CHAPTER 4

SOME GENERALIZATIONS OF BOOLEAN RINGS
4.1 INTRODUCTION: It is well-known that the Boolean condition, namely \( x^2 = x \) for all ring elements \( x \) renders a ring commutative. Trivially, in a Boolean ring \( R \) we have

(1) \( (xy)^2 = xy \) and (ii) \( (xy)^2 = yx \) for all \( x, y \in R \), but there exist non-Boolean rings satisfying the identities (1) and (ii). The present chapter concerns the commutativity of this larger class of rings. Recently the author jointly with Quadri and Ashraf [94] established that a semi prime ring \( R \) with \( (xy)^2 - yx \) central must be commutative. Section 4.2 begins with the same result but here we assume in the hypothesis that \( (xy)^2 - yx \) is central. In Theorem 4.2.2 we extend the mentioned result as follows: "Let \( n > 1 \) be a positive integer. If a semi prime ring \( R \) satisfies the polynomial identity \( [(xy)^n - yx, xy] = 0 \), then \( R \) is commutative". The last two sections deal with the commutativity of rings in which \( (xy)^n - xy \) or \( (xy)^n - yx \) is central.

In the end we remark that the following conjecture should also be possibly true: "Let \( n > 1 \) be a positive integer and \( R \) be an \( n \)-torsion free ring with unity satisfying one of the identities (i) \( [xy - (xy)^n, x] = 0 \), (ii) \( [xy - (yx)^n, x] = 0 \) for all \( x, y \in R \). Then \( R \) is commutative".
4.2 We know that a Boolean ring is necessarily commutative. Thus in a Boolean ring $R$ for any pair of elements $x, y \in R$, we have

\[(P_1) \quad (xy)^2 = xy. \]
\[(P_2) \quad (xy)^2 = yx. \]

Conditions $(P_1)$ and $(P_2)$ are weaker than the Boolean condition, namely $x^2 = x$. As an example, consider an abelian group $G$ having at least two elements and define trivial product, namely $xy = 0$ for all $x, y$ in $G$. It is easy to show that a ring satisfying $(P_1)$ also turns out to be commutative. The author with Quadri and Ashraf [94] recently established that a semi prime ring $R$ in which $(xy)^2 - xy \in Z(R)$ is commutative. The non-commutative ring of $3 \times 3$ strictly upper triangular matrices over the ring of integers satisfies the condition $(xy)^2 - xy \in Z(R)$ as well as $(xy)^2 - yx \in Z(R)$. Thus certainly arbitrary rings with either of the above conditions need not be commutative.

Though we shall attempt to prove a more general result but it would be worthwhile to begin with the following particular case.

**Theorem 4.2.1:** Let $R$ be a semi prime ring in which $(xy)^2 - yx$ is central for every $x$ and $y$ in $R$. Then $R$ is
In preparation for the proof of our theorem we need the following lemmas:

**Lemma 4.2.1**: Let \( R \) be a semi prime ring such that \((xy)^2 - yx\) is in \( Z(R) \), for all \( x, y \in R \). Then \( R \) has no non-zero nilpotent elements.

**Proof**: Let \( x \in R \) such that \( x^2 = 0 \). By our hypothesis, we have for any \( y \in R \),

\[(1) \quad ((xy)^2 - yx)y = y((xy)^2 - yx).\]

Now setting \( y = xy \) and using \( x^2 = 0 \) (1) yields \( (xy)^2 x = 0 \) i.e. \( (xy)^3 = 0 \).

If \( xR \neq (0) \), then this shows that \( xR \) is a non zero nil right ideal satisfying \( t^3 = 0 \) for all \( t \in xR \) and by Proposition 1.3.5, \( R \) has a non zero nilpotent ideal. This is a contradiction, since \( R \) is semi prime. Thus \( xR = (0) \) which implies that \( xRx = (0) \) and hence \( x = 0 \). This proves our lemma.

Further the above result together with Proposition 1.3.6, proves that:

**Lemma 4.2.2**: Let \( R \) be a prime ring satisfying the hypothesis of our theorem. If there exist integers \( m, n > 1 \)
such that \([a^m, b^n] = 0\) for all \(a, b\) in \(R\). Then \(R\) is commutative.

PROOF OF THEOREM 4.2.1: We may start with a prime ring \(R\) such that \((xy)^2 - yx\) is in \(Z(R)\) for all \(x, y\) in \(R\).

