_{i=1}^\ell | \sum_{j \in K_i} a_{nj} | \leq 1, \text{ whenever } \{K_1, K_2, \ldots, K_\ell\} \text{ is a sup}\text{-partition of } \mathbb{N}. \]

### 3.3. Main Results

In this section we give alternative conditions for the above core Theorems 3.2.6 and 3.2.7.

**Theorem 3.3.1.** If \( \|A\| < \infty \), then for every \( x \in \ell_\infty \)

\[ (3.3.1.1) \quad st - core\{Ax\} \subseteq K - core\{x\} \]

if and only if

\[ (3.3.1.2) \quad st - \lim_n \sum_{k \in E} a_{nk} = 1, \text{ whenever } \mathbb{N} \setminus E \text{ is finite, where } E \subseteq \mathbb{N}; \]

\[ (3.3.1.3) \quad A \in (c, st \cap \ell_\infty)_{reg}. \]

**Proof. Necessity.** Suppose that

\[ st - core\{Ax\} \subseteq K - core\{x\} \]

and \( x \in \ell_\infty \) has a limit point \( l \). Then
\{\ell\} = K - \text{core}\{x\} \supset st - \text{core}\{Ax\}.

Since \(\|A\| < \infty\), \(Ax \in \ell_\infty\) for \(x \in \ell_\infty\) by Lemma 2.1.3. Hence

\[\liminf x \leq st - \liminf Ax \leq st - \limsup Ax \leq \limsup x.\]

But \(\liminf x = \limsup x = \ell\), so that

\[st - \liminf Ax = st - \limsup Ax = \ell.\]

That is \(st - \lim Ax = \lim x = \ell\) and so \(st - \text{core}\{Ax\} = \{\ell\}\). Hence \(A \in (c, st \cap \ell_\infty)_{\text{reg}}\), i.e. condition (3.3.1.3).

To prove (3.3.1.2), let us define \(x = (x_k)\) by

\[x_k = \begin{cases} 1 & \text{if } k \in E, \\ 0 & \text{otherwise}; \end{cases}\]

where \(E \subseteq \mathbb{N}\) such that \(\mathbb{N}\setminus E\) is finite. Then

\[K - \text{core}\{x\} = \{1\}.\]

Since \(\|A\| < \infty\) implies \(Ax \in \ell_\infty\) for \(x \in \ell_\infty\), \(Ax\) has at least one statistical cluster point by Proposition 1.4.1. Now, by proposition 1.4.2, the set of statistical cluster points is in \(st - \text{core}\{Ax\}\). Therefore, \(st - \text{core}\{Ax\} \neq \emptyset\). Since

\[st - \text{core}\{Ax\} \subseteq K - \text{core}\{x\} = \{1\},\]

we have \(st - \text{core}\{Ax\} = \{1\}\) and 1 is the only statistical cluster point of \(Ax\).

Using proposition 1.4.4, we have \(st - \lim Ax = 1\), i.e.

\[st - \lim_n \sum_{k \in E} a_{nk} = 1, \text{ whenever } \mathbb{N}\setminus E\text{ is finite.}\]

Hence (3.3.1.2) holds.
Sufficiency. Let conditions (3.3.1.2) and (3.3.1.3) hold and that 
\( w \in \text{st} - \text{core}\{Ax\} \). Then for any \( z \in C \), we have

\[
| w - z | \leq \text{st} - \limsup_n | z - A_n(x) |
\]

\[
= \text{st} - \limsup_n | z - \sum_{k=1}^{\infty} a_{nk} x_k |
\]

\[
\leq \text{st} - \limsup_n | \sum_{k=1}^{\infty} a_{nk} (z - x_k) |
\]

\[
+ \text{st} - \limsup_n | z | | 1 - \sum_{k=1}^{\infty} a_{nk} |
\]

(3.3.1.4) \( = \text{st} - \limsup_n | \sum_{k=1}^{\infty} a_{nk} (z - x_k) | \), by (3.3.1.2).

Let \( r = \lim sup_k | z - x_k | \) and \( E := \{ k : | z - x_k | > r + \epsilon \} \) for \( \epsilon > 0 \). Then \( \delta(E) = 0 \) as \( E \) is finite and we have

(3.3.1.5) \[
| \sum_{k} a_{nk} (z - x_k) | \leq \sup_k | z - x_k | | \sum_{k \in E} a_{nk} | + (r + \epsilon) | \sum_{k \notin E} a_{nk} |.
\]

Therefore, by conditions (3.3.1.2) and (3.3.1.3), we obtain

\[
\text{st} - \limsup_n | \sum_{k=1}^{\infty} a_{nk} (z - x_k) | \leq r + \epsilon.
\]

Hence by (3.3.1.4) we have

\[
| w - z | \leq r + \epsilon
\]

and since \( \epsilon \) is arbitrary,

\[
| w - z | \leq r = \limsup_k | z - x_k |,
\]

i.e. \( w \in K^*_x(z) \). Hence \( w \in K - \text{core}\{x\} \) and so

\[
\text{st} - \text{core}\{Ax\} \subseteq K - \text{core}\{x\}.
\]

This completes the proof of the theorem.
Remark 3.3.2. Condition (3.3.1.2) can not be replaced by

\[(3.3.2.1) \quad \text{st-} \lim_{n} \sum_{k \in E} a_{nk} = 1, \text{ for any set } E \subseteq \mathbb{N} \text{ such that } \delta(E) = 1.\]

Consider the following example.

