CHAPTER 3

SOME COMMUTATIVITY THEOREMS FOR SEMI PRIME RINGS
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3.1 INTRODUCTION: In this chapter we consider some polynomial identities which do not imply commutativity in arbitrary rings. However, we succeed to establish the commutativity of semi prime rings in all these cases. In the following chapters we shall again turn to some commutativity conditions for semi prime rings. Those results are not included in this chapter because their setting is considerably more appropriate at those places only. Most of the proofs here fall into two distinct parts. First, we study the prime ring cases of our theorems. For this purpose we make use of the following:

(1) In a prime ring if the product of two elements is central of which one is a nonzero central element, then the other element is also central.

(2) Centre of a prime ring with a polynomial identity is non-trivial (which, we demonstrate in each case).

Finally, we obtain elementary reductions of the results for semi prime rings to the corresponding results for prime rings, using structure theory of subdirect sums. But at some places (Theorem 3.35 and 3.36) we achieve this goal through division rings.

The first half of the chapter contains the results
whose developments depend mainly on Chapter 2. In section 3.2 we establish that if \( R \) is a semi prime ring satisfying either of the identities (i) \( x^2oy = xoy^2 \in Z(R) \)
(ii) \( [x^2, y] + [x, y^2] \in Z(R) \)
(iii) \( [x^2, y] - [x, y^2] \in Z(R) \), then in each case \( R \) must be commutative. In section 3.3 we discuss the commutativity of rings with \( (xy)^n = xy^n x \) for all ring elements \( x, y \) and a fixed positive integer \( n > 1 \).

The result obtained in section 3.3 is further generalized in section 3.4 as follows: Let \( n \) be a fixed positive integer larger than 1 and \( R \) be a semi prime ring in which \( (xy)^n - xy^n x \in Z(R) \) for all \( x, y \in R \). Then \( R \) is commutative."

Although Theorem 3.3.5 should have been avoided in view of its direct generalization in Theorem 3.3.6 but the proofs given in two cases are so different in techniques that we feel it would be of some interest to present them one by another. Moreover the situation might be considered necessary for the natural development of the study as well.

In section 3.5 we reexamine the identity \( (xy)^2 = x^2 y^2 \) of the well-known theorem due to Johnson, Outcalt and Yaqub [61] expressing it alternatively as \( x[y, xy] = 0 \). In fact, we replace the polynomial \( x[y, xy] \) by \( x(yoxy) \) and prove the following theorem: "If \( R \) is a semi prime ring satisfying \( x(yoxy) \in Z(R) \), for all \( x, y \in R \), then \( R \) is commutative." The main result proved in section 3.6 stems from a theorem of Bell [11] that for a
fixed positive integer $n > 1$ if a ring $R$ is generated by $n$th powers of its elements and satisfies the identity $[x^n, y] = [y, x^n]$, then $R$ is commutative.

3.2 : In section 2.4 of the previous chapter, we have proved that any 2-torsion free ring $R$ with unity in which $x^2oy = xoy^2$, for every $x$ and $y$ in $R$, is necessarily commutative [Theorem 2.4.3 (b)]. We generalize the mentioned result as follows for semi prime ring :

**THEOREM 3.2.1** : Let $R$ be a semi prime ring in which $x^2oy - xoy^2$ is central for every $x$ and $y$ in $R$. Then $R$ is commutative.

We begin with the following lemmas :

**LEMMA 3.2.1** : Let $R$ be a prime ring and $x \neq 0$ be an element in $Z(R)$. If for any $y \in R$, $xycZ(R)$, then $ycZ(R)$.

**PROOF** : Since $x$ and $xy \in Z(R)$, we have for any $zcZ(R)$.

\begin{align*}
(1) \quad & xz = zx \\
(2) \quad & (xy)z = z(xy) \\
(2) \text{ together with (1) yields that } \quad & xyz = xzy, \text{ for all } zcR.
\end{align*}

Let $rsR$ be an arbitrary element. Replacing $z$ by $rz$ in (3) we get,
\[
x r z y = (x y r) z \\
= (x r y) z, \text{ from (3)}
\]
whence we obtain
\[
x r(z y - y z) = 0, \text{ for all } r, z \in R.
\]
Consequently \(x R(z y - y z) = 0\) for all \(z \in R\).