First we assert that \(Z(R) \neq (0)\). Assume on contrary that \(Z(R) = (0)\). Then we have

\[(1) \quad (xy)^2 - yx = 0 \quad \text{for all } x \text{ and } y \text{ in } R.\]

Replacing \(x\) by \(x+y\) in (1), we get

\[(2) \quad (xy^2 + y^2x)y = 0.\]

Putting \(x\) by \(x^r\) in (2), we obtain

\[(3) \quad (x^ry^2 + y^2xr)y = 0.\]

Since by (2) \(ry^2y = -y^2ry\), (3) becomes:

\[(xy^2 - y^2x)ry = 0 \quad \text{which gives } (xy^2 - y^2x)Ry = (0).\]

Now since \(R\) is prime, so either \(y = 0\) or \(xy^2 - y^2x = 0\).

But if \(y = 0\), then also \(xy^2 - y^2x = 0\). This implies that \(y^2\) is in \(Z(R) = (0)\), i.e. \(y^2 = 0\) for every \(y \in R\) which gives that \((x+y)^2y = 0\) for all \(x \in R\) thus \(yRy = (0)\). Again primeness of \(R\) yields that \(y = 0\) and consequently \(R = (0)\), a contra-
dictation. Hence \( Z(R) \neq (0) \).

Now let \( c \) be a nonzero element of \( Z(R) \). Replacing \( y \) by \( y+c \) in \((xy)^2 - yxyZ(R)\) we get \( c(x^2y + xyx) \in Z(R) \) and so by Lemma 3.2.1.

\[ x^2y + xyx \in Z(R) \text{ for every } x \text{ and } y \text{ in } R. \]

This gives

\[ x(x^2y + xyx) = (x^2y + xyx)x \]

or

\[ x(x^2y - yx^2) = 0 \text{ for all } x, y \in R. \]

Replacing \( ry \) for \( y \) in (4) we obtain

\[ x(x^2ry - ryx^2) = 0 \text{ for all } x, y \text{ and } r \in R. \]

But since (4) yields \( x^3r = xrx^2 \), we get from (5)

\[ xR(x^2y - yx^2) = (0) \text{ which gives } x = 0 \text{ or } x^2y = yx^2. \]

But \( x = 0 \) also gives \( x^2y = yx^2 \). Thus in every case \([x^2, y] = 0\). Hence by Lemma 4.2.2, \( R \) is commutative.

The above theorem is in fact a particular case of the following result:

**THEOREM 4.2.2**: Let \( R \) be a semi prime ring. If there exists a positive integer \( n > 1 \) such that \([(xy)^n - yx, xy] = 0 \) for all \( x, y \in R \), then \( R \) is commutative.

**PROOF OF THEOREM 4.2.2**: Simplifying the identity of
our theorem we get \([xy, yx] = 0\). We may assume that \(R\) is prime.

We first show that \(R\) has no nonzero nilpotent elements. Let \(a\) be a nonzero element of \(R\) such that \(a^2 = 0\).

By our hypothesis, we have

\[
a x^2 a = xa^2 x = 0 \quad \text{for all } x \in R.
\]

Thus we have

\[
(ax)^3 = (ax)^2 a + axaxa = (axa)^2 a + axa + axa
\]

\[
(\text{axa} (\text{xa} + xa) + ax(\text{xa} + xa))x
\]

\[
= a(xa + xa + xa + xa)x
\]

\[
= 0.
\]

If \(aR \neq (0)\), then the above shows that \(aR\) is a nonzero nil right ideal satisfying the identity \(t^3 = 0\) for all \(t\) in \(aR\). So by Proposition 1.3.5, \(R\) has a nonzero nilpotent ideal which is a contradiction since \(R\) is prime. Thus \(aR = (0)\) and primeness of \(R\) implies that \(a = 0\).

Further we claim that \(R\) has no proper zero divisors. Let \(xy = 0\). Then \((yx)^2 = y(xy)x = 0\) and by the fact proved
just now that \( R \) contains no nonzero nilpotent elements we have \( yx = o \). However \( xy = o \) implies \( x(yr) = o \) for all \( r \in R \) whence by above \( yrx = o \) forcing \( yRx = o \). Since \( R \) is prime we must have \( x = o \) or \( y = o \). Replacing \( y + y^2 \) for \( y \) in \([xy, yx] = o \) we obtain

\[
(1) \quad 2xy^3x = yx^2y^2 + y^2x^2y, \text{ for all } x, y \text{ in } R.
\]

Case 1: If \( \text{char } (R) \neq 2 \), (1) yields

\[
(2) \quad yx^2y^2 + y^2x^2y = o \text{ for all } x, y \in R.
\]

This together with \( xy^2x = yx^2y \) reduces to

\[
(3) \quad yx (xy^2 + y^2x) = o.
\]