Example 3.3.3. Let \( A = (a_{nk}) \) be an infinite matrix defined as

\[
a_{nk} = \begin{cases} 
1, & \text{if } n \text{ is a nonsquare and } k = n^{2}, \\
0, & \text{otherwise.}
\end{cases}
\]

Then

\[
\sum_{k} a_{nk} = \begin{cases} 
1, & \text{if } n \text{ is a nonsquare,} \\
0, & \text{otherwise.}
\end{cases}
\]

We see that \( A \in (c, st \cap \ell_{\infty})_{\text{reg}} \) but \( A \) is not regular. Further, for any set \( E \subseteq \mathbb{N} \) such that \( \mathbb{N} \setminus E \) is finite we have

\[
\text{st-} \lim_{n} \sum_{k \in E} a_{nk} = 1.
\]

So that for any bounded sequence \( x \) we have

\[
\text{st-} \text{core}\{Ax\} \subseteq K - \text{core}\{x\}.
\]

Now, let \( E = \{k \neq n^{2} : k \in \mathbb{N}\} \). Then \( \delta(E) = 1 \) and we have

\[
\sum_{k \in E} a_{nk} = 0, \text{ for all } n.
\]

Hence

\[
\text{st-} \lim_{n} \sum_{k \in E} a_{nk} = 0.
\]

Further, for any bounded sequence, e.g. \( x = (1,1,\cdots) \), we have

\( K - \text{core}\{x\} = \{1\} \) and

\[
\sum_{k} a_{nk}x_{k} = \begin{cases} 
1, & \text{if } n \text{ is a nonsquare,} \\
0, & \text{otherwise.}
\end{cases}
\]
So that \( st - \lim Ax = 1 \), i.e.

\[
st - \text{core}\{Ax\} = \{1\} = K - \text{core}\{x\}.
\]

Therefore, we see that (3.3.1.2) hold but (3.3.2.1) does not hold. Hence condition (3.3.1.2) is necessary.

**Theorem 3.3.4.** If \( \|A\| < \infty \), then for every \( x \in \ell_\infty \)

(3.3.4.1) \( st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} \)

if and only if

(3.3.4.2) \( st - \lim_{n} \sum_{k \in E} a_{nk} = 1 \), for every \( E \subseteq \mathbb{N} \) such that \( \delta(E) = 1; \)

(3.3.4.3) \( A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{\text{reg}}. \)

**Proof.** Necessity. Let \( st - \text{core}\{Ax\} \subseteq st - \text{core}\{x\} \) and \( x \) be statistically convergent to the number \( \ell \). Then

\[
\{\ell\} = st - \text{core}\{x\} \supseteq st - \text{core}\{Ax\}.
\]

Since \( \|A\| < \infty \) implies \( Ax \in \ell_\infty \) for \( x \in \ell_\infty \), we have

\[
st - \lim \inf x \leq st - \lim \inf Ax \leq st - \lim \sup Ax \leq st - \lim \sup x.
\]

But \( st - \lim \inf x = st - \lim \sup x = \ell \), so that

\[
st - \lim Ax = st - \lim x = \ell,
\]

i.e. \( st - \text{core}\{Ax\} = \{\ell\} \). Hence \( A \in (st \cap \ell_\infty, st \cap \ell_\infty)_{\text{reg}}. \)

To prove (3.3.4.2), let \( E \subseteq \mathbb{N} \) such that \( \delta(E) = 1 \). Let \( \chi_E \) be the characteristic function of \( E \). Then

\[
st - \text{core}\{\chi_E\} = \{1\}.
\]
Since \( \|A\| < \infty \) implies \( A\chi_E \in \ell_\infty \) for \( \chi_E \in \ell_\infty \), we have that \( A\chi_E \) has at least one statistical cluster point. Therefore, \( \text{st} - \text{core}\{A\chi_E\} \neq \emptyset \). Also \( \text{st} - \text{core}\{A\chi_E\} = \{1\} \), since

\[
\text{st} - \text{core}\{A\chi_E\} \subseteq \text{st} - \text{core}\{\chi_E\} = \{1\}.
\]

Hence

\[
\text{st} - \lim A\chi_E = \text{st} - \lim \sum_{k \in E} a_{nk} = 1, \text{ where } \delta(E) = 1,
\]

i.e. condition (3.3.4.2).

