CHAPTER VII

STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

7.1. Introduction

Two dimensional analogue of natural density have been introduced by Christopher [5] which we will use to define statistical convergence of double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two dimensional set of positive integers and let $K(n, m)$ be the numbers $(i, j)$ in $K$ such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of $K$ is defined as

$$\liminf_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

In case the sequence $\left( \frac{K(n,m)}{nm} \right)$ has a limit then we say that $K$ has a double natural density and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n} \sqrt{m}}{nm} = 0,$$

i.e. the set $K$ has double natural density zero.

In this chapter we define and study statistical analogue of convergence and Cauchy for double sequences using the idea of double natural density due to Christopher [5]. We also establish the relation between statistical convergence and strongly Cecàro summable sequences.
7.2. Statistical Convergence

We define the statistical analogue for double sequences $x = (x_{jk})$ as follows:

**Definition 7.2.1.** A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number $\ell$ if for each $\varepsilon > 0$, the set

$$\{(j, k), j \leq n \text{ and } k \leq m : |x_{jk} - \ell| \geq \varepsilon\}$$

has double natural density zero. In this case we write $st_2 - \lim_{n,m} x_{nm} = \ell$ and we denote the set of all statistically convergent double sequences by $st_2$.

**Remark 7.2.2.** (a) If $x$ is a convergent double sequence then it is also statistically convergent to the same number. As if $x$ is bounded then the set

$$\{(j, k), j \leq n, k \leq m : |x_{jk} - \ell| \geq \varepsilon\}$$

is finite for each $\varepsilon > 0$ and hence of natural density zero.

In case $x$ is convergent but not bounded, then there are only a finite number of unbounded rows and (or) columns and hence

$$K(n, m) \leq s_1 m + s_2 n$$

where $s_1$ and $s_2$ are finite numbers, which we can conclude that $x$ is statistically convergent.

(b) If $x$ is statistically convergent to the number $\ell$, then $\ell$ is determined uniquely.

(c) If $x$ is statistically convergent, then $x$ need not be convergent. Also it is not necessary bounded. For example, let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} \sqrt{j} \sqrt{k}, & \text{if } j \text{ and } k \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_2 - \lim x_{jk} = 1$, since

$$\delta_2\{(j, k) : x_{jk} \neq 1\} \leq \lim_{j,k} \frac{\sqrt{j} \sqrt{k}}{j k} = 0.$$ 

But $x$ is neither convergent nor bounded.
We prove some analogues for double sequences. For single sequences such results have been proved by Šalát [49].

**Theorem 7.2.3.** A real double sequence $x = (x_{jk})$ is statistically convergent to a number $\ell$ if and only if there exists a subset $K = \{(i, s)\} \subseteq \mathbb{N} \times \mathbb{N}$, $i, s = 1, 2, \cdots$ such that $\delta(K) = 1$ and

$$\lim_{i, s} x_{j, k} = \ell.$$

**Proof.** Let $x$ be statistically convergent to $\ell$. Then for every $\varepsilon > 0$ the set

$$K = \{(j, k) : |x_{jk} - \ell| \geq \varepsilon\}$$

has natural density zero and hence its complement

$$M = \{(i, s) : |x_{is} - \ell| < \varepsilon\}$$

has natural density 1, i.e.

$$\delta_2(M) = \delta_2(\mathbb{N} \times \mathbb{N}) - \delta_2(K) = 1 - 0 = 1,$$

and

$$K \cap M = \emptyset.$$

Now we have to show that $(x_{is})$, $i, s \in M$ is convergent to $\ell$. Suppose that $(x_{is})$ is not convergent to $\ell$. Then there exists $\varepsilon_0 > 0$ such that

$$|x_{is} - \ell| \geq \varepsilon_0$$

for infinitely many terms.