But since \(x \neq 0\), primeness of \(R\) forces that \(z y - y z = 0\) for all \(z \in R\). Hence \(y z \in Z(R)\).

**LEMMA 3.2.2**: Let \(R\) be a prime ring with \(x^2 o y - x o y^2 \in Z(R)\) for every \(x\) and \(y\) in \(R\). Then \(R\) is commutative.

**PROOF**: Replacing \(x\) by \(x + y\) in \(x^2 o y - x o y^2 \in Z(R)\), we get
\[
(1) \quad x y^2 + y^2 x + 2 y x z \in Z(R), \text{ for every } x \text{ and } y \text{ in } R.
\]
Putting \(y - x\) in place of \(y\) in (1), this gives
\[
(2) \quad x^2 y + y x^2 - 2 x y x z \in Z(R).
\]
Now we assert that \(Z(R) \neq (0)\). Assume on contrary that \(Z(R) = (0)\). In that case,
\[
(3) \quad x^2 y + y x^2 - 2 x y x = 0 \text{ for every } x \text{ and } y \text{ in } R.
\]
whence it follows \(x(y y - y x) = (y y - y x)x\). If \(\text{char } (R) \neq 2\), then by Proposition 1.3.7, we have \(x z \in Z(R)\) which forces \(R = (0)\),
a contradiction. If char (R) is 2, then (3) yields $x^2 \in Z(R)$, for every $x \in R$. Now putting $x = x + y$, we get $x \in Z(R) = (0)$, forcing $R = (0)$, again a contradiction. Hence we find $Z(R) \neq (0)$.

Now replacing $y$ by $xy$ in (2), we get $x(x^2 y + yx^2 - 2xyx)$ in $Z(R)$ and by Lemma 3.2.1 $x \in Z(R)$ unless $x^2 y + yx^2 - 2xyx = 0$. But if $x \in Z(R)$ then also $x^2 y + yx^2 - 2xyx = 0$ and so in every case

(4) $x^2 y + yx^2 - 2xyx = 0$, for all $x$ and $y$ in $R$.

If $R$ is 2-torsion free, then proceeding as above we get $x \in Z(R)$, and thus $R$ is commutative.

Suppose that char (R) = 2. Then (4) gives $x^2 \in Z(R)$ for every $x \in R$. Thus $(x + y)^2 \in Z(R)$ for any $x$ and $y$ in $R$ which implies that

(5) $xy + yx \in Z(R)$

Replacing $y$ by $xy$ in (5), we get

$x(xy + yx) \in Z(R)$

Then by Lemma 3.2.1 $x$ is in $Z(R)$ unless $xy + yx = 0$. But if $xy + yx = 0$ also implies $x \in Z(R)$, as char (R) is 2. Hence $R$ is commutative.

Further if $R$ is a semi prime ring satisfying
\(x^2oy - xoy^2 \in Z(R)\) for all \(x\) and \(y\) in \(R\), then \(R\) is isomorphic to a subdirect sum of prime rings \(R_x\), each of which as a homomorphic image of \(R\) satisfies the hypothesis placed on \(R\), and by Lemma 3.2.2 each of \(R_x\) is commutative. Hence \(R\) is commutative which proves our theorem.

The following example demonstrates that the Theorem 3.2.1 cannot be generalized for arbitrary rings.

**Example 3.2.1:** Let \(M_n\) be the ring of \(n \times n\) matrices over a division ring \(M\). Consider \(A_3 = \{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \} \). Then \(A_3\) is a non-commutative nilpotent ring of index 3 in which \(x^2oy - xoy^2\) is central for every \(x, y \in A_3\).

We also prove the following result:

**Theorem 3.2.2:** Let \(R\) be a semi prime ring satisfying the polynomial identity \([([x, y] + [x, y^2], Z] = 0\). Then \(R\) is commutative.

**Proof of Theorem 3.2.2:** We may assume that \(R\) is a prime ring with \([x^2, y] + [x, y^2] \in Z(R)\), for every \(x\) and \(y\) in \(R\). First we assert that \(Z(R) \neq (0)\). Suppose on contrary that \(Z(R) = (0)\), then we have

\[(1) \quad [x^2, y] + [x, y^2] = 0, \quad \text{for all} \ x \text{and} \ y \text{in} \ R.\]

Replacing \(x\) by \(x + y\) in (1) we get
\[\quad [x^2, y] + 2[x, y^2] = 0\]
which yields \([x, y^2] = o\), for all \(x, y\) in \(R\) and so \(y^2\) in \(Z(R)\) for all \(y\) in \(R\). Then \((x+y)^2 x = o\) i.e. \(xyx = o\), for all \(y\) in \(R\). This implies that \(xRx = (o)\). Now since \(R\) is prime, we get \(R = (o)\), a contradiction. Hence \(Z(R) \neq (o)\).

