CHAPTER VIII

TAUBERIAN THEOREMS FOR STATISTICAL CONVERGENT DOUBLE SEQUENCES

8.1. Introduction

In [39], Moricz has proved some Tauberian theorems for Cesàro summable double sequences and deduced the Tauberian theorems of Landau [33] and Hardy [26] type. Friday and Khan [20] have proved statistical extensions of such classical Tauberian theorems.

Let \((x_k)\) be a sequence of real numbers. Landau [33] gave a classical one-side Tauberian theorem as follows:

Lemma 8.1.1. If \((x_k)\) is summable \(C_1\) to a finite number \(\ell\) and there exists a constant \(M\) such that

\[(*) \quad k(x_k - x_{k-1}) \geq -M \quad (k = 1, 2, \cdots)\]

then \((x_k)\) converges to \(\ell\).

We say that \((x_k)\) is slowly decreasing (cf. Schmidt [50]) if for each \(\epsilon > 0\) there exist \(n_1 > 0\) and \(\lambda > 1\) such that

\[(**) \quad x_k - x_n \geq -\epsilon \quad \text{whenever} \quad n_1 < n < k \leq \lambda n.\]

Clearly \((*)\) is a particular case of \((**).\)

Lemma 8.1.2 (Hardy [26]). If a sequence \((x_k)\) is summable \(C_1\) to a finite number \(\ell\) and \((x_k)\) is slowly decreasing, then \((x_k)\) converges to \(\ell\).
Landau's theorem remains valid if condition (*) is replaced by (cf. Zygmund [55])

\[ k \mid \nabla x_k \mid \leq H \quad (k = 1, 2, \ldots) \]

where the "backward difference" \( \nabla x_k = x_k - x_{k-1} \) with \( x_{-1} = 0 \).

Moricz [39] have proved the following results for double sequences.

**Theorem 8.1.3.** If \( \{x_{jk}\} \) is summable \( C_{n_1} \) to a finite limit and there exist constants \( n_1 \) and \( M \) such that the following conditions hold:

1. \( jk(x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}) \geq -M \) whenever \( j, k > n_1 \);
2. \( j(x_{jk} - x_{j-1,k}) \geq -M \) whenever \( j, k > n_1 \);
3. \( k(x_{jk} - x_{j,k-1}) \geq -M \) whenever \( j, k > n_1 \).

Then \( \{x_{jk}\} \) converges.

**Theorem 8.1.4.** If \( \{x_{jk}\} \) is summable \( C_{n_1} \) to a finite limit and there exist constants \( n_1 \) and \( M \) such that the following conditions hold:

1. \( jk \mid x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1} \mid \leq M \) whenever \( j, k > n_1 \);
2. \( j \mid x_{jk} - x_{j-1,k} \mid \leq M \) whenever \( j, k > n_1 \);
3. \( k \mid x_{jk} - x_{j,k-1} \mid \leq M \) whenever \( j, k > n_1 \).

Then \( \{x_{jk}\} \) converges.

Fridy [18] established a Tauberian theorem for statistical convergence of single sequences.

**Theorem 8.1.5.** If \( st - \lim x = \ell \) and \( \nabla x_k = O(\frac{1}{k}) \), then \( \lim x = \ell \).

In [20], Fridy and Khan have extended this idea as follows:
Theorem 8.1.6. If \( st - \lim C_1 x = \ell \) and \( \nabla x_k = O\left(\frac{1}{k}\right) \), then \( \lim x = \ell \).

Lemma 8.1.7. If \( \nabla x_k = O\left(\frac{1}{k}\right) \), then \( \langle \nabla C_1 x \rangle_n = O\left(\frac{1}{n}\right) \).

Theorem 8.1.8. If \( st - \lim C_1 x = \ell \) and \( \nabla x_k = O\left(\frac{1}{k}\right) \), then \( \lim x = \ell \).

Theorem 8.1.9. If \( st - \lim x = \ell \) and \( k \nabla x_{k+1} \geq -c \), for some \( c > 0 \) and for every \( k \), then \( \lim x = \ell \).

