CHAPTER VI

ALMOST CONVERGENCE AND A CORE THEOREM FOR DOUBLE SEQUENCES

6.1. Introduction

By the convergence of a double sequence we mean the convergence in Pringsheim’s sense [47]. A double sequence \( x = (x_{jk})_{j,k=0}^{\infty} \) is said to be convergent in the Pringsheim’s sense or \( P \)-convergent if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{jk} - \ell| < \varepsilon \) whenever \( j, k > N \) and we denote by \( P - \lim x = \ell \). The number \( \ell \) is called the Pringsheim limit of \( x \).

More exactly we say that a double sequence \( (x_{jk}) \) converges to a finite number \( \ell \) if \( x_{jk} \) tends to \( \ell \) as both \( j \) and \( k \) tend to \( \infty \) independently of one another.

We denote the space of \( P \)-convergent sequences by \( C_2 \).

A double sequence \( x = (x_{jk}) \) is said to be Cauchy sequence if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{pq} - x_{jk}| < \varepsilon \) for all \( p \geq j \geq N \) and \( q \geq k \geq N \).

A double sequence \( x \) is bounded if there exists a positive number \( M \) such that \( |x_{jk}| < M \) for all \( j \) and \( k \), i.e. if

\[
\|x\|_{(\infty,2)} = \sup_{j,k} |x_{jk}| < \infty.
\]

We denote the set of all bounded double sequences by \( \ell_\infty^2 \).

Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded but every convergent real (or complex) double sequence is Cauchy.

Let \( A = (a_{jk}^{mn})_{j,k=0}^{\infty} \) be a doubly infinite matrix of real numbers for all \( m, n = 0, 1, \ldots \). Forming the sums

\[
y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk},
\]
called the \textit{A-means} of the double sequence \(x\), yields a method of summability.  

We say that a sequence \(x\) is \textit{A-summable} to the limit \(\ell\) if the A-means exist for all \(m, n = 0, 1, \cdots\) in the sense of Pringsheim’s convergence:

\[
\lim_{p,q \to \infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn},
\]

and

\[
\lim_{m,n \to \infty} y_{mn} = \ell.
\]

A two dimensional matrix transformation is said to be \textit{regular} if it maps every convergent sequence into a convergent sequence with the same limit. In 1926 Robinson [48] presented a four dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness:

A four dimensional matrix \(A\) is said to be \textit{bounded-regular} or \(RH\)-regular if it maps every bounded \(P\)-convergent sequence into a \(P\)-convergent sequence with the same \(P\)-limit.

The following is a four dimensional analogue of the well-known Silverman-Toeplitz theorem:

\textbf{Theorem 6.1.1} (Hamilton [25], Robinson [48]). The four dimensional matrix \(A\) is bounded-regular or \(RH\)-regular if and only if

\(RH_1\) : \(P - \lim_{m,n} a_{jk}^{mn} = 0 \ (j, k = 0, 1, \cdots)\);

\(RH_2\) : \(P - \lim_{m,n} \sum_{j,k=0}^{\infty} a_{jk}^{mn} = 1\);

\(RH_3\) : \(P - \lim_{m,n} \sum_{j=0}^{\infty} |a_{jk}^{mn}| = 0 \ (k = 0, 1, \cdots)\);

\(RH_4\) : \(P - \lim_{m,n} \sum_{k=0}^{\infty} |a_{jk}^{mn}| = 0 \ (j = 0, 1, \cdots)\);

\(RH_5\) : \(\sum_{j,k=0}^{\infty} |a_{jk}^{mn}| \leq C < \infty \ (m, n = 0, 1, \cdots)\), where \(C\) is constant.

Note that \(RH_1\) is a consequence of each of \(RH_3\) and \(RH_4\).
Recently in [45], Patterson extended the idea of Knopp’s core theorem for double sequences by defining the Pringsheim core as follows:

Let \( P - C_n\{x\} \) be the least closed convex set that includes all points \( x_{jk} \) for \( j, k > n \); then the Pringsheim core of the double sequence \( x = (x_{jk}) \) is the set

\[
P - C\{x\} = \bigcap_{n=1}^{\infty} [P - C_n\{x\}].
\]

Note that the Pringsheim core of a real-valued bounded double sequence is the closed interval \( [P - \lim \inf x, P - \lim \sup x] \).