Let \(\text{char } (R) \neq 2\) and \(c\) be a nonzero element of \(Z(R)\). Replacing \(x\) by \(x+c\) in the identity of the hypothesis:

\[
(2) \quad [x^2, y] + [x, y^2] \in Z(R)
\]

we get

\[
2c [x, y] \in Z(R)
\]

which gives readily that \(c[x, y] \in Z(R)\), since \(R\) is 2-torsion free and so by Lemma 3.2.1, we have

\[
(3) \quad [x, y] \in Z(R), \text{ for all } x \text{ and } y \text{ in } R.
\]

Now with \(x\) by \(xy\) (4) gives

\[
(4) \quad [x, y] yz \in Z(R) \text{ for all } x \text{ and } y \text{ in } R.
\]

Using Lemma 3.2.1 again, (3) and (4) yield \(yz \in Z(R)\), unless \([x, y] = o\), but if \(yz \in Z(R)\) then also \([x, y] = o\) and so in every case \([x, y] = o\) for all \(x, y\) in \(R\). Hence \(R\) is commutative.

Next if \(\text{char } (R) = 2\), then replacing \(x\) by \(x+y\) in (2), we get

\[
(5) \quad [x, y^2] \in Z(R).
\]
Now with \( x = xy \), (5) gives

\[
(6) \quad [x, y^2] y \in Z(R).
\]

By Lemma 3.2.2, (5) and (6) yield that \( y \in Z(R) \), unless

\([x, y^2] = 0\) which implies that \( y^2 \in Z(R) \), for every \( y \) in \( R \),

and so \((x+y)^2 \in Z(R)\) that is \( xy +yx \in Z(R) \). Since \( \text{char} (R) \)

is 2 we get \([x, y] \in Z(R)\) which is (3) and so by the same

arguments as in previous case, we have \( R \) is commutative.

Now we can derive the following corollary which is,
of course, a result due to Gupta [32].

**COROLLARY 3.2.1**: Let \( R \) be a semi prime ring in which

\([x^2, y] - [x, y^2] \in Z(R)\) for all \( x \) and \( y \) in \( R \). Then \( R \) is commu-
tative.

**PROOF**: If \( \text{char} (R) = 2 \), we are through. Let us assume

that \( \text{char} (R) \neq 2 \). For \( o \neq c \in Z(R) \), replacing \( x \) by \( x + c \)

in identity of Corollary 3.2.1 and using the identity itself

we get \( 2c [x, y] \in Z(R) \). Now proceeding on the same lines as

in case of the proof of the Theorem 3.2.2, we get that \( R \) is

commutative.

The existence of non-commutative ring \( M_3 \) of \( 3 \times 3 \)
upper triangular matrices over the ring \( Z \) of integers in which

the identity of the Theorem 3.2.2 and that of Corollary 3.2.1
are satisfied, rules out the possibility of extending these
results to arbitrary rings.

3.3 In the beginning of our chapter 2, we have proved that a ring with unity 1, satisfying the polynomial identity \((xy)^2 = xy^2x\) must be commutative (Theorem 2.2.1). We have also seen that the restriction of unity on the hypothesis cannot be dropped in case of arbitrary rings. In a natural way, one can generalize the mentioned identity as follows:

\[(A) \quad (xy)^n = x^p y^n x^q, \text{ where } p, q \text{ are positive integers such that } p + q = n \geq 3.\]

The following example shows that a ring (even with unity) satisfying \((A)\) may be badly non-commutative.

**Example 3.3.2**: Let \( R = \{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} / a, b, c, d \in GF(p) \} \)

where \( p \) is a prime such that \( p \) divides \( n \) if \( n \) is odd and \( p \) divides \( n/2 \) if \( n \) is even, for each integer \( n \geq 3 \) and all \( x, y \in R \), it can be verified that \((xy)^n = x^p y^n x^q\) holds for all \( p + q = n \) but \( R \) is not commutative. However \( R \) contains unity.