Theorem 8.1.10. If \( st - \lim C_1 x = \ell \) and \( k \nabla x_{k+1} \geq -c \), for some \( c > 0 \) and for every \( k \), then \( \lim x = \ell \).

In this chapter we obtain Tauberian results for statistically convergent double sequences.

8.2. Main Results

Here we denote the backward differences of \( x_{jk} \) as follows:

\[
\nabla_{11} x_{jk} = x_{jk} - x_{j-1,k} - x_{j,k-1} + x_{j-1,k-1}
\]

\[
\nabla_{10} x_{jk} = x_{jk} - x_{j-1,k}
\]

\[
\nabla_{01} x_{jk} = x_{jk} - x_{j,k-1}
\]

We can also easily compute the following differences:

\[
D_1. \quad x_{jk} - x_{j+p,k+q} = \sum_{i=j+1}^{j+p} \sum_{s=k+1}^{k+q} \nabla_{11} x_{is}
\]

\[
= - \sum_{i=j+1}^{j+p} \nabla_{10} x_{i,k+q} - \sum_{s=k+1}^{k+q} \nabla_{01} x_{j+p,s}
\]

\[
= - \sum_{i=j+1}^{j+p} \nabla_{10} x_{ik} - \sum_{s=k+1}^{k+q} \nabla_{01} x_{j+p,s}
\]


\[ D_2. \quad x_{nj} - x_{nk} = \sum_{s=k+1}^{j} \nabla_{01} x_{ns} \]

\[ D_3. \quad x_{nm} - x_{km} = \sum_{i=k+1}^{n} \nabla_{10} x_{im} \]

\[ D_4. \quad x_{jk} - x_{nm} = \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11} x_{is} \]

\[ - \sum_{i=j+1}^{n} \nabla_{10} x_{im} - \sum_{s=k+1}^{m} \nabla_{01} x_{ns} \]

\[ D_5. \quad x_{jk} - x_{nm} - x_{nk} + x_{nm} = \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11} x_{is}. \]

Our first Tauberian theorem is as follows:

**Theorem 8.2.1.** Let \( x = (x_{jk}) \) be a double sequence with \( \text{st}_2 - \lim x = \ell \) and there exist a constant \( n_1 \) such that

\[ (8.2.1.1) \quad \nabla_{11} x_{jk} = O(\frac{1}{j}); \]

\[ (8.2.1.2) \quad \nabla_{10} x_{jk} = O(\frac{1}{j}); \]

\[ (8.2.1.3) \quad \nabla_{01} x_{jk} = O(\frac{1}{k}), \]

whenever \( j, k > n_1 \). Then \( \lim_{j,k} x_{jk} = \ell \).

**Proof.** Since \( \text{st}_2 - \lim x = \ell \), by Corollary 7.3.4, there exists a sequence \( y = (y_{jk}) \) such that \( \lim_{j,k} y_{jk} = \ell \) and

\[ (8.2.1.4) \quad \delta_\varepsilon\{(j, k), j \leq n, k \leq m : y_{jk} = x_{jk}\} = 1. \]

For each \( j \) and \( k \) we write

\[ j = \lambda(j) + \mu(j) \quad \text{and} \quad k = \lambda(k) + \mu(k), \]

where \( \lambda(j) = \max\{i \leq j : y_{ik} = x_{ik}\} \) and \( \lambda(k) = \max\{s \leq k : y_{js} = x_{js}\} \). In case the sets \( \{i \leq j : y_{ik} = x_{ik}\} \) and/or \( \{s \leq k : y_{js} = x_{js}\} \) are empty, we take \( \lambda(j) = -1, \lambda(k) = -1 \). Note that this can happen only for a finite number of \( j \)
and $k$.