In this regard, Patterson [45] proved the following:

**Theorem 6.1.2.** If \( A \) is a four dimensional matrix, then for all real-valued double sequences \( x \),

\[
(6.1.2.1) \quad P - \lim \sup Ax \leq P - \lim \sup x
\]

if and only if

\[
(6.1.2.2) \quad A \text{ is } RH\text{-regular};
\]

\[
(6.1.2.3) \quad P - \lim_{mn} \sum_{j,k=0,0}^{\infty,\infty} |a_{jk}^{mn}| = 1.
\]

In the present chapter we define the MR-core of a double sequence by using the idea of almost convergence introduced and studied by Moricz and Rhoades [40], and then proved an analogue of Theorem 6.1.2.

**6.2. Almost Convergence and MR-Core**

The notion of almost convergence for single sequences was introduced by Lorentz [35]. Recently Moricz and Rhoades [40] extended this idea for double sequences.
A double sequence \( x = (x_{jk})_{j,k=0}^{\infty} \) of real numbers is said to be **almost convergent** to a limit \( L \) if

\[
\lim_{p,q \to \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} |x_{jk} - L| = 0,
\]

that is, the average value of \((x_{jk})\) taken over any rectangle \(\{(j,k) : m \leq j \leq m + p - 1; n \leq k \leq n + q - 1\}\) tends to \(L\) as both \(p\) and \(q\) tend to \(\infty\), and this convergence is uniform in \(m\) and \(n\).

Note that a convergent single sequence is also almost convergent but for a double sequence this is not the case, that is, a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

Using the idea of almost convergence, Lorentz [35] introduced and characterized strongly regular matrices.

We say that a four dimensional matrix \(A\) is **strongly regular** if every almost convergent double sequence \(x\) is \(A\)-summable to the same limit, and the \(A\)-means are also bounded.

If a double sequence \(x\) is almost convergent to \(L\), then we write \(f_2 - \lim x = L\) and \(f_2\) for the space of almost convergent double sequences.

In [40], Moricz and Rhoades gave four dimensional analogue of strongly regular matrices as follows:

**Theorem 6.2.1.** Necessary and sufficient conditions for a matrix \(A = (a_{jk}^{mn})\) to be strongly regular are that \(A\) is bounded-regular and satisfies the following two conditions:

\[
MR_1 : \quad \lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} |\Delta_{10} a_{jk}^{mn}| = 0;
\]

\[
MR_2 : \quad \lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} |\Delta_{01} a_{jk}^{mn}| = 0,
\]

where \(\Delta_{10} a_{jk}^{mn} = a_{jk}^{mn} - a_{j+1,k}^{mn}\) and \(\Delta_{01} a_{jk}^{mn} = a_{jk}^{mn} - a_{j,k+1}^{mn}\), \((j,k = 0,1,\ldots)\).
We quote here the following useful lemma:

**Lemma 6.2.2** (Patterson [45]). If $A$ is a real or complex-valued four dimensional matrix such that $RH_3, RH_4,$ and

$$P - \limsup_{m,n} \sum_{j,k=0}^{\infty} |a_{jk}^{mn}| = M$$

hold, then for any bounded double sequence $x$ we have

$$P - \limsup |Ax| \leq M(P - \limsup |x|).$$

**6.3. Main Result**

We define the following:

Let us write

$$L^*(x) = \limsup_{p,q \to \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}. $$

Then we define the $MR$-core of a real-valued bounded double sequence $x$ to be the closed interval $[-L^*(-x), L^*(x)].$