Let

$$M' = \{(i', s'), i' \leq i, s' \leq s : |x_{i's'} - \ell| \geq \varepsilon_0\}.$$

Clearly $\emptyset \neq M' \subseteq M$. Then

$$K = \{(j, k) : |x_{jk} - \ell| \geq \varepsilon_0\} \supseteq \{(i', s') : |x_{i's'} - \ell| \geq \varepsilon_0\}.$$

Hence $\delta_2(M') = 0$, i.e. $M' \subseteq K$ which contradicts the fact that $K \cap M = \emptyset$. Hence $(x_{is})$ is convergent to $\ell$. 
Conversely, Suppose that there exists a subset $K = \{(j_i, k_s)\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $\lim_{i,s} x_{j_i,k_s} = \ell$, i.e. there exists $N \in \mathbb{N}$ such that

$$| x_{j_i,k_s} - \ell | < \varepsilon, \quad \forall j_i, k_s > N.$$ 

Now

$$\{(j, k) : | x_{jk} - \ell | \geq \varepsilon\} \subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), \cdots \}.$$ 

Therefore

$$\delta_2\{(j, k) : | x_{jk} - \ell | \geq \varepsilon\} \leq 1 - 1 = 0.$$ 

Hence $x$ is statistically convergent to $\ell$.

**Theorem 7.2.4.** The set $st_2 \cap l_\infty^2$ is a closed linear subspace of the normed linear space $l_\infty^2$.

**Proof.** Let $x^{(nm)} = (x_{jk}^{(nm)}) \in st_2 \cap l_\infty^2$ and $x^{(nm)} \rightarrow x \in l_\infty^2$. Since $x^{(nm)} \in st_2 \cap l_\infty^2$, there exists a real number $a_{nm}$ such that

$$st_2 - \lim_{j,k} x_{jk}^{(nm)} = a_{nm} \quad (n, m = 1, 2, \cdots).$$

Also as $x^{(nm)} \rightarrow x$. Therefore for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$(7.2.4.1) \quad | x^{(pq)} - x^{(nm)} | < \varepsilon/3$$

for every $p \geq n \geq N$, $q \geq m \geq N$.

From Theorem 7.2.3, there exist subsets $K_{pq}$ and $K_{nm}$ of $\mathbb{N} \times \mathbb{N}$ with $\delta_2(K_{pq}) = \delta_2(K_{nm}) = 1$ and

(1) \ldots \ldots \quad \lim_{j,k, j,k \in K_{nm}} x_{jk}^{(nm)} = a_{nm};

(2) \ldots \ldots \quad \lim_{j,k, j,k \in K_{pq}} x_{jk}^{(pq)} = a_{pq}.

Now the set $K_{pq} \cap K_{nm}$ is infinite since $\delta_2(K_{pq} \cap K_{nm}) = 1$. 

Choose \((k_1, k_2) \in K_{pq} \cap K_{nm}\). We have from (1) and (2) that,

\[
(7.2.4.2) \quad |x_{k_1,k_2}^{(pq)} - a_{pq}| < \varepsilon/3,
\]

and

\[
(7.2.4.3) \quad |x_{k_1,k_2}^{(nm)} - a_{nm}| < \varepsilon/3.
\]

Therefore for each \(p \geq n \geq N\) and \(q \geq m \geq N\) we have from (7.2.4.1), (7.2.4.2) and (7.2.4.3)

\[
|a_{pq} - a_{nm}| \leq |a_{pq} - x_{k_1,k_2}^{pq}| + |x_{k_1,k_2}^{pq} - x_{k_1,k_2}^{nm}| + |x_{k_1,k_2}^{nm} - a_{nm}|
\]

\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\]

\[
= \varepsilon.
\]

That is the sequence \((a_{nm})\) is a Cauchy sequence and hence convergent. Let

\[
(3) \quad \lim_{n,m} a_{nm} = a.
\]

We need to show that \(x\) is statistically convergent to \(a\). Since \(x^{(nm)}\) is convergent to \(x\), there exists \(N_0 \in \mathbb{N}\) such that for every \(\varepsilon > 0\) and \(n, m \geq N_0\)

\[
|x^{(nm)} - x| < \varepsilon/3.
\]

Also from (3) we have for every \(\varepsilon > 0\) there exists \(N_1 \in \mathbb{N}\) such that for all \(n, m \geq N_1\)

\[
|a_{nm} - a| < \varepsilon/3.
\]