Now we will show that

$$\lim_{j} \frac{\mu(j)}{\lambda(j)} = 0 = \lim_{k} \frac{\mu(k)}{\lambda(k)}.$$  

Suppose that

$$\frac{\mu(j)}{\lambda(j)} > \varepsilon > 0 \text{ and } \frac{\mu(k)}{\lambda(k)} > \varepsilon > 0.$$  

Then

$$\frac{1}{jk} \mid \{(i, s), i \leq j, s \leq k : x_{is} \neq y_{is}\} \mid$$

$$\leq \frac{\lambda(j)\mu(k) + \lambda(k)\mu(j) + \mu(j)\mu(k)}{\lambda(j) + \mu(j))\lambda(k) + \mu(k))}$$

$$= \frac{\lambda(j)\mu(k) + \lambda(k)\mu(j)}{\lambda(j)\lambda(k) + \mu(j)\lambda(k)} \leq \frac{\varepsilon + \varepsilon + \varepsilon^{2}}{\varepsilon^{2} + \varepsilon + 1} = \frac{\varepsilon^{2} + 2\varepsilon}{\varepsilon^{2} + 2\varepsilon + 1} \text{ by (8.2.1.6)},$$

i.e. if (8.2.1.6) holds for infinitely many terms $j$ and $k$ we get a contradiction to (8.2.1.4).

Hence (8.2.1.5) holds.

Now by the conditions (8.2.1.1) — (8.2.1.3) and $D_1$, we have

$$\mid y_{\lambda(j),\lambda(k)} - x_{jk} \mid = \mid x_{\lambda(j),\lambda(k)} - x_{\lambda(j)+\mu(j),\lambda(k)+\mu(k)} \mid$$

$$\leq \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \sum_{s=\mu(k)+1}^{\lambda(k)+\mu(k)} \mid \nabla_{11} x_{is} \mid + \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \mid \nabla_{10} x_{is,\lambda(k)+\mu(k)} \mid$$
\[ Afc + \sum_{s=\mu(k)+1}^{\lambda(k)+\mu(k)} | \nabla_0 x_{\lambda(j)+\mu(j),s} | \]

\[ \leq \frac{A\lambda(j)\mu(k)}{(\lambda(j) + 1)(\lambda(k) + 1)} + \frac{B\mu(j)}{\lambda(j) + 1} + \frac{C\mu(k)}{\mu(k) + 1} \]

\[ \rightarrow 0 \text{ as } j, k \rightarrow \infty \text{ by (8.2.1.5)} , \]

where \( A, B \) and \( C \) are constants.

Hence \( \lim_{j,k} x_{jk} = \ell \), since \( \lim_{j,k} y_{jk} = \ell \).

**Lemma 8.2.2.** Let \( x = (x_{jk}) \) be a double sequence. If there exists a constant \( n_1 \) such that the following conditions hold

(8.2.2.1) \( \nabla_{11} x_{jk} = O\left(\frac{1}{j^k}\right) \);

(8.2.2.2) \( \nabla_{10} x_{jk} = O\left(\frac{1}{j}\right) \);

(8.2.2.3) \( \nabla_{01} x_{jk} = O\left(\frac{1}{k}\right) \),

whenever \( j, k > n_1 \). Then \( (\nabla_{11} \sum x_{nm})_{nm} = O\left(\frac{1}{nm}\right) \).

**Proof.** For all \( n, m > 1 \)

\[ nm(\nabla_{11} \sum x_{nm})_{nm} = nm \left[ \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - \frac{1}{(n-1)m} \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} \right] \]

\[ - \frac{1}{n(m-1)} \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} + \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \]

\[ = \frac{1}{(n-1)(m-1)} \left[ (n-1)(m-1) \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} \right. \]

\[ - n(m-1) \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} - m(n-1) \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} + nm \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} \]