Since every bounded convergent double sequence is almost convergent, we have

$$L^*(x) \leq P - \limsup x = L(x), \text{ say},$$

and hence it follows that

$$MR\text{-core}\{x\} \subseteq P\text{-core}\{x\}$$

for a bounded double sequence $x = (x_{jk})_{j,k=0}^{\infty}.$

Here we prove a core theorem for double sequences making use of four dimensional strongly regular matrices due to Moricz and Rhoades [40].
Theorem 6.3.1. For every bounded double sequence \( x \),

\[
\tag{6.3.1.1} L(Ax) \leq L^*(x)
\]

(or \( P\text{-core}\{Ax\} \subseteq MR\text{-core}\{x\} \)) if and only if

\[
\tag{6.3.1.2} A = (a_{jk}^{mn}) \text{ is strongly regular;}
\]

\[
\tag{6.3.1.3} P - \lim_{m,n \to \infty} \sum_{j,k=0,0}^{\infty} |a_{jk}^{mn}| = 1.
\]

Proof. Necessity. Let us consider a bounded double sequence \( x \) to be almost convergent to \( \ell \). Then we have \( L^*(x) = -L^*(-x) \). By (6.3.1.1), we get

\[
\ell = -L^*(-x) \leq -L(-Ax) \leq L(Ax) \leq L^*(x) = \ell.
\]

Hence \( Ax \) is \( P \)-convergent and \( P - \lim Ax = f_2 - \lim x = \ell \), and so \( A \) is strongly regular, i.e condition (6.3.1.2) holds.

Since every strongly regular matrix is also bounded-regular, by Lemma 6.2.2 there exists a bounded double sequence \( x \) such that \( \limsup |x| = 1 \) and \( P - \limsup Ax = C \), where \( C \) is defined by \( RH_5 \). Therefore we have

\[
1 \leq P - \liminf_{m,n \to \infty} \sum_{j,k=0,0}^{\infty} |a_{jk}^{mn}| \leq P - \limsup_{m,n \to \infty} \sum_{j,k=0,0}^{\infty} |a_{jk}^{mn}| \leq 1,
\]

i.e. condition (6.3.1.3) holds.

Sufficiency. Given \( \varepsilon > 0 \), we can find fixed integers \( p, q \geq 2 \) such that

\[
\tag{6.3.1.4} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} < L^*(x) + \varepsilon.
\]

Now as in [40], we can write

\[
\tag{6.3.1.5} y_{MN} = \sum_{j,k=0,0}^{\infty} a_{jk}^{MN} x_{jk} = \sum_{1} + \sum_{2} + \sum_{3} + \sum_{4} + \sum_{5} + \sum_{6} + \sum_{7} + \sum_{8},
\]
where

\[ \sum_1 = \frac{1}{pq} \sum_{m,n=0}^{\infty} a_{mn}^{MN} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} ^x j k \]

\[ \sum_2 = -\frac{1}{pq} \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} x_j k \sum_{m=0}^{j-p+1} \sum_{n=0}^{k} a_{mn}^{MN} \]

\[ \sum_3 = -\frac{1}{pq} \sum_{j=p-1}^{\infty} \sum_{k=q-1}^{\infty} x_j k \sum_{m=0}^{j-p+1} \sum_{n=0}^{k} a_{mn}^{MN} \]

\[ \sum_4 = -\frac{1}{pq} \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} x_j k \sum_{m=0}^{j-p+1} \sum_{n=0}^{k} a_{mn}^{MN} \]

\[ \sum_5 = -\sum_{j=p-1}^{\infty} \sum_{k=q-1}^{\infty} x_j k \left( \frac{1}{pq} \sum_{m=0}^{j-p+1} \sum_{n=0}^{k} a_{mn}^{MN} - a_{j k}^{MN} \right) \]

\[ \sum_6 = \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} a_{j k}^{MN} x_j k \]

\[ \sum_7 = \sum_{j=p-1}^{\infty} \sum_{k=q-1}^{\infty} a_{j k}^{MN} x_j k \]

\[ \sum_8 = -\sum_{j=0}^{p-2} \sum_{k=q-1}^{\infty} a_{j k}^{MN} x_j k . \]