Again, since \(x^{(nm)}\) is statistically convergent to \(a_{nm}\), there exists a set \(K_{jk} \subseteq \mathbb{N} \times \mathbb{N}\) such that \(\delta_2(K_{jk}) = 1\) and for every \(\varepsilon > 0\) there exists \(N_2 \in \mathbb{N}\) such that for all \(n, m \geq N_2\)

\[
|x_{jk}^{(nm)} - a_{nm}| < \varepsilon/3, \quad \forall j, k \in K_{jk}.
\]
Let \( \max\{N_0, N_1, N_2\} = N_3 \). Then for a given \( \varepsilon > 0 \), for every \( j, k \in K_{jk} \) and for all \( n, m \geq N_3 \)

\[
|x_{jk} - a| \leq |x_{jk} - x_{njm}| + |x_{njm}^{(nm)} - a_{nm}| + |a_{nm} - a| = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Therefore \( x \) is statistically convergent to \( a \), i.e. \( x \in st_2 \cap \ell_2 \).

Hence \( st_2 \cap \ell_2^2 \) is a closed linear subspace of \( \ell_2^2 \).

**Theorem 7.2.5.** The set \( st_2 \cap \ell_2^2 \) is nowhere dense in \( \ell_2^2 \).

**Proof.** Since every closed linear subspace of an arbitrary linear normed space \( S \) different from \( S \) is a nowhere dense set in \( S \) (cf. Neubrum, Smítal and Šalát [43]). From Theorem 7.2.4 we need only to show that \( st_2 \cap \ell_2^2 \neq \ell_2^2 \).

Let the sequence \( x = (x_{jk}) \) be defined by

\[
x_{jk} = \begin{cases} 
1 & \text{if } j \text{ and } k \text{ are even,} \\
0 & \text{otherwise.}
\end{cases}
\]

It is clear that \( x \) is not statistically convergent but \( x \) is bounded. Hence \( st_2 \cap \ell_2^2 \neq \ell_2^2 \).

**7.3. Statistically Cauchy Sequences**

In [18], Fridy has defined the concept of statistically Cauchy single sequences. In this section we define statistically Cauchy double sequences and prove some analogues.

**Definition 7.3.1.** A real double sequence \( x = (x_{jk}) \) is said to be statistically Cauchy if for every \( \varepsilon > 0 \) there exist \( N = N(\varepsilon) \) and \( M = M(\varepsilon) \) such that the set

\[
\{(j, k), j \leq n, k \leq m : |x_{jk} - x_{nm}| \geq \varepsilon\}
\]

has double natural density zero.
Theorem 7.3.2. A real double sequence \( x = (x_{jk}) \) is statistically convergent if and only if \( x \) is statistically Cauchy.

Proof. Let \( x \) be statistically convergent to a number \( \ell \). Then for every \( \varepsilon > 0 \), the set
\[
\{(j,k), j \leq n, k \leq m : |x_{jk} - \ell| \geq \varepsilon\}
\]
has natural density zero. We can choose two numbers \( N \) and \( M \) such that \( |x_{NM} - \ell| \geq \varepsilon \).

Now
\[
A = \{(j,k), j \leq n, k \leq m : |x_{jk} - x_{NM}| \geq \varepsilon\} \subseteq B \cup C,
\]
where
\[
B = \{(j,k), j \leq n, k \leq m : |x_{jk} - \ell| \geq \varepsilon\}
\]
and
\[
C = \{(j,k), j \leq n, k \leq m : |x_{NM} - \ell| \geq \varepsilon\}.
\]
Therefore \( \delta_2(A) \leq \delta_2(B) + \delta_2(C) = 0 \). Hence \( x \) is statistically Cauchy.