\[ - n(m-1) \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} - m(n-1) \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} + nm \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} \]
\[ \begin{align*}
&= \frac{1}{(n - 1)(m - 1)} \left[ nm \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - n \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - m \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} 
\right. \\
&\quad + \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - nm \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} + n \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} \\
&\quad \left. - nm \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} + m \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} + nm \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \right] \\
&= \frac{1}{(n - 1)(m - 1)} \left[ nm \left( \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} - \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \right) \right. \\
&\quad \left. - n \left( \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} \right) \right. \\
&\quad \left. - m \left( \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} - \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} + \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} \right) \right. \\
&\quad \left. \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} + \sum_{j=1}^{n} \sum_{k=1}^{m-1} x_{jk} \right] \\
&= \frac{1}{(n - 1)(m - 1)} \left[ nm x_{nm} - n \sum_{k=1}^{m} x_{nk} - m \sum_{j=1}^{n} x_{jm} + \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m} x_{jk} \right] \\
&= \frac{1}{(n - 1)(m - 1)} \left[ nm x_{nm} - n \sum_{k=1}^{m-1} x_{nk} - n x_{nm} - m \sum_{j=1}^{n} x_{jm} \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \right. \\
&\quad \left. + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \right] \\
\end{align*} \]
\[ \frac{1}{(n-1)(m-1)} \left[ (n-1)(m-1) x_{nm} - n \sum_{k=1}^{m-1} x_{nk} - m \sum_{j=1}^{n-1} x_{jm} + \sum_{j=1}^{n-1} x_{jm} + \sum_{k=1}^{m-1} x_{nk} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \right] \]

\[ = \frac{1}{(n-1)(m-1)} \left[ (n-1)(m-1) x_{nm} - (n-1) \sum_{k=1}^{m-1} x_{nk} - (m-1) \sum_{j=1}^{n-1} x_{jm} + \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} x_{jk} \right] \]

\[ = \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} \left[ \sum_{i=k+1}^{m} \nabla_{01} x_{ni} - \sum_{s=k+1}^{m} \nabla_{01} x_{js} \right], \text{ by } D_2 \]

\[ = \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \left[ \sum_{k=1}^{m-1} \sum_{i=k+1}^{m} \nabla_{01} x_{ni} - \sum_{k=1}^{m-1} \sum_{s=k+1}^{m} \nabla_{01} x_{js} \right] \]

\[ = \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \left[ \sum_{i=2}^{m} (i-1) \nabla_{01} x_{ni} - \sum_{s=2}^{m} (s-1) \nabla_{01} x_{js} \right] \]

\[ = O(1). \]

This completes the proof of the theorem.
Corollary 8.2.3. Let \( x = (x_{jk}) \) be a double sequence and there exist a constant \( n_1 \) such that

(a) if \( \mathbf{v}_{10} x_{jk} = O(\frac{1}{j}) \), then \( \mathbf{v}_{10} C_{10} x_n = O(\frac{1}{n}) \)

(b) if \( \mathbf{v}_{01} x_{jk} = O(\frac{1}{k}) \), then \( \mathbf{v}_{01} C_{01} x_m = O(\frac{1}{m}) \),

whenever \( j, k > n_1 \).

Theorem 8.2.4. For a double sequence \( x = (x_{jk}) \) if \( s_{t_2} - \lim C_{11} x = \ell \) and there exists a constant \( n_1 \) such that

\[
(8.2.4.1) \quad \mathbf{v}_{11} x_{jk} = O(\frac{1}{jk}) ; \\
(8.2.4.2) \quad \mathbf{v}_{10} x_{jk} = O(\frac{1}{j}) ; \\
(8.2.4.3) \quad \mathbf{v}_{01} x_{jk} = O(\frac{1}{k}) ,
\]

whenever \( j, k > n_1 \). Then \( \lim_{j,k} x_{jk} = \ell \).

Proof. Using conditions (8.2.4.1) - (8.2.4.3) and Lemma 8.2.2 we have \( \lim C_{11} x = \ell \). Now by Theorem 8.1.4 we get

\[
\lim_{j,k} x_{jk} = \ell .
\]

Corollary 8.2.5. Let \( x = (x_{jk}) \) be a double sequence and there exist a constant \( n_1 \) such that

(a) if \( s_{t_2} - \lim C_{10} x = \ell \) and \( \mathbf{v}_{10} x_{jk} = O(\frac{1}{j}) \) then \( \lim_{j,k} x_{jk} = \ell \)

(b) if \( s_{t_2} - \lim C_{01} x = \ell \) and \( \mathbf{v}_{01} x_{jk} = O(\frac{1}{k}) \) then \( \lim_{j,k} x_{jk} = \ell \),

whenever \( j, k > n_1 \).