Putting \( x = x - y \) in (3), we get

\[
(4) \quad y^2(xy^2 + y^2x) = o \text{ for all } x, y \in R.
\]

As established just now, \( R \) is free from proper zero divisors and since \( \text{char } (R) = 2 \) we have from (4),

\[
[x, y^2] = o \text{ for all } x \in R.
\]

Hence by Proposition 1.3.6 together with the fact that \( R \) contains no nonzero nilpotent elements this proves that \( R \) is commutative.
Case 2: If \( \text{char} (R) \neq 2 \). Then we replace \( y \) by \( x+y \) in the identity \( xy^2x = yx^2y \) to get

\[
(5) \quad [x^2, [x, y]] = 0 \quad \text{for every } x, y \in R.
\]

Using (5) and the fact that \( [x^2, y] = x[x, y] + [x, y]x \), we get

\[
[x^2, [x^2, y]] = x^2[x^2, y] - [x^2, y]x^2
\]

\[
= x^2(x[x, y] + [x, y]x) - (x[x, y] + [x, y]x)x^2
\]

\[
= x^3[x, y] + [x, y]x^3 - x^3[x, y] - [x, y]x^3
\]

\[
= 0.
\]

Hence

\[
(6) \quad [x^2, [x^2, y^2]] = 0 \quad \text{for all } x, y \text{ in } R.
\]

Now we have

\[
[x^2, [x^2, y^2]] - 2[x^2, y]^2 = x^4y^2 + y^2x^4 - 2(x^2y)^2 - 2(y^2x)^2 + 2yx^4y.
\]

\[
= (x^4y + yx^4 - 2x^2yx^2)y + y(x^4y + yx^4 - 2x^2yx^2)
\]

\[
= [x^2, [x^2, y]]y + y[x^2, [x^2, y]]
\]

\[
= 0.
\]

Thus,
Since $R$ has no nonzero zero divisors and $\text{char } (R) \neq 2$
(7) forces that $[x^2, y] = 0$. Hence by Proposition 1.5.6 $R$
is commutative.

4.3 In [36] Herstein proved a theorem which at the
same time generalizes the famous theorem of Jacobson
[Proposition 1.3.3] and also the result that any Boolean
ring is commutative. The theorem to which we refer is,
namely: any ring $R$ in which there exists a positive integer
$n > 1$ such that $x^n - x \in Z(R)$ for all ring elements $x$, is
necessarily commutative. As we have remarked earlier, a ring
$R$ with $(xy)^n - xy \in Z(R)$ need not be in general commutative,
even if $n = 2$. However we prove the following result:

**Theorem 4.3.3**: Let $R$ be a semi prime ring. Then
the following conditions are equivalent:

(a) $R$ is commutative.

(b) There exists a positive integer $n > 1$ such that
for every $x, y \in R$, $[(xy)^n - xy, x] = 0$.

(c) There exists a positive integer $m > 1$ such that
for every $x, y \in R$ $[(xy)^m - xy, y] = 0$.

We begin with the following:
LEMMA 4.3.3: A division ring R satisfying either (b) or (c) must be a field.

PROOF: First let us take the case when R satisfies (b). If x = 0, trivially we have \([x^n - y, x] = 0\). Let x be a nonzero element of R. Applying (b) for the elements \(x^{-1}y\) and \(x\) where \(y\) is an arbitrary element of R we get,

\[
[(x.x^{-1}y)^n - x.x^{-1}y, x] = 0
\]
or
\[
[y^n - y, x] = 0.
\]

Thus \(y^n - y \in Z(R)\) for all \(y \in R\). Hence R is commutative by [36, Theorem 18].

Similarly we conclude the result when R satisfies (c).

PROOF OF THEOREM 4.3.3: We may assume that R is a prime ring.

(a) implies (b) and (c) is obvious.

(b) \(\Rightarrow\) (a): Let R satisfy (b). Then proceeding as in previous theorem, we may prove that R contains no proper zero divisors. Thus by Proposition 1.3.11, R can be embedded in a division ring D satisfying (b) which is commutative by Lemma 4.3.2. Hence R is commutative.

(c) \(\Rightarrow\) (a), can be proved by the similar arguments.
We can also establish the following result:

**Theorem 4.3.4:** Let \( n > 1 \) be a fixed positive integer. If \( R \) is a semi prime ring satisfying \([xy]^n - yx, x] = 0\) for all \( x, y \in R \), then \( R \) is commutative.