Conversely, let \( x \) be statistically Cauchy but not statistically convergent. Then there exist \( N \) and \( M \) such that the set
\[
K_{NM} = \{(j,k), j \leq n, k \leq m : |x_{jk} - x_{NM}| \geq \varepsilon\}
\]
has natural density zero. Hence the set
\[
E_{NM} = \{(i,s), i \leq n, s \leq m : |x_{is} - x_{NM}| < \varepsilon\}
\]
has natural density 1. Also
\[
E_{NM} \cap K_{NM} = \emptyset.
\]
Now, since $x$ is not statistically convergent and $\delta(E_{NM}) = 1$ so that from Theorem 7.2.3, the subsequence $(x_{is})$; $i, s \in E_{NM}$ does not converge to any number. Therefore there exists $\varepsilon_o > 0$ such that for all $N_o \in \mathbb{N}$

$$|x_{pq} - x_{is}| \geq \varepsilon_o, \forall p \geq i \geq N_o, \ q \geq s \geq N_o.$$ 

Now, consider the set

$$E'_NM = \{(p, q) : |x_{pq} - x_{NM}| \geq \varepsilon_o \} \neq \emptyset.$$ 

Then

$$\{(j, k) : |x_{jk} - x_{NM}| \geq \varepsilon_o \} \supseteq \{(p, q) : |x_{pq} - x_{NM}| \geq \varepsilon_o \}.$$ 

This implies that $\delta_2(E'_{NM}) = 0$, i.e. $E'_{NM} \subseteq K_{NM}$ which is a contradiction. Hence $x$ is statistically convergent.

This completes the proof of the theorem.

From Theorems 7.2.3 and 7.3.2 we can state the following for double sequences analogous to the result of Fridy [18].

**Theorem 7.3.3.** The following statements are equivalent:

1. $x$ is statistically convergent
2. $x$ is statistically Cauchy
3. there exists a set $K_{nm} = \{(j_1, k_1), \ldots, (j_n, k_m)\}$ such that $\delta_2(K_{nm}) = 1$ and

$$\lim_{n,m} x_{j_n,k_m} = 1.$$ 

**Corollary 7.3.4.** If $x$ is statistically convergent to $\ell$ then there exists a subsequence $y$ of $x$ such that

$$\lim y = \ell \text{ and } \delta_2\{(j, k) : x_{jk} = y_{jk}\} = 1.$$
7.4. Relation Between Statistical Convergence and Strongly Cesàro Summable Sequences

In [39], Moricz defined the means \( C_{11}, C_{10} \) and \( C_{01} \) of \( x = (x_{jk}) \) respectively by

\[
\sigma_{mn}^{11} = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk},
\]

\[
\sigma_{mn}^{10} = \frac{1}{m} \sum_{j=1}^{m} x_{jn},
\]

and

\[
\sigma_{mn}^{01} = \frac{1}{n} \sum_{k=1}^{n} x_{mk}.
\]

We say that a double sequence \( x = (x_{jk}) \) is \( C_{11} \)-summable or Cesàro summable to a finite limit \( \ell \) if the sequence \( (\sigma_{mn}^{11}) \) is convergent to \( \ell \) in Pringsheim’s sense, i.e.

\[
\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk} = \ell.
\]

Similarly \( C_{10} \) and \( C_{01} \) summable sequences are defined.

We can define the following as in case of single sequences.

**Definition 7.4.1.** Let \( x = (x_{jk}) \) be a double sequence and \( p \) be a positive real number. Then the double sequence \( x \) is said to be **strongly** \( p \)-Cesàro summable to \( \ell \) if

\[
\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{m} \sum_{k=1}^{n} | x_{jk} - \ell |^p = 0.
\]

We denote the space of all strongly \( p \)-Cesàro summable double sequences by \( w_p^2 \).
Remark 7.4.2. (i) If $0 < p < q < \infty$, then $w^2_q \subseteq w^2_p$ (by Hölder's inequality) and

$$w^2_p \cap l^2_\infty = w^2_1 \cap l^2_\infty \subseteq C_{11} \cap l^2_\infty.$$  

(ii) If $x$ is convergent but unbounded then $x$ is statistically convergent but $x$ need not be Cesàro nor strongly Cesàro.