Using the conditions of strong regularity of \( A \), we observe that \( (M, N \rightarrow \infty) \),

\[ | \Sigma_2 | \leq \| x \|_{(\infty, 2)} \sum_{m=0}^{p-2} \sum_{n=0}^{q-2} | a_{mn}^{MN} | \rightarrow 0 , \]

and

\[ | \Sigma_6 | \leq \| x \|_{(\infty, 2)} \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} | a_{j k}^{MN} | \rightarrow 0 , \text{ by } RH_1 ; \]

\[ | \Sigma_3 | \leq \| x \|_{(\infty, 2)} \sum_{m=0}^{\infty} \sum_{n=0}^{q-2} | a_{mn}^{MN} | \rightarrow 0 , \]

and

\[ | \Sigma_7 | \leq \| x \|_{(\infty, 2)} \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} | a_{j k}^{MN} | \rightarrow 0 , \text{ by } RH_3 ; \]

\[ | \Sigma_4 | \rightarrow 0 \text{ and } | \Sigma_8 | \rightarrow 0 \text{ by } RH_4 . \]
Now we have

\[ |\sum_{s}| \leq \frac{\|h\|_{(\infty, 2)}}{pq} \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} ((p - r - 1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{10} a_{jk}^{MN}| \]

\[ + (q - s - 1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\Delta_{01} a_{jk}^{MN}| \rightarrow 0 \quad \text{by } MR_{1} \text{ and } MR_{2}. \]

Therefore we have by (6.3.1.5)

\[ L(Ax) \leq \limsup_{M,N} \sum_{m,n=0,0}^{\infty,\infty} a_{mn}^{MN} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} \]

\[ \leq \limsup_{M,N} \sum_{m,n=0,0}^{\infty,\infty} \left( \frac{|a_{mn}^{MN}|}{2} + \frac{|a_{mn}^{MN}|}{2} - \frac{|a_{mn}^{MN}|}{2} \right) \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} \]

\[ \leq \limsup_{M,N} \left\{ \sum_{m,n=0,0}^{\infty,\infty} \left( \frac{|a_{mn}^{MN}|}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} \right) \right\} \]

\[ + \|x\|_{(\infty, 2)} \sum_{m,n=0,0}^{\infty,\infty} \left( |a_{mn}^{MN}| - \frac{a_{mn}^{MN}}{2} \right) \].

Now conditions \(RH_{1}, RH_{5}\) and (6.3.1.3) yield

\[ L(Ax) \leq L^{*}(x) + \epsilon. \]

Since \(\epsilon\) is arbitrary we finally have

\[ L(Ax) \leq L^{*}(x). \]

This completes the proof of the theorem.
6.4. Examples

6.4.1. Almost convergent sequences

(i) Define the double sequence \( x = (x_{jk}) \) by

\[
x_{jk} = \begin{cases} 1, & \text{if } j \text{ is odd, for all } k, \\ 0, & \text{otherwise.} \end{cases}
\]

Then \( x \) is almost convergent to \( \frac{1}{2} \).

(ii) Define \( x = (x_{jk}) \) by

\[
x_{jk} = (-1)^{j} \text{ for all } k.
\]

Then \( x \) is almost convergent to \( 0 \).

6.4.2. Strongly regular matrix

Define \( A = (a_{jk}) \) by

\[
a_{jk}^{mn} = \begin{cases} \frac{1}{m^2}, & \text{if } m = n \text{ and } j,k \leq m \text{ (even)}, \\ \frac{1}{m^2 - m}, & \text{if } m = n, j \neq k \text{ and } j,k \leq m \text{ (odd)}, \\ 0, & \text{otherwise.} \end{cases}
\]

We can easily verify that \( A \) is strongly regular, that is, conditions \( RH_1 - RH_5, MR_1 \) and \( MR_2 \) hold. Moreover, for the sequence in 6.4.1(i), we have