Theorem 8.2.6. Let \( x = (x_{jk}) \) be a double sequence with \( s_{t_2} - \lim x = \ell \) and there exist constants \( n_1 \) and \( M \) such that

\[
(8.2.6.1) \quad jk v_{11} x_{j+1,k+1} \geq -M ;
\]
\[(8.2.6.2)\] \( j \nabla_{10} x_{j+1,k} \geq -M; \)

\[(8.2.6.3)\] \( k \nabla_{01} x_{j,k+1} \geq -M, \)

whenever \( j, k > n_1. \) Then \( \lim_{j,k} x_{jk} = \ell. \)

**Proof.** From Theorem 8.2.1 we have

\[(8.2.6.4)\] \( \lim_j \frac{\mu(j)}{\lambda(j)} = 0, \quad \lim_k \frac{\mu(k)}{\lambda(k)} = 0. \)

and

\[
\delta_2 \{(j, k), j \leq n, k \leq m : y_{jk} = x_{jk}\} = 1
\]

where \( j = \lambda(j) + \mu(j) \) and \( k = \lambda(k) + \mu(k). \)

Now

\[
y_{\lambda(j), \lambda(k)} - x_{jk} = x_{\lambda(j), \lambda(k)} - x_{\lambda(j) + \mu(j), \lambda(k) + \mu(k)}
\]

\[
= - \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \nabla_{10} x_{i,\lambda(k)} - \sum_{s=\lambda(k)+1}^{\lambda(k)+\mu(k)} \nabla_{01} x_{\lambda(j) + \mu(j), s}\] (by \( D_1 \))

\[
\leq \sum_{i=\lambda(j)+1}^{\lambda(j)+\mu(j)} \frac{b}{i-1} + \sum_{s=\lambda(k)+1}^{\lambda(k)+\mu(k)} \frac{c}{s-1}
\]

\[
|y_{\lambda(j), \lambda(k)} - x_{jk}| \leq \frac{b\mu(j)}{\lambda(j)} + \frac{c\mu(k)}{\lambda(k)}
\]

\[\rightarrow 0\] as \( j, k \rightarrow \infty \) by \( (8.2.6.4) \).

Hence \( \lim_{j,k} x_{jk} = \ell. \)
Lemma 8.2.7. Let $x = (x_{jk})$ be a double sequence and there exist constants $n_1$ and $M$ such that

(8.2.7.1) $jk \nabla_{11} x_{j+1,k+1} \geq -M$;
(8.2.7.2) $j \nabla_{10} x_{j+1,k} \geq -M$;
(8.2.7.3) $k \nabla_{01} x_{j,k+1} \geq -M$,

whenever $j, k > n_1$. Then $(\nabla_{11} C_{11} x)_{nm} \geq -M$.

Proof. As in Lemma 8.2.2 we have

\[
(\nabla_{11} C_{11} x)_{nm} = \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} [x_{nm} - x_{nk} - x_{jm} - x_{jk}]
\]

\[
= \frac{1}{(n-1)(m-1)} \sum_{j=1}^{n-1} \sum_{k=1}^{m-1} \left[ \sum_{i=j+1}^{n} \sum_{s=k+1}^{m} \nabla_{11} x_{is} \right] \quad \text{(by D5)}
\]

\[
= \frac{1}{(n-1)(m-1)} \sum_{i=2}^{n} \sum_{s=2}^{m} (i-1)(s-1) \nabla_{11} x_{is}
\]

\[
\geq \frac{1}{(n-1)(m-1)} \sum_{i=2}^{n} \sum_{s=2}^{m} (i-1)(s-1) \frac{-M}{(i-1)(s-1)} = -M.
\]

Corollary 8.2.8. Let $x = (x_{jk})$ be a double sequence and there exist constants $n_1$ and $M$ such that

(a) if $j \nabla_{10} x_{j+1,k} \geq -M$, then $(\nabla_{10} C_{10} x)_n \geq -M$

(b) if $k \nabla_{01} x_{j,k+1} \geq -M$, then $(\nabla_{01} C_{01} x)_m \geq -M$,

whenever $j, k > n_1$. 