Now we extend the identity \((xy)^2 = xy^2x\) as follows:

\[(B) \quad (xy)^n = xy^n x, \text{ } n \text{ a positive integer greater than 1.}\]

We can easily prove the following result:
THEOREM 3.3.3: Let \( n \) be a fixed positive integer larger than 1. If \( D \) is a division ring satisfying (B), then \( D \) is commutative.

PROOF OF THEOREM 3.3.3: Let \( x \) and \( y \) be any nonzero elements of \( D \). Then \((xy)^n = xy^n\) yields that

\[(1) \quad (xy)^{n-1} = y^{n-1}x\]

Putting \( xy^{-1} \) in place of \( x \) in the given identity we get

\[x^n = xy^{n-1}xy^{-1}\]

i.e.

\[x^ny - xy^{n-1}x = 0\]

or

\[x(x^{n-1}y - y^{n-1}x) = 0\]

This implies that

\[(2) \quad x^{n-1}y - y^{n-1}x = 0\]

From (1) and (2) we obtain

\[(3) \quad (yx)^{n-1} - (xy)^{n-1} = 0\]

Again, replacing \( y \) by \( x^{-1}y \) in (3) yields:

\[x^{-1}y^{n-1}x - y^{n-1} = 0\]

or
Hence by Proposition 1.3.6, $D$ is commutative.

We are now in a position to start step-by-step climb to establish the result for semi prime rings using the structure theory for rings as our ladder. In preparation to do so, we first establish the following lemma:

**Lemma 3.3.3:** A prime ring $R$ satisfying (B) for a fixed positive integer $n > 1$ contains no nonzero zero divisors.

**Proof:** It suffices to show that $R$ is a reduced ring. Let $a$ be an element of $R$ such that $a^2 = 0$. In the identity of the hypothesis, putting $a$ in place of $x$, we have

$$(ay)^n a = ay^n a^2 = 0,$$

which implies that $(ay)^{n+1} = 0$, for all $y$ in $R$. Thus $aR$ is a right ideal of $R$ in which each element is nilpotent and hence by Proposition 1.3.5, $aR = 0$, which forces $a = 0$, since $R$ is a prime ring.

Now since a prime ring $R$ satisfying (B) contains no zero divisor, by Proposition 1.3.11, $R$ can be embedded in a division ring $D$ satisfying (B). But $D$ is commutative by Theorem 3.3.3 so the prime ring $R$ with (B) is also commutative. Thus we obtain:

**Theorem 3.3.4:** Let $R$ be a prime ring satisfying (B) for a fixed positive integer $n > 1$. Then $R$ is commutative.
Let us now turn to the case that $R$ is semi prime and satisfies (B). Then, $R$ is a subdirect sum of prime rings $R_i$, each of which as a homomorphic image of $R$ satisfies the polynomial identity (B) and Theorem 3.5.5 forces each of $R_i$ to be commutative. So we have proved:

**Theorem 3.3.5**: Let $R$ be a semi prime ring in which there exists a fixed positive integer $n$ larger than 1 such that $(xy)^n = xy^nx$ for all $x,y \in R$. Then $R$ is commutative.

3.4 In an attempt to further generalize our Theorem 3.5.5, we first prove the following lemma:

**Lemma 3.4.4**: Let $R$ be a prime ring satisfying

$$(xy)^n - xy^nx \in Z(R),$$

for a fixed integer $n > 1$ and all $x,y \in R$. Then $R$ is a reduced ring.

**Proof**: Let $a$ be an element of $R$ such that $a^2 = 0$. In our hypothesis, we replace $x$ by $a$ and $y$ by $x$ to get

$$(ax)^n - ax^n a \in Z(R).$$

This implies that,

$$a((ax)^n - ax^n a) = ((ax)^n - ax^n a)a$$

which gives,

$$(ax)^n a = 0 \text{ i.e. } (ax)^{n+1} = 0 \text{ for all } x \text{ in } R.$$  

It follows that $aR$ is a right ideal of $R$ in which $z^{n+1} = 0$ for each $z \in aR$. Thus $aR = (0)$, by Proposition 1.3.5. This
implies that $x = 0$, since $R$ is prime. This proves the assertion in our lemma.