Example 1. Let $x = (x_{jk})$ be defined as

$$x_{jk} = \begin{cases} k, & j = 1, \text{ for all } k, \\ j, & k = 1, \text{ for all } j, \\ 1, & \text{otherwise}. \end{cases}$$

Then $\lim_{j,k} x_{jk} = 1$ but

$$\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = \lim_{n,m} \frac{1}{nm} \left( \frac{1}{2} m + \frac{1}{2} m^2 + (m + 1) + \cdots + (m + (n - 1)) \right)$$

$$= \lim_{n,m} \frac{1}{nm} \left[ \frac{1}{2} m + \frac{1}{2} m^2 + (n - 1)m + \frac{1}{2} n^2 - \frac{1}{2} n \right]$$

which does not tend to a finite limit. Hence $x$ is not Cesàro. Also $x$ is not strongly Cesàro but

$$\lim_{n,m} \frac{1}{nm} \sum_{j} | (j,k) : | x_{jk} - 1 | \geq \varepsilon | = \lim_{n,m} \frac{m + n - 1}{nm} = 0,$$

i.e. $x$ is statistically convergent to 1.

(iii) If $x$ is a bounded convergent double sequence then it is also $C_{11}$, $w^2_p$ and $st_2$.

The following result is analogue of Theorem 2.1 [6].

Theorem 7.4.3. Let $x = (x_{jk})$ be a double sequence and $p$ be a positive real number. Then

(a) if $x$ is strongly $p$-Cesàro summable to $\ell$, then it is also statistically convergent to $\ell$,
Proof. (a) Let 

$$K_{nm} = \{(j, k), j \leq n, k \leq m : |x_{jk} - \ell|^p \geq \varepsilon\}.$$ 

Now since $x$ is strongly $p$-Cesàro summable to $\ell$ then

$$0 \leq \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} |x_{jk} - \ell|^p = \frac{1}{nm} \left\{ \sum_{j \in K_{nm}} \sum_{k \in K_{nm}} |x_{jk} - \ell|^p + \sum_{j \notin K_{nm}} \sum_{k \notin K_{nm}} |x_{jk} - \ell|^p \right\} + \sum_{j \notin K_{nm}} \sum_{k \in K_{nm}} |x_{jk} - \ell|^p$$

$$\geq \frac{1}{nm} \left\{ \sum_{j \in K_{nm}} \sum_{k \in K_{nm}} |x_{jk} - \ell|^p + \sum_{j \notin K_{nm}} \sum_{k \notin K_{nm}} |x_{jk} - \ell|^p \right\}$$

Hence $x$ is statistically convergent to $\ell$.

(b) Let 

$$I_{nm} = \{(j, k), j \leq n, k \leq m : |x_{jk} - \ell| \geq \left(\frac{\varepsilon}{4}\right)^{1/p}\}$$

and $M = \|x\|_{(\infty, 2)} + |\ell|$, where $\|x\|_{(\infty, 2)}$ is the sup-norm for bounded double sequences $x = (x_{jk})$.

Since $x$ is a bounded statistically convergent, we can choose $N = N(\varepsilon)$ such that for all $n, m > N$

$$\frac{1}{nm} \left\{ \{(j, k), j \leq n, k \leq m : |x_{jk} - \ell| \geq \left(\frac{\varepsilon}{4}\right)^{1/p}\} \right\} < \frac{\varepsilon}{4M^p}.$$
Now for all \( n, m > N \) we have

\[
\frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} |x_{jk} - \ell|^p = \frac{1}{nm} \left\{ \sum_{j \in I_{nm}} \sum_{k \in I_{nm}} |x_{jk} - \ell|^p + \sum_{j \in I_{nm}} \sum_{k \notin I_{nm}} |x_{jk} - \ell|^p + \sum_{j \notin I_{nm}} \sum_{k \in I_{nm}} |x_{jk} - \ell|^p \right\} < \frac{1}{nm} \frac{\varepsilon}{4M^p} K^p + \frac{1}{nm} \frac{\varepsilon}{4} + \frac{1}{nm} \frac{\varepsilon}{4} + \frac{1}{nm} \frac{\varepsilon}{4} = \varepsilon.
\]

Hence \( x \) is strongly \( p \)-Cesàro summable to \( \ell \).

**Remark 7.4.4.** Note that if a bounded sequence \( x \) is statistically convergent then it is also \( C_{11} \) summable but not conversely.

**Example 2.** Let \( x = (x_{jk}) \) be defined by

\[
x_{jk} = (-1)^j, \quad \forall k
\]

then

\[
\lim_{n,m} \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} = 0,
\]

but obviously \( x \) is not statistically convergent.