**Sufficiency.** It follows on the same lines as in Theorem 3.3.1, i.e. for \( w \in \text{st} - \text{core}\{Ax\} \), we arrived at

\[
|w - z| \leq r, \text{ where } r = \text{st} - \limsup_{k} |z - x_k|, \text{ for any } z \in \mathbb{C},
\]

by using conditions (3.3.4.2) and (3.3.4.3). So that \( w \in S_x^*(z) \). Hence \( w \in \text{st} - \text{core}\{x\} \), i.e.

\[
\text{st} - \text{core}\{Ax\} \subseteq \text{st} - \text{core}\{x\}.
\]

This completes the proof of the theorem.

**Remark 3.3.5.** If \( \text{st} - \lim \sum_{k \in E} a_{nk} = 1, \text{ where } \delta(E) = 1 \) then

\[
\text{st} - \limsup_n \sum_{k \in E} a_{nk} = 0, \text{ whenever } \delta(E) = 0.
\]

Similarly, if \( \text{st} - \lim_n \sum_{k \in E} a_{nk} = 1 \), whenever \( \mathcal{N} \setminus E \) is finite then

\[
\text{st} - \lim_n \sum_{k \in E} a_{nk} = 0. \text{ whenever } E \text{ is finite.}
\]

**Remark 3.3.6.** Note that relaxing the condition on the matrix \( A \) we get stronger necessary condition than condition (3.2.6.3) (i.e. Theorem 3 of Li and Fridy [34]). To show this we prove the following proposition.
Proposition 3.3.7. Let $A = (a_{nk})$ be an infinite matrix such that $\|A\| < \infty$. If

\((3.3.7.1)\) \quad A \in (st \cap l_\infty, st \cap l_\infty)_{\text{reg}},

then

\((3.3.7.2)\) \quad st - \lim \sup_n \sum_{i=1}^\ell | \sum_{j \in K_i} a_{nj} | \leq 1,

whenever $\{K_1, K_2, \ldots, K_\ell\}$ is a st-partition of $\mathbb{N}$, but not conversely.

Proof. Let $A \in (st \cap l_\infty, st \cap l_\infty)_{\text{reg}}$ and suppose that \((3.3.7.2)\) does not hold. Then

$$st - \lim \sup_n \sum_{i=1}^\ell | \sum_{j \in K_i} a_{nj} | = \beta > 1,$$

where $\{K_1, K_2, \ldots, K_\ell\}$ is a st-partition of $\mathbb{N}$.

Since $A$ is bounded which implies that it is statistically bounded. Then by Theorem 1.4.3 we can find $\ell$ real numbers $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ such that

\((3.3.7.3)\) \quad \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = \beta,

and

\((3.3.7.4)\) \quad \delta^* \{ n \in \mathbb{N} : \sum_{i=1}^\ell | \sum_{j \in K_i} a_{nj} - \alpha_i | < \varepsilon \} > 0.

Let $K = \bigcup_{i=1}^\ell K_i$ and let us define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & \text{if } k \in K, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to see that $st - \lim x = 1$, since $\delta(K) = 1$. 
Now
\[ | \sum_{j=1}^{\infty} a_{nj}x_j | = | \sum_{i=1}^\ell \left( \sum_{j \in K_i} a_{nj}x_j \right) + \sum_{j \notin K} a_{nj}x_j | \]
\[ = | \sum_{i=1}^\ell \left( \sum_{j \in K_i} a_{nj} - \alpha_i \right) + \sum_{i=1}^\ell \alpha_i | \]
\[ \geq | \sum_{i=1}^\ell \alpha_i | - | \sum_{i=1}^\ell \left( \sum_{j \in K_i} a_{nj} - \alpha_i \right) |. \]
\[ > \beta - \epsilon \quad \text{by (3.3.7.3) and (3.3.7.4)}. \]

Now
\[ F = \{ n \in \mathbb{N} : \sum_{i=1}^\ell | \sum_{j \in K_i} a_{nj} - \alpha_i | < \epsilon \} \subseteq \{ n \in \mathbb{N} : \sum_{j=1}^{\infty} a_{nj}x_j | > \beta - \epsilon \} = G. \]
Therefore
\[ \delta^*(G) \geq \delta^*(F) > 0. \]

This implies that
\[ st - \limsup A x \geq \beta > 1 = st - \lim x \]
which contradicts our hypothesis that \( A \in (st \cap \ell_{\infty}, st \cap \ell_{\infty})_{reg}. \)
Hence (3.3.7.2) must hold.

For the converse part see Example 2 of Li and Fridy [34]. For the sake of completeness we would like to include it here.