Further, if a prime ring $R$ satisfies the polynomial identity

$$q(x, y, z) = [(xy)^n - xy^nx, z]$$

$$= (xy)^nz - xy^n xz - z(xy)^n + zxy^nx = 0.$$  

whose coefficients are coprime integers, then no $2 \times 2$ matrix ring over $GF(p)$, $p$ a prime satisfies the identity $q(x, y, z) = 0$. As a consideration we may take $x = (\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} )$, $y = (\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} )$ and $z = (\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} )$. Thus by Proposition 1.3.1, $R$ has a nil commutator ideal. But by Lemma 3.4.4 in a prime ring $R$ with identity $q(x, y, z) = 0$, $R$ has no non zero nilpotent elements which forces that the commutator ideal of $R$ must be zero. Hence prime ring $R$ is commutative.

Now using structure theory for semi prime rings, we conclude as follows:

**THEOREM 3.4.6**: Let $n$ be a fixed positive integer larger than $1$ and $R$ be a semi prime ring in which $(xy)^n - xy^nx \in Z(R)$, for all $x$ and $y$ in $R$. Then $R$ is commutative.

3.5 The identity of the result of Johnsen, Outcalt and Yaqub [61] namely, $(xy)^2 = x^2y^2$ can be rewritten as
In section 2.3 of chapter 2 we have discussed the induced Lie product \([x,y]\) and the Jordan product \(xoy\). It may, therefore, be natural to question whether a ring satisfying \(x(yoxy) = 0\) is also commutative. There are examples of rings which show that in general the rings with \(x(yoxy) = 0\) need not be commutative. However we prove the following:

**Theorem 3.5.7:** Let \(R\) be a semi prime ring (may not have unity) in which \(x(yoxy)\) is central for all \(x,y \in R\). Then \(R\) is commutative.

**Proof of Theorem 3.5.7:** Let \(R\) be a prime ring. We assert that \(Z(R) \neq \{0\}\). Assume on contrary that \(Z(R) = \{0\}\). In this case \([z, x(yoxy)] = 0\) gives,

\[
(1) \quad x(yoxy) = 0 \quad \text{for all } x \text{ and } y \text{ in } R.
\]

Replacing \(x\) by \(x+y\) in (1), we get

\[
(2) \quad 2xy^3 + y^2xy + yxy^2 = 0
\]

\[
(3) \quad x(2xy^2 + y^2x + yxy)y = 0 \quad \text{for all } x \text{ and } y \text{ in } R.
\]

Also (1) gives \((xy)^2 = -x^2y^2\) and thus (3) yields,

\[
(4) \quad xy(yx - xy)y = 0.
\]

Again replacing \(x+y\) for \(x\), in (4) we obtain
\[ xy(yx - xy)y + y^2(yx - xy)y = 0 \]

Now, using (4) this becomes

\[ y^2(yx - xy) y = 0 \]

If \( y^2(yx - xy) \neq 0 \), then \( y^2(yx - xy) \) is nonzero nilpotent element of index 2. Now (1) gives

\[ [y^2(yx - xy)r]^2 + [y^2(yx - xy)]^2r^2 = 0 \]

i.e.

\[ [y^2(yx - xy)r]^2 = -[y^2(yx - xy)]^2r^2 \]

\[ = 0. \]

Hence \( y^2(yx - xy) \) is a right ideal in which the square of every element is zero. By Proposition 1.3.5, \( R \) has a nonzero nilpotent ideal. But primeness of \( R \) follows that

\[ y^2(yx - xy) = 0 \]

Putting \( x = ry \) in (6), we get \( y^2(ryx - rxy) = 0 \) which together with (6) gives \( y^2r(yx - xy) = 0 \). Hence \( y^2R(yx - xy) = (0) \).

Now \( R \) is a prime ring and so we have \( y^2 = 0 \) or \( yx - xy = 0 \) for all \( x \) and \( y \) in \( R \).