**Proof of Theorem 4.3.4:** We first show that \( R \) is a reduced ring. Let \( a \in R \) such that \( a^2 = 0 \). Applying the identity \([xy]^n - yx, x] = 0\), to the elements \( x = xa \) and \( y = a \) we get \([ax]^n - axa, xa] = 0\) i.e. \((ax)^2a = 0\). This implies that \((ax)^3 = 0\) for all \( x \in R \). Hence \( aR \) is a nil right ideal in which for any \( z \in aR \), \( z^3 = 0 \) and so by Proposition 1.3.5, \( R \) contains a nonzero nilpotent ideal. This is not possible, because \( R \) is prime and so \( aR = 0 \) which forces \( a = 0 \).

Further through the same arguments as in Theorem 4.2.2 we conclude that \( R \) contains no nonzero zero divisors. Thus by Proposition 1.3.11, \( R \) can be embedded in a division ring satisfying the same polynomial identity. So we may assume that \( R \) is a division ring. Let \( x, y \) be any nonzero elements of \( R \). By our hypothesis, we have

\[
[(x, x^{-1} y)^n - x^{-1} y, x, x] = 0
\]

or

\[
[y^n - x^{-1} y x, x] = 0
\]

i.e.

\[
xy^n - yx = y^n x - x^{-1} y x^2
\]
Multiplying by $x$ on left we obtain

$$x^2y^n - xy^nx = xyx - yx^2$$

or

(1) \[ x[x, y^n] = [x, y]x. \]

Replacing $x+y$ for $x$ in (1), we get

$$(x+y)[x, y^n] = [x, y] (x+y).$$

Using (1) this becomes

(2) \[ y[x, y^n] = [x, y] y. \]

Replacing $y^2 + x$ for $x$ in (1), we obtain

(3) \[ y^2[x, y^n] = [x, y] y^2. \]

Multiply by $y$ on left (2) gives

(4) \[ y^2[x, y^n] = y[x, y] y. \]

(3) and (4) yield,

$$y[x, y]y = [x, y] y^2$$

or

$$(y[x, y] = [x, y]y)y = 0.$$ Since $R$ has no zero divisors, we have

(5) \[ y[x, y] = [x, y]y. \]
If char \((R) \neq 2\), then by Proposition 1.3.7, \(R\) is commutative.

If char \((R) = 2\), then (5) gives \([x,y^2] = 0\) and \(R\) is commutative by Proposition 1.3.6.

4.4 We now go to the next step of extending the results of Theorem 4.3.3 and Theorem 4.3.4 to the case when ring \(R\) is not necessarily semi prime. In this direction we prove the following:

**Theorem 4.4.5:** Let \(n > 1\) be a positive integer and \(R\) be a ring with unity \(1\) in which the additive group \(R^+\) is nil torsion free. Suppose that \(R\) satisfies one of the following identities:

(i) \([x(y)^n - xy,x] = 0\), for all \(x,y \in R\)

(ii) \([x(y)^n - yx,x] = 0\), for all \(x,y \in R\).

Then \(R\) is commutative.

**Proof of Theorem 4.4.5:** (i) Let \(\lambda\) be an integer.

Replace \(x\) by \(x + \lambda \cdot 1\) in the identity (i). Then on expanding we get polynomials

\[P_i(x,y), \ i = 1,2, \ldots, n.\]

with integral coefficients such that

\[\lambda P_1(x,y) + \lambda^2 P_2(x,y) + \ldots + \lambda^n P_n(x,y), x] = 0.\]
Put \( \lambda = 1, 2, \ldots, n \) and note that the polynomials \( P_i \)'s do not depend on \( \lambda \). Thus we get

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^n \\
n & n^2 & \cdots & n^n \\
\end{bmatrix}
\begin{bmatrix}
[P_1(x, y), x] \\
[P_2(x, y), x] \\
[P_n(x, y), x] \\
\end{bmatrix}
= 
\begin{bmatrix}
o \\
o \\
o \\
\end{bmatrix}
\]

Since \( R \) is \( n \)-torsion free, the matrix on the left is invertible and so we have \([P_i(x, y), x] = 0\) for \( i = 1, 2, \ldots, n \) and all \( x, y \in R \). But \( P_n(x, y) = y^n \) forcing that \([y^n, x] = 0\) for all \( x, y \in R \). Hence by [16, Theorem 4] \( R \) is commutative.

(ii) The proof for identity (ii) is essentially the same.

Before closing, the author should like to point out a possible generalization of the above theorem as follows:

**CONJECTURE:** If \( R \) is \( n \)-torsion free ring with unity 1 satisfying one of the condition (i) and (ii), then \( R \) is commutative.