Define the matrix \( A \) by
\[ a_{nk} = \begin{cases} 1, & \text{if } n = k, \\ \frac{1}{2}, & \text{if } k \text{ is a square and } (\sqrt{k} - 1)^2 \leq n < k, \\ 0, & \text{otherwise}; \end{cases} \]
and \( x = (x_k) \) defined by
\[
x_k = \begin{cases} 
2 & \text{if } k \text{ is a square}, \\
1 & \text{otherwise}.
\end{cases}
\]

Then \( st - \lim x = 1 \), \( st - \lim Ax = \frac{3}{2} \) and condition (3.3.7.2) holds but \( A \notin (st \cap \ell_\infty, st \cap \ell_\infty)_{\text{reg}} \).

Similarly we can show that

**Proposition 3.3.8.** Let \( A = (a_{nk}) \) be an infinite matrix such that \( \|A\| < \infty \).

If

\[(3.3.8.1) \quad A \in (c, st \cap \ell_\infty)_{\text{reg}},\]

then

\[(3.3.8.2) \quad st - \lim sup_n \sum_{i=1}^{\ell} | \sum_{j \in K_i} a_{nj} | \leq 1,
\]

whenever \( \{K_1, K_2, \ldots, K_\ell\} \) is a sup-partition of \( \mathbb{N} \),

but not conversely.

**Theorem 3.3.9.** Let \( T = (t_{jk}) \) be a normal matrix and \( A = (a_{nj}) \) be any matrix. In order that whenever \( Tx \) is bounded \( Ax \) should exist and be bounded and satisfy

\[(3.3.9.1) \quad st - \text{core}\{Ax\} \subseteq K - \text{core}\{Tx\}\]

it is necessary and sufficient that

\[(3.3.9.2) \quad (c_{nk}) = C = AT^{-1} \text{ exists};\]

\[(3.3.9.3) \quad C \in (c, st \cap \ell_\infty)_{\text{reg}};\]

\[(3.3.9.4) \quad st - \lim_n \sum_{k \in E} c_{nk} = 1, \text{ whenever } \mathbb{N} \setminus E \text{ is finite};\]

\[(3.3.9.5) \quad \text{for any fixed } n, \quad \sum_{k=0}^{m} \sum_{j=m+1}^{\infty} a_{nj} t_{jk}^{-1} \to 0 \text{ as } m \to \infty.\]
Proof. Necessity. Let us suppose that $A_n(x)$ exist for every $n$, whenever $Tx \in \ell_\infty$. Then by Lemma 2.1.5, it follows that conditions (3.3.9.2) and (3.3.9.5) hold, and $Ax = Cy$, where $y = Tx$. Since $Ax \in \ell_\infty$, (3.3.9.1) implies that

$$\text{(3.3.9.6)} \quad st - \text{core} \{Cy\} \subseteq K - \text{core}\{y\}. $$

Now, by Theorem 3.3.1, (3.3.9.6) implies that $C \in (c, st \cap \ell_\infty)_\text{reg}$ and $st - \lim_n \sum_{k \in E} c_{nk} = 1$, whenever $\mathcal{N}\setminus E$ is finite, i.e. conditions (3.3.9.3) and (3.3.9.4) hold.

Sufficiency. Let the conditions (3.3.9.2)-(3.3.9.5) hold. Then by Theorem 3.3.1, $C \in (c, st \cap \ell_\infty)_\text{reg}$. Therefore by Theorem 2.2.2, $\sup_n \sum_k |c_{nk}| < \infty$ which implies that $\sum_k |c_{nk}| < \infty$ holds for every $n$. So that by Lemma 2.1.5, the conditions of Theorem 3.2.3 are satisfied and consequently

$$A_n(x) = \sum_k c_{nk}y_k$$

exists. Further use of Lemma 2.1.5 yields $Ax = Cy \in \ell_\infty$, where $y = Tx$ which implies by Theorem 3.3.1 that

$$st - \text{core} \{Cy\} \subseteq K - \text{core}\{y\}. $$

Hence (3.3.9.1) holds.

This completes the proof of the theorem.

On the same lines by using Theorem 3.3.4, we can easily prove the following:

Theorem 3.3.10. Let $A$ and $T$ be same as in Theorem 3.3.9. Then

$$\text{(3.3.10.1)} \quad st - \text{core} \{Ax\} \subseteq st - \text{core} \{Tx\}$$

if and only if (3.3.9.2) and (3.3.9.5) hold and

$$\text{(3.3.10.2)} \quad C \in (st \cap \ell_\infty, st \cap \ell_\infty)_\text{reg};$$

$$\text{(3.3.10.3)} \quad st - \lim_n \sum_{k \in E} c_{nk} = 1, \text{ for every } E \subseteq \mathcal{N} \text{ such that } \delta(E) = 1.$$