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} = a_{11}^{mm} x_{11} + a_{12}^{mm} x_{12} + \cdots + a_{1m}^{mm} x_{1m} + a_{21}^{mm} x_{21} + a_{22}^{mm} x_{22} + \cdots + a_{2m}^{mm} x_{2m}.
\]
\[ + a_{31}^{mm} x_{31} + a_{32}^{mm} x_{32} + a_{33}^{mm} x_{33} + \cdots + a_{3m}^{mm} x_{3m} \]

\[ + \left( a_{m-1,1} x_{m-1,1} + \cdots + a_{m-1,m} x_{m-1,m} \right) \]

\[ + a_{m1}^{mm} x_{m1} + \cdots + a_{mm}^{mm} x_{mm} \]

\[ = \frac{m}{m^2} \cdot \frac{m}{2} \quad \text{if } m \text{ is even,} \]

\[ \rightarrow \frac{1}{2} \quad \text{as } m, n \rightarrow \infty . \]

Similarly

\[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} = \frac{m - 1}{m^2 - m} \cdot \frac{m + 1}{2} \quad \text{if } m \text{ is odd,} \]

\[ \rightarrow \frac{1}{2} \quad \text{as } m, n \rightarrow \infty . \]

That is

\[ P - \lim Ax = \frac{1}{2} = f_2 - \lim x, \]

and so \( A \) transforms almost convergent sequence into convergent (\( P \)-convergent) to the same limit.
6.4.3. Bounded-regular matrix which is not strongly regular

In 6.4.2, \( A \) is strongly regular and so bounded regular. Let us define \( A = (a_{jm}^{mn}) \) as

\[
a_{jm}^{mn} = \begin{cases} 
\frac{2}{m^2}, & \text{if } m = n, j + k = \text{even}, \text{ and } j, k \leq m \text{ (even)}, \\
\frac{1}{m^2 - m}, & \text{if } m = n, j \neq k \text{ and } j, k \leq m \text{ (odd)}, \\
0, & \text{otherwise}.
\end{cases}
\]

Then \( A \) is bounded-regular but not strongly regular. Conditions \( RH_1 - RH_5 \) can easily be verified. But

\[
\lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{jm}^{mn} - a_{j+1,k}^{mn}| = \begin{cases} 
2, & \text{if } m \text{ is even}, \\
0, & \text{if } m \text{ is odd},
\end{cases}
\]

and also

\[
\lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{jm}^{mn} - a_{j,k+1}^{mn}| = \begin{cases} 
2, & \text{if } m \text{ is even}, \\
0, & \text{if } m \text{ is odd}.
\end{cases}
\]

Therefore conditions \( MR_1 \) and \( MR_2 \) do not hold and so \( A \) is not strongly regular.

6.4.4. In Theorem 6.3.1, strong regularity of \( A \) can not be replaced by bounded-regularity.

Consider the matrix \( A = (a_{jm}^{mn}) \) as defined in 6.4.3. This is bounded-regular but not strongly regular, and also

\[
P - \lim_{m,n} \sum_{j,k=0}^{\infty} |a_{jm}^{mn}| = 1,
\]

i.e condition (6.3.1.3) of Theorem 6.3.1 holds. Take the bounded double sequence \( x = (x_{jk}) \) defined by \( x_{jk} = (-1)^{j+k} \), which is almost convergent to zero, that is, \( L^*(x) = 0 \).
Now

\[ \sum_{j,k} a_{jk}^{mn} x_{jk} = \begin{cases} \frac{2}{m^2} \cdot \frac{m}{2} \cdot m, & \text{if } m \text{ is even}, \\ \frac{1}{m^2 - m} \cdot m, & \text{if } m \text{ is odd}. \end{cases} \]

Therefore

\[ \limsup_{m,n} \sum_{j,k} a_{jk}^{mn} x_{jk} = 1 \]

and

\[ \liminf_{m,n} \sum_{j,k} a_{jk}^{mn} x_{jk} = 0 \]

i.e. \( L(Ax) = 1 \). Hence \( L(Ax) > L^*(x) \), that is (6.3.1.1) does not hold.