Theorem 8.2.9. Let \( x = (x_{jk}) \) be a double sequence with \( s^2 - \lim C_{11} x = \ell \) and there exist constants \( n_1 \) and \( M \) such that

\[(8.2.9.1) \quad j \nabla_{11} x_{j+1,k+1} \geq -M; \]
\[(8.2.9.2) \quad j \nabla_{10} x_{j+1,k} \geq -M; \]
\[(8.2.9.3) \quad k \nabla_{01} x_{j,k+1} \geq -M, \]
whenever \( j, k > n_1 \). Then \( \lim_{j,k} x_{jk} = \ell \).

Proof. Replacing \( x \) by \( C_{11} x \) in Theorem 8.2.6, we get \( \lim C_{11} x = \ell \). Now using Theorem 8.1.3 we get \( \lim_{j,k} x_{jk} = \ell \).

Corollary 8.2.10. Let \( x = (x_{jk}) \) be a double sequence and there exist constants \( n_1 \) and \( M \) such that

(a) if \( s^2 - \lim C_{10} x = \ell \) and \( j \nabla_{10} x_{j+1,k} \geq -M \) then \( \lim_{j,k} x_{jk} = \ell \)

(b) if \( s^2 - \lim C_{01} x = \ell \) and \( k \nabla_{01} x_{j,k+1} \geq -M \) then \( \lim_{j,k} x_{jk} = \ell \),
whenever \( j, k > n_1 \).

8.3. Examples

In this section we give examples (i) to show that all the conditions in Theorem 8.2.1 and 8.2.4 must be satisfied, (ii) as suggested by Moricz [39, Problem 1].

Example 8.3.1. Let \( x = (x_{jk}) \) be a double sequence defined by

\[ x_{jk} = \begin{cases} k & \text{if } k \text{ is square}, \\ k^{-1} & \text{otherwise}. \end{cases} \]

We see that

\[ | \{ (j, k) : | x_{jk} - 0 | \geq \varepsilon \} | \leq n \sqrt{m}. \]
Hence \( st_2 - \lim_{j,k} x_{j,k} = 0 \). Also \( jk \nabla_{11} x_{j,k} = 0 = j \nabla_{10} x_{j,k}, \quad \forall j, k > 1 \) but \( k \nabla_{01} x_{j,k} \) is unbounded. Further \( \lim_{j,k} x_{j,k} \) does not exist. Therefore all the conditions in Theorem 8.2.1 and 8.2.4 must hold.

Note that \( \lim C_{11} x \) does not exist but \( st_2 - \lim C_{11} x = 0 \).

**Example 8.3.2.** Let \( x = (x_{jk}) \) be a double sequence defined by

\[
 x_{jk} = \begin{cases} 
 1, & \text{if } j \text{ is odd, for all } k, \\
 0, & \text{otherwise.}
\end{cases}
\]

It is clear that \( \lim_{j,k} x_{j,k} \) and \( st_2 - \lim_{j,k} x_{j,k} \) do not exist but \( \lim C_{11} x = \frac{1}{2} = st_2 - \lim C_{11} x \).

Now

\[
 jk \nabla_{11} x_{j,k} = 0, \quad k \nabla_{01} x_{j,k} = 0
\]

but \( j \nabla_{10} x_{j,k} \) is unbounded. Hence we can not drop any condition of our theorems.

This example also provides the solution of Problem 1 of Moricz, i.e. there exists a double sequence \( x = (x_{jk}) \) which is summable \( C_{11} \) to a finite limit, conditions

\[
 jk \nabla_{11} x_{j+1,k+1} \geq -M
\]

and

\[
 k \nabla_{01} x_{j,k-1} \geq -M
\]

hold but the condition

\[
 j \nabla_{10} x_{j+1,k} \geq -M
\]

does not hold and \( x \) fails to converge.