But if for some \( y^2 \neq 0 \), then \( yx - xy = 0 \) for all \( x \) in \( R \) and so \( y \) in \( Z(R) = (0) \), a contradiction and thus \( y^2 = 0 \) for all \( y \) whence \( R \) is nil of bounded index and therefore not prime [78], a contradiction. Hence we conclude that \( Z(R) \neq (0) \).
Now let \( r \) be a nonzero element in \( Z(R) \). Replacing \( x \) by \( x+r \) in \( x(yoxy) \in Z(R) \), we get \( r(3xy^2 + yxy) \in Z(R) \). By Lemma 3.2.1 we have

\[
(7) \quad 3xy^2 + yxy \in Z(R)
\]

If \( \text{char}(R) \neq 2 \), then (7) yields

\[
(8) \quad (xy - yx)y \in Z(R).
\]

Putting \( x = xy \) in (8), we obtain

\[
(9) \quad (xy - yx)y^2 \in Z(R).
\]

Now since \( R \) is prime (8) and (9) give \( y \in Z(R) \), by Lemma 3.2.1 unless \( (xy - yx)y = 0 \). But if \( y \in Z(R) \) then also \( (xy - yx)y = 0 \). Hence in every case we have

\[
(10) \quad (xy - yx)y = 0 \quad \text{for all } x \text{ and } y \text{ in } R.
\]

Now replacing \( x \) by \( z \) and \( xz \) in (10) we get respectively

\[
(zy - yz)y = 0 \quad \text{and} \quad (xzy - yxz)y = 0,
\]

which in turn give

\[
(xy - yx)zy = 0 \quad \text{for all } x, y, z \in R \text{ and we have } (xy - yx)Ry = (0).
\]

Now \( R \) is nonzero prime ring and so we have \( xy - yx = 0 \) for all \( x \) and \( y \) in \( R \). Hence \( R \) is commutative.

If \( \text{char}(R) \neq 2 \), then with \( x = y \) in (7) we get \( 4y^2 \in Z(R) \), so \( 4[y^3, z] = 0 \) for all \( z \) and thus \( [y^3, z] = 0 \) whence \( y^3 \in Z(R) \). Now putting \( x = y \) in \([z, x(yoxy)] = 0, \)

we get \( 2y^4 = y(y^3 + y^3) = y(y^2)y^2 \in \mathbb{Z}(R) \) so \( 2[y^4, w] = 0 \) for all \( w \) and hence \( [y^4, w] = 0 \) whence \( y^4 \in \mathbb{Z}(R) \). Thus we have \( y^3 \) and \( y^4 \in \mathbb{Z}(R) \), and by lemma 3.2.1, either

\[(11) \quad y^3 = 0 \text{ or } y \in \mathbb{Z}(R).\]

Now we assert that \( y^3 = 0 \) implies \( y^2 = 0 \). Suppose on contrary that \( y^2 \neq 0 \) but \( y^3 = 0 \). Then \( \mathbb{Z}(R) \) contains

\[y^2(xy^2x) = y^2(xy^2x + y^2x^2) = y^2x \cdot y^2x + y^4x^2 = (y^2x)^2.\]

Thus in particular, \( (y^2x)^2y = y(y^2x)^2 = 0 \) or \( (y^2x)^3 = 0 \). Let \( J = y^2R \). Then \( J \neq 0 \) as \( y^2 = 0 \) and \( R \) is prime. But \( J \) is a nonzero right ideal in which for each \( j \in J \), \( j^3 = 0 \) and consequently by Proposition 1.3.5 \( R \) is not prime, a contradiction. Hence \( y^2 = 0 \). Proceeding on the same lines we can show that \( y = 0 \) whenever \( y^2 = 0 \). Thus we conclude that if \( y^3 = 0 \), then \( y = 0 \). Hence in every case \( y \in \mathbb{Z}(R) \) and \( R \) is commutative.

Further if \( R \) is semi prime ring satisfying

\([z, x(yoxy)] = 0\), then by using the same argument as used in the proof of theorem 3.2.1, we may assume that \( R \) is prime ring satisfying \([z, x(yoxy)] = 0\). Hence \( R \) is commutative.

3.6 In his paper [11] Bell proved that "for a fixed positive integer \( n > 1 \), a ring \( R \) generated by the \( n \)-th power of its elements and satisfying the identity \([x^n, y] = [x, y^n]\) is commutative." Also recently Hermanci [34] proved that:
THEOREM 3.6.8 [34]: If R is a ring with unity and satisfying the identities:

\[[x^n, y] = [x, y^n] \text{ and } [x^{n+1}, y] = [x, y^{n+1}] \text{ for all } x, y \in R\]

where \( n > 1 \) is a fixed integer, then R is commutative.

We extend the above result of Hermanci [34] for semi prime rings as follows:

THEOREM 3.6.9: Let R be a semi prime ring with unity satisfying:

(i) \([x^n, y] + [x, y^n] \in \mathcal{Z}(R)\)

(ii) \([x^{n+1}, y] + [x, y^{n+1}] \in \mathcal{Z}(R)\) for all \( x, y \in R \)

where \( n > 1 \) is a fixed integer, then R is commutative.

PROOF OF THEOREM 3.6.9: Without loss of generality we may assume that R is a prime ring with unity satisfying:

(i) \([x^n, y] + [x, y^n] \in \mathcal{Z}(R)\)

(ii) \([x^{n+1}, y] + [x, y^{n+1}] \in \mathcal{Z}(R)\)

for all \( x, y \in R \) and a fixed integer \( n > 1 \)

and replacing \( 1+x \) for \( x \) in condition (i), we get

\([ (1+x)^n, y] + [1+x, y^n] \in \mathcal{Z}(R)\)
or
\[ [x^n, y] + \text{n}[x, y] + \sum_{k=2}^{n-1} \binom{n}{k} [x^k, y] + [x, y^n] \in \mathbb{Z}(R). \]

Using (i) this becomes

\[ (1) \quad n[x, y] + \sum_{k=2}^{n-1} \binom{n}{k} [x^k, y] \in \mathbb{Z}(R). \]

Now replacing \( x \) by \( l+x \) in the condition (ii) of the hypothesis we get

\[ [(l+x)^{n+1}, y] + [l+x, y^{n+1}] \in \mathbb{Z}(R) \]

i.e.

\[ [x^{n+1}, y] + (n+1)[x, y] + \sum_{j=2}^{n} \binom{n+1}{j} [x^j, y] + [x, y^{n+1}] \in \mathbb{Z}(R). \]

Again using condition (ii) we obtain,

\[ (2) \quad (n+1)[x, y] + \sum_{j=2}^{n} \binom{n+1}{j} [x^j, y] \in \mathbb{Z}(R). \]

(1) and (2) yield,

\[ \sum_{k=2}^{n-1} \binom{n}{k} [x^k, y] - \sum_{j=2}^{n} \binom{n+1}{j} [x^j, y] - [x, y] \in \mathbb{Z}(R) \]

or

\[ \left[ \sum_{k=2}^{n-1} \binom{n}{k} x^k - \sum_{j=2}^{n} \binom{n+1}{j} x^j, y \right] - [x, y] \in \mathbb{Z}(R) \]

or

\[ \left[ \sum_{k=2}^{n-1} \binom{n}{k} x^k - \sum_{j=2}^{n} \binom{n+1}{j} x^j - x, y \right] \in \mathbb{Z}(R). \]
Hence for all $x$ and $y$ in $R$, there is a polynomial $p(x)$ with integer coefficients, such that

\[(4) \quad [x^2 p(x) - x, y] \in \mathbb{Z}(R)\]

With $y = yx$ in (4) we get

\[(5) \quad [x^2 p(x) - x, y] x \in \mathbb{Z}(R)\]

(4) and (5) together with Lemma 3.2.1 force that $x \in \mathbb{Z}(R)$, unless $[x^2 p(x) - x, y] = 0$. But if $x \in \mathbb{Z}(R)$, then also $[x^2 p(x) - x, y] = 0$ and so in every case $[x^2 p(x) - x, y] = 0$. Hence by [41, Theorem 3], $R$ is commutative.

The following example shows that the condition for ring to be semi prime in the hypothesis of Theorem 3.6.9 is not superfluous.

**Example 3.6.3**: Let $T$ be a non-commutative ring such that $T^3 = 0$ and $R = T \times \mathbb{Z}/(3)$. Define addition and multiplication in $R$ as follows:

\[(a, n) + (b, m) = (a + b, n + m)\]

and

\[(a, n)(b, m) = (ab + am + bn, nm)\]

Then the Jacobson radical $J(R)$ of $R$ consists of all elements
$(a, o)$ where $ae T$ and $R/J(R) = Z/(3)$, so $R$ is completely primary ring. It can be verified easily that

\begin{enumerate}
  \item $[x^2, y] + [x, y^2] \in Z(R)$
  \item $[x^3, y] + [x, y^3] \in Z(R)$
\end{enumerate}

But $R$ is non